CONVERGENCE OF TWO-STEP ITERATIVE SCHEME WITH ERRORS FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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A two-step iterative scheme with errors has been studied to approximate the common fixed points of two asymptotically nonexpansive mappings through weak and strong convergence in Banach spaces.

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(a) For a nonempty subset \( C \) of a normed space \( E \) and \( T : C \to C \), the sequence \( \{x_n\} \) in \( C \), iteratively defined by

\[
\begin{align*}
x_1 &= x \in C, \\
x_{n+1} &= (1-a_n)x_n + a_n T y_n + u_n, \\
y_n &= (1-b_n)x_n + b_n T x_n + v_n, \quad n \geq 1,
\end{align*}
\]

(1.1)

where \( \{a_n\}, \{b_n\} \) are sequences in \([0,1]\) and \( \{u_n\}, \{v_n\} \) are sequences in \( E \) satisfying \( \sum_{n=1}^{\infty} \|u_n\| < \infty \), \( \sum_{n=1}^{\infty} \|v_n\| < \infty \), is known as Ishikawa iterative scheme with errors.

(b) With \( E, C, \) and \( T \) as in (a), the sequence \( \{x_n\} \), iteratively defined by

\[
\begin{align*}
x_1 &= x \in C, \\
x_{n+1} &= (1-a_n)x_n + a_n T x_n + u_n, \quad n \geq 1,
\end{align*}
\]

(1.2)

where \( \{a_n\} \) is a sequence in \([0,1]\) and \( \{u_n\} \) a sequence in \( E \) satisfying \( \sum_{n=1}^{\infty} \|u_n\| < \infty \), is known as Mann iterative scheme with errors.

In 1999, Huang [2] studied the above schemes for asymptotically nonexpansive mappings. Recall that a mapping \( T : C \to C \) is asymptotically nonexpansive if there is a sequence \( \{k_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} k_n = 1 \) and \( \|T^n x - T^n y\| \leq k_n \|x - y\| \) for all \( x, y \in C \) and for all \( n \in \mathbb{N} \), where \( \mathbb{N} \) denotes the set of positive integers.

Moreover, in 2001, Khan and Takahashi [3] approximated the fixed points of two asymptotically nonexpansive mappings \( S, T : C \to C \) through the sequence \( \{x_n\} \) given by

\[
\begin{align*}
x_1 &= x \in C, \\
x_{n+1} &= (1-a_n)x_n + a_n S y_n, \\
y_n &= (1-b_n)x_n + b_n T x_n,
\end{align*}
\]

(1.3)

where \( \{a_n\}, \{b_n\} \) are sequences in \([0,1]\) satisfying certain conditions.
Inspired and motivated by the study of the above schemes, we suggest a new iterative scheme \( \{x_n\} \) in \( C \) constructed through a pair of asymptotically nonexpansive mappings \( S, T : C \to C \) given by

\[
x_1 = x \in C, \\
x_{n+1} = (1-a_n)x_n + a_nS^n y_n + u_n, \\
y_n = (1-b_n)x_n + b_nT^n x_n + v_n, \quad n \geq 1,
\]

where \( \{a_n\}, \{b_n\} \) are sequences in \([0, 1]\) with appropriate conditions and \( \{u_n\}, \{v_n\} \) are sequences in \( E \) with \( \sum_{n=1}^{\infty} \|u_n\| < \infty, \sum_{n=1}^{\infty} \|v_n\| < \infty \).

It is to be noted here that each of the above schemes follows as a special case of our scheme.

2. Preliminaries. Let \( E \) be a Banach space with \( C \) as its nonempty convex subset. Throughout this paper, \( \mathbb{N} \) denotes the set of positive integers and \( F(T) \) the set of fixed points of the mapping \( T \). Now we list the following definitions and results used to prove the results in the next section.

**Definition 2.1.** A mapping \( T : C \to C \) is uniformly \( k \)-Lipschitzian if for some \( k > 0 \),

\[
\|T^n x - T^n y\| \leq k\|x - y\| \quad \text{for all } x, y \in C \text{ and for all } n \in \mathbb{N}.
\]

**Definition 2.2.** A mapping \( T : C \to C \) is completely continuous if and only if \( \{T x_n\} \) has a convergent subsequence for every bounded sequence \( \{x_n\} \) in \( C \).

**Definition 2.3.** \( E \) is said to satisfy Opial’s condition [5] if for any sequence \( \{x_n\} \) in \( E \), \( x_n \rightharpoonup x \) implies that \( \limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\| \) for all \( y \in E \) with \( y \neq x \).

**Definition 2.4.** A mapping \( T : C \to E \) is called demiclosed with respect to \( y \in E \) if for each sequence \( \{x_n\} \) in \( C \) and each \( x \in E \), \( x_n \rightharpoonup x \) and \( Tx_n \to y \) imply that \( x \in C \) and \( Tx = y \).

**Lemma 2.5** [6]. Suppose that \( E \) is a uniformly convex Banach space and \( 0 < p \leq t_n \leq q < 1 \) for all \( n \in \mathbb{N} \). Also, suppose that \( \{x_n\} \) and \( \{y_n\} \) are two sequences of \( E \) such that \( \limsup_{n \to \infty} \|x_n\| \leq r, \limsup_{n \to \infty} \|y_n\| \leq r, \text{ and } \lim_{n \to \infty} \|T_n x_n + (1-t_n) y_n\| = r \) hold for some \( r \geq 0 \). Then \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).

**Lemma 2.6** [7]. Let \( \{r_n\}, \{s_n\}, \{t_n\} \) be three nonnegative sequences satisfying

\[
r_{n+1} \leq (1 + s_n) r_n + t_n \quad \forall n \geq 1.
\]

If \( \sum_{n=1}^{\infty} s_n < \infty \) and \( \sum_{n=1}^{\infty} t_n < \infty \), then \( \lim_{n \to \infty} r_n \) exists.

**Lemma 2.7** [1]. Let \( E \) be a uniformly convex Banach space satisfying Opial’s condition and let \( C \) be a nonempty closed convex subset of \( E \). Let \( T \) be an asymptotically nonexpansive mapping of \( C \) into itself. Then \( I - T \) is demiclosed with respect to zero.
3. Approximating common fixed points. We start with the following lemma.

**Lemma 3.1.** Let $E$ be a normed space and $C$ a nonempty bounded closed convex subset of $E$. Let, for $k > 0$, $S$ and $T$ be uniformly $k$-Lipschitzian mappings of $C$ into itself. Let $\{x_n\}$ be a sequence as defined in (1.4), where $\{u_n\}, \{v_n\}$ are sequences in $E$ such that $\lim_{n \to \infty} \|u_n\| = 0 = \lim_{n \to \infty} \|v_n\|$ and

$$\lim_{n \to \infty} \|x_n - S^n x_n\| = 0 = \lim_{n \to \infty} \|x_n - T^n x_n\|. \quad (3.1)$$

Then

$$\lim_{n \to \infty} \|x_n - S x_n\| = 0 = \lim_{n \to \infty} \|x_n - T x_n\|. \quad (3.2)$$

**Proof.** Take $c_n = \|x_n - T^n x_n\|$ and $d_n = \|x_n - S^n x_n\|$. Consider

$$\|x_{n+1} - x_n\| = \|a_n (S^n y_n - x_n) + u_n\|
\leq a_n \|S^n y_n - x_n + (S^n x_n - x_n)\| + \|u_n\|
\leq a_n k \|(1 - b_n)x_n + b_n T^n x_n + v_n - x_n\| + a_n d_n + \|u_n\|
= a_n k \|b_n (T^n x_n - x_n) + v_n\| + a_n d_n + \|u_n\|
\leq a_n b_n c_n k + a_n k \|v_n\| + a_n d_n + \|u_n\|
\leq c_n k + d_n + k \|v_n\| + \|u_n\|. \quad (3.3)$$

That is,

$$\|x_{n+1} - x_n\| \leq c_n k + d_n + k \|v_n\| + \|u_n\|. \quad (3.4)$$

Next, consider

$$\|x_{n+1} - S x_{n+1}\| = \|(x_{n+1} - S^{n+1} x_{n+1}) + (S^{n+1} x_{n+1} - S x_{n+1})\|
\leq d_{n+1} + k \|(x_{n+1} - x_n) + (x_n - S^n x_n) + (S^n x_n - S^n x_{n+1})\|
\leq d_{n+1} + k d_n + k (k + 1) \|x_{n+1} - x_n\| \quad (3.5)$$

by (3.4). Taking limsup on both sides in the above inequality, we obtain

$$\limsup_{n \to \infty} \|x_{n+1} - S x_{n+1}\| \leq 0. \quad (3.6)$$

That is,

$$\lim_{n \to \infty} \|x_n - S x_n\| = 0. \quad (3.7)$$

Similarly, we can prove that

$$\lim_{n \to \infty} \|x_n - T x_n\| = 0. \quad (3.8)$$

This completes the proof of the lemma.
**Lemma 3.2.** Let $E$ be a uniformly convex Banach space and $C$ its nonempty bounded closed convex subset. Let $S$ and $T$ be self-mappings of $C$ satisfying

$$
||S^n x - S^n y|| \leq k_n ||x - y||, \\
||T^n x - T^n y|| \leq k_n ||x - y||, 
$$

(3.9)

for all $n \in \mathbb{N}$, where $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{x_n\}$ be as in (1.4) with $\{a_n\}, \{b_n\}$ in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$ and $\{u_n\}, \{v_n\}$ in $E$ with $\sum_{n=1}^{\infty} ||u_n|| < \infty$, $\sum_{n=1}^{\infty} ||v_n|| < \infty$. If $F(S) \cap F(T) \neq \emptyset$, then $\lim_{n \to \infty} ||x_n - Sx_n|| = 0 = \lim_{n \to \infty} ||x_n - Tx_n||.$

**Proof.** Let $p \in F(S) \cap F(T)$. Then

$$
||x_{n+1} - p|| \\
= ||a_n (S^n y_n - p) + (1 - a_n) (x_n - p) + u_n|| \\
\leq a_n k_n ||y_n - p|| + (1 - a_n) ||x_n - p|| + ||u_n|| \\
= a_n k_n ||(1 - b_n) x_n + b_n T^n x_n + v_n - p|| + (1 - a_n) ||x_n - p|| + ||u_n|| \\
= a_n k_n ||b_n (T^n x_n - p) + (1 - b_n) (x_n - p) + v_n|| + (1 - a_n) ||x_n - p|| + ||u_n|| \\
\leq a_n b_n k_n^2 ||x_n - p|| + a_n k_n ||v_n|| + a_n (1 - b_n) k_n ||x_n - p|| + (1 - a_n) ||x_n - p|| + ||u_n|| \\
= (1 + a_n b_n k_n^2 + a_n (1 - b_n) k_n - a_n) ||x_n - p|| + a_n k_n ||v_n|| + ||u_n||. 
$$

(3.10)

Since $\{k_n\}$ is a bounded sequence, therefore there exists $h > 0$ such that $k_n \leq h$ for all $n \geq 1$ so that

$$
||x_{n+1} - p|| \leq [1 + a_n b_n h (k_n - 1) + a_n (k_n - 1)] ||x_n - p|| + a_n h ||v_n|| + ||u_n||. 
$$

(3.11)

Take $s_n = a_n b_n h (k_n - 1) + a_n (k_n - 1)$, $t_n = a_n h ||v_n|| + ||u_n||$, and $r_n = ||x_n - p||$. As $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, so $\lim_{n \to \infty} ||x_n - p||$ exists by Lemma 2.6. Let $\lim_{n \to \infty} ||x_n - p|| = c$, where $c \geq 0$ is a real number. Assume that $c > 0$, as the result for the case $c = 0$ is obviously true. Now $||T^n x_n - p|| \leq k_n ||x_n - p||$ for all $n \in \mathbb{N}$ gives $\limsup_{n \to \infty} ||T^n x_n - p|| \leq c$. Also,

$$
||y_n - p|| = ||b_n (T^n x_n - p) + (1 - b_n) (x_n - p) + v_n|| \\
\leq ||x_n - p|| + (k_n - 1) b_n ||x_n - p|| + ||v_n|| 
$$

(3.12)

gives

$$
\limsup_{n \to \infty} ||y_n - p|| \leq c. 
$$

(3.13)

Next, consider

$$
||S^n y_n - p + a_n^{-1} u_n|| \leq k_n ||y_n - p|| + a_n^{-1} ||u_n|| \leq k_n ||y_n - p|| + \frac{1}{\delta} ||u_n||. 
$$

(3.14)
By the above inequality and by virtue of \( \|u_n\| \to 0 \) and \( k_n \to 1 \) as \( n \to \infty \), we get
\[
\limsup_{n \to \infty} \|S^n y_n - p + a_n^{-1} u_n\| \leq c. \tag{3.15}
\]
Moreover, \( c = \lim_{n \to \infty} \|x_{n+1} - p\| \) means that
\[
\lim_{n \to \infty} \|a_n (S^n y_n - p + a_n^{-1} u_n) + (1 - a_n) (x_n - p)\| = c. \tag{3.16}
\]
Applying Lemma 2.5,
\[
\lim_{n \to \infty} \|S^n y_n - x_n + a_n^{-1} u_n\| = 0. \tag{3.17}
\]
Thus
\[
\|S^n y_n - x_n\| \leq \|S^n y_n - x_n + a_n^{-1} u_n\| + \frac{1}{\delta} \|u_n\| \tag{3.18}
\]
yields that
\[
\lim_{n \to \infty} \|S^n y_n - x_n\| = 0. \tag{3.19}
\]
Also, then
\[
\|x_n - p\| \leq \|x_n - S^n y_n\| + \|S^n y_n - p\| \leq \|x_n - S^n y_n\| + k_n \|y_n - p\| \tag{3.20}
\]
implies that
\[
c \leq \liminf_{n \to \infty} \|y_n - p\|. \tag{3.21}
\]
By (3.13) and (3.21), we obtain
\[
\lim_{n \to \infty} \|y_n - p\| = c. \tag{3.22}
\]
That is,
\[
\lim_{n \to \infty} \|b_n (T^n x_n - p + b_n^{-1} v_n) + (1 - b_n) (x_n - p)\| = c. \tag{3.23}
\]
Again by Lemma 2.5, we get
\[
\lim_{n \to \infty} \|T^n x_n - x_n + b_n^{-1} v_n\| = 0, \tag{3.24}
\]
which finally gives that
\[
\lim_{n \to \infty} \|T^n x_n - x_n\| = 0. \tag{3.25}
\]
Now
\[
\|S^n x_n - x_n\| \leq \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\|
\leq k_n b_n \|T^n x_n - x_n\| + \|v_n\| + \|S^n y_n - x_n\| \tag{3.26}
\]
implies, together with (3.19) and (3.25), that
\[
\lim_{n \to \infty} \| S^n x_n - x_n \| = 0 = \lim_{n \to \infty} \| T^n x_n - x_n \|.
\] (3.27)

Lemma 3.1 now reveals that
\[
\lim_{n \to \infty} \| S x_n - x_n \| = \lim_{n \to \infty} \| T x_n - x_n \|,
\] (3.28)
which is as desired.

**Theorem 3.3.** Let \( E \) be a uniformly convex Banach space satisfying Opial’s condition and let \( C, S, T, \) and \( \{ x_n \} \) be as taken in Lemma 3.2. If \( F(S) \cap F(T) \neq \emptyset \), then \( \{ x_n \} \) converges weakly to a common fixed point of \( S \) and \( T \).

**Proof.** Let \( p \in F(S) \cap F(T) \). Then, as proved in Lemma 3.2, \( \lim_{n \to \infty} \| x_n - p \| \) exists. Now we prove that \( \{ x_n \} \) has a unique weak subsequential limit in \( F(S) \cap F(T) \). To prove this, let \( w_1 \) and \( w_2 \) be weak limits of the subsequences \( \{ x_{n_i} \} \) and \( \{ x_{n_j} \} \) of \( \{ x_n \} \), respectively. By Lemma 3.2, \( \lim_{n \to \infty} \| x_n - S x_n \| = 0 \) and \( I - S \) is demiclosed with respect to zero by Lemma 2.7; therefore, we obtain \( S w_1 = w_1 \). Similarly, \( T w_1 = w_1 \). Again, in the same way, we can prove that \( w_2 \in F(S) \cap F(T) \). Next, we prove the uniqueness. For this, suppose that \( w_1 \neq w_2 \); then by Opial’s condition,
\[
\lim_{n \to \infty} \| x_n - w_1 \| = \lim_{n_i \to \infty} \| x_{n_i} - w_1 \| < \lim_{n_i \to \infty} \| x_{n_i} - w_2 \| \\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (3.29)
\]
a contradiction. Hence the proof is over.

**Remark 3.4.** If we take \( u_n = v_n = 0 \) for all \( n \in \mathbb{N} \), the above theorem reduces to [3, Theorem 1] of Khan and Takahashi. Moreover, [6, Theorem 2.1] of Schu becomes a special case of the above theorem when \( u_n = v_n = 0 \) as well as \( T = I \), the identity mapping.

Finally, we approximate common fixed points by the following strong convergence theorem.

**Theorem 3.5.** Let \( E \) be a uniformly convex Banach space and \( C \) its bounded closed convex subset. Let \( S, T, \) and \( \{ x_n \} \) be as taken in Lemma 3.2. If \( F(S) \cap F(T) \neq \emptyset \) and either \( S \) or \( T \) is completely continuous, then \( \{ x_n \} \) converges strongly to a common fixed point of \( S \) and \( T \).

**Proof.** Assume that \( T : C \to C \) is completely continuous. Since \( \{ x_n \} \) is a bounded sequence and \( T \) is completely continuous, therefore \( \{ T x_n \} \) must have a convergent subsequence \( \{ T x_{n_i} \} \). Hence by (3.28), \( \{ x_n \} \) must have a subsequence \( \{ x_{n_i} \} \) such that \( x_{n_i} \to q \) (say) in \( C \) as \( n_i \to \infty \). Now continuity of \( S \) and \( T \) gives that \( S x_{n_i} \to S q \) and \( T x_{n_i} \to T q \) as \( n_i \to \infty \). Then, again by (3.28), \( \| S q - q \| = 0 = \| T q - q \| \). This yields that \( q \in F(S) \cap F(T) \) so that \( \{ x_{n_i} \} \) converges strongly to \( q \) in \( F(S) \cap F(T) \). As proved in
Lemma 3.2, $\lim_{n \to \infty} \|x_n - p\|$ exists for all $p \in F(S) \cap F(T)$; therefore, $\{x_n\}$ must itself converge to $q \in F(S) \cap F(T)$. Hence the proof.

**Remark 3.6.** If we put $T = I$, $v_n = 0$ in the above theorem, then [2, Theorem 1] of Huang is obtained. When we take $S = T$ in the above theorem, then [2, Theorem 2] of Huang follows except when $b_n = 0$. Since a self-mapping with compact domain is completely continuous, therefore [3, Theorem 2] of Khan and Takahashi can also be obtained by putting $u_n = v_n = 0$. It is also worth mentioning that the results presented in this paper are for two mappings while the results in Huang [2] are for one mapping only. Meanwhile, calculations in this paper are made much simpler as compared to Huang [2].

**References**


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