IJMMS 2004:37, 1965–1971
PII. S0161171204308161
http://ijmms.hindawi.com

CONVERGENCE OF TWO-STEP ITERATIVE SCHEME WITH ERRORS FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Received 28 August 2003

A two-step iterative scheme with errors has been studied to approximate the common fixed points of two asymptotically nonexpansive mappings through weak and strong convergence in Banach spaces.

2000 Mathematics Subject Classification: 47H09, 49M05.

1. Introduction. In 1995, Liu [4] introduced iterative schemes with errors as follows.

(a) For a nonempty subset C of a normed space E and $T : C \to C$, the sequence $\{x_n\}$ in C, iteratively defined by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-a_n)x_n + a_n T y_n + u_n, \\ y_n &= (1-b_n)x_n + b_n T x_n + v_n, \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$ are sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ are sequences in E satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$, is known as Ishikawa iterative scheme with errors.

(b) With E, C, and T as in (a), the sequence $\{x_n\}$, iteratively defined by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-a_n)x_n + a_n T x_n + u_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

where $\{a_n\}$ is a sequence in $[0,1]$ and $\{u_n\}$ a sequence in E satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$, is known as Mann iterative scheme with errors.

In 1999, Huang [2] studied the above schemes for asymptotically nonexpansive mappings. Recall that a mapping $T : C \to C$ is asymptotically nonexpansive if there is a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers.

Moreover, in 2001, Khan and Takahashi [3] approximated the fixed points of two asymptotically nonexpansive mappings $S, T : C \to C$ through the sequence $\{x_n\}$ given by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-a_n)x_n + a_n S^n y_n, \\ y_n &= (1-b_n)x_n + b_n T^n x_n, \end{aligned} \tag{1.3}$$

where $\{a_n\}$, $\{b_n\}$ are sequences in $[0,1]$ satisfying certain conditions.

Inspired and motivated by the study of the above schemes, we suggest a new iterative scheme $\{x_n\}$ in C constructed through a pair of asymtotically nonexpansive mappings $S,T : C \to C$ given by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-a_n)x_n + a_n S^n y_n + u_n, \\ y_n &= (1-b_n)x_n + b_n T^n x_n + v_n, \quad n \geq 1, \end{aligned} \tag{1.4}$$

where $\{a_n\}$, $\{b_n\}$ are sequences in $[0,1]$ with appropriate conditions and $\{u_n\}$, $\{v_n\}$ are sequences in E with $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

It is to be noted here that each of the above schemes follows as a special case of our scheme.

2. Preliminaries. Let E be a Banach space with C as its nonempty convex subset. Throughout this paper, $\mathbb{N}$ denotes the set of positive integers and $F(T)$ the set of fixed points of the mapping T. Now we list the following definitions and results used to prove the results in the next section.

Definition 2.1. A mapping $T : C \to C$ is uniformly k-Lipschitzian if for some $k > 0$, $\|T^n x - T^n y\| \leq k\|x - y\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$.

Definition 2.2. A mapping $T : C \to C$ is completely continuous if and only if $\{Tx_n\}$ has a convergent subsequence for every bounded sequence $\{x_n\}$ in C.

Definition 2.3. E is said to satisfy Opial's condition [5] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.

Definition 2.4. A mapping $T : C \to E$ is called demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Lemma 2.5 [6]. *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.6 [7]. *Let $\{r_n\}$, $\{s_n\}$, $\{t_n\}$ be three nonnegative sequences satisfying*

$$r_{n+1} \leq (1+s_n)r_n + t_n \quad \forall n \geq 1. \tag{2.1}$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

Lemma 2.7 [1]. *Let E be a uniformly convex Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let T be an asymptotically nonexpansive mapping of C into itself. Then $I - T$ is demiclosed with respect to zero.*

3. Approximating common fixed points. We start with the following lemma.

Lemma 3.1. *Let E be a normed space and C a nonempty bounded closed convex subset of E. Let, for $k > 0$, S and T be uniformly k-Lipschitzian mappings of C into itself. Let $\{x_n\}$ be a sequence as defined in (1.4), where $\{u_n\}$, $\{v_n\}$ are sequences in E such that $\lim_{n\to\infty}\|u_n\| = 0 = \lim_{n\to\infty}\|v_n\|$ and*

$$\lim_{n\to\infty}\|x_n - S^n x_n\| = 0 = \lim_{n\to\infty}\|x_n - T^n x_n\|. \tag{3.1}$$

Then

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0 = \lim_{n\to\infty}\|x_n - Tx_n\|. \tag{3.2}$$

Proof. Take $c_n = \|x_n - T^n x_n\|$ and $d_n = \|x_n - S^n x_n\|$. Consider

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|a_n(S^n y_n - x_n) + u_n\| \\
&\le a_n\|(S^n y_n - S^n x_n) + (S^n x_n - x_n)\| + \|u_n\| \\
&\le a_n k\|(1-b_n)x_n + b_n T^n x_n + v_n - x_n\| + a_n d_n + \|u_n\| \\
&= a_n k\|b_n(T^n x_n - x_n) + v_n\| + a_n d_n + \|u_n\| \\
&\le a_n b_n c_n k + a_n k\|v_n\| + a_n d_n + \|u_n\| \\
&\le c_n k + d_n + k\|v_n\| + \|u_n\|.
\end{aligned} \tag{3.3}$$

That is,

$$\|x_{n+1} - x_n\| \le c_n k + d_n + k\|v_n\| + \|u_n\|. \tag{3.4}$$

Next, consider

$$\begin{aligned}
\|x_{n+1} - Sx_{n+1}\| &= \|(x_{n+1} - S^{n+1}x_{n+1}) + (S^{n+1}x_{n+1} - Sx_{n+1})\| \\
&\le d_{n+1} + k\|(x_{n+1} - x_n) + (x_n - S^n x_n) + (S^n x_n - S^n x_{n+1})\| \\
&\le d_{n+1} + kd_n + k(k+1)\|x_{n+1} - x_n\| \\
&\le d_{n+1} + kd_n + k(k+1)[c_n k + d_n + k\|v_n\| + \|u_n\|]
\end{aligned} \tag{3.5}$$

by (3.4). Taking limsup on both sides in the above inequality, we obtain

$$\limsup_{n\to\infty}\|x_{n+1} - Sx_{n+1}\| \le 0. \tag{3.6}$$

That is,

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{3.7}$$

Similarly, we can prove that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.8}$$

This completes the proof of the lemma. □

Lemma 3.2. *Let E be a uniformly convex Banach space and C its nonempty bounded closed convex subset. Let S and T be self-mappings of C satisfying*

$$\begin{aligned} \|S^n x - S^n y\| &\le k_n \|x-y\|, \\ \|T^n x - T^n y\| &\le k_n \|x-y\|, \end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}$, where $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$. Let $\{x_n\}$ be as in (1.4) with $\{a_n\}$, $\{b_n\}$ in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $\{u_n\}$, $\{v_n\}$ in E with $\sum_{n=1}^{\infty}\|u_n\| < \infty$, $\sum_{n=1}^{\infty}\|v_n\| < \infty$. If $F(S)\cap F(T) \neq \phi$, then $\lim_{n\to\infty}\|x_n - Sx_n\| = 0 = \lim_{n\to\infty}\|x_n - Tx_n\|$.

Proof. Let $p \in F(S)\cap F(T)$. Then

$$\begin{aligned} &\|x_{n+1}-p\| \\ &\quad = \|a_n(S^n y_n - p) + (1-a_n)(x_n - p) + u_n\| \\ &\quad \le a_n k_n \|y_n - p\| + (1-a_n)\|x_n - p\| + \|u_n\| \\ &\quad = a_n k_n \|(1-b_n)x_n + b_n T^n x_n + v_n - p\| + (1-a_n)\|x_n - p\| + \|u_n\| \\ &\quad = a_n k_n \|b_n(T^n x_n - p) + (1-b_n)(x_n - p) + v_n\| + (1-a_n)\|x_n - p\| + \|u_n\| \\ &\quad \le a_n b_n k_n^2 \|x_n - p\| + a_n k_n \|v_n\| + a_n(1-b_n)k_n\|x_n - p\| + (1-a_n)\|x_n - p\| + \|u_n\| \\ &\quad = (1 + a_n b_n k_n^2 + a_n(1-b_n)k_n - a_n)\|x_n - p\| + a_n k_n \|v_n\| + \|u_n\|. \end{aligned} \tag{3.10}$$

Since $\{k_n\}$ is a bounded sequence, therefore there exists $h > 0$ such that $k_n \le h$ for all $n \ge 1$ so that

$$\|x_{n+1}-p\| \le [1 + a_n b_n h(k_n - 1) + a_n(k_n - 1)]\|x_n - p\| + a_n h\|v_n\| + \|u_n\|. \tag{3.11}$$

Take $s_n = a_n b_n h(k_n-1) + a_n(k_n-1)$, $t_n = a_n h\|v_n\| + \|u_n\|$, and $r_n = \|x_n - p\|$. As $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, so $\lim_{n\to\infty}\|x_n - p\|$ exists by Lemma 2.6. Let $\lim_{n\to\infty}\|x_n - p\| = c$, where $c \ge 0$ is a real number. Assume that $c > 0$, as the result for the case $c = 0$ is obviously true. Now $\|T^n x_n - p\| \le k_n\|x_n - p\|$ for all $n \in \mathbb{N}$ gives $\limsup_{n\to\infty}\|T^n x_n - p\| \le c$. Also,

$$\begin{aligned} \|y_n - p\| &= \|b_n(T^n x_n - p) + (1-b_n)(x_n - p) + v_n\| \\ &\le \|x_n - p\| + (k_n - 1)b_n\|x_n - p\| + \|v_n\| \end{aligned} \tag{3.12}$$

gives

$$\limsup_{n\to\infty}\|y_n - p\| \le c. \tag{3.13}$$

Next, consider

$$\|S^n y_n - p + a_n^{-1}u_n\| \le k_n\|y_n - p\| + a_n^{-1}\|u_n\| \le k_n\|y_n - p\| + \frac{1}{\delta}\|u_n\|. \tag{3.14}$$

By the above inequality and by virtue of $\|u_n\| \to 0$ and $k_n \to 1$ as $n \to \infty$, we get

$$\limsup_{n\to\infty} \|S^n y_n - p + a_n^{-1} u_n\| \le c. \tag{3.15}$$

Moreover, $c = \lim_{n\to\infty} \|x_{n+1} - p\|$ means that

$$\lim_{n\to\infty} \|a_n (S^n y_n - p + a_n^{-1} u_n) + (1 - a_n)(x_n - p)\| = c. \tag{3.16}$$

Applying Lemma 2.5,

$$\lim_{n\to\infty} \|S^n y_n - x_n + a_n^{-1} u_n\| = 0. \tag{3.17}$$

Thus

$$\|S^n y_n - x_n\| \le \|S^n y_n - x_n + a_n^{-1} u_n\| + \frac{1}{\delta}\|u_n\| \tag{3.18}$$

yields that

$$\lim_{n\to\infty} \|S^n y_n - x_n\| = 0. \tag{3.19}$$

Also, then

$$\|x_n - p\| \le \|x_n - S^n y_n\| + \|S^n y_n - p\| \le \|x_n - S^n y_n\| + k_n \|y_n - p\| \tag{3.20}$$

implies that

$$c \le \liminf_{n\to\infty} \|y_n - p\|. \tag{3.21}$$

By (3.13) and (3.21), we obtain

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{3.22}$$

That is,

$$\lim_{n\to\infty} \|b_n (T^n x_n - p + b_n^{-1} v_n) + (1 - b_n)(x_n - p)\| = c. \tag{3.23}$$

Again by Lemma 2.5, we get

$$\lim_{n\to\infty} \|T^n x_n - x_n + b_n^{-1} v_n\| = 0, \tag{3.24}$$

which finally gives that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.25}$$

Now

$$\begin{aligned} \|S^n x_n - x_n\| &\le \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\| \\ &\le k_n b_n \|T^n x_n - x_n\| + \|v_n\| + \|S^n y_n - x_n\| \end{aligned} \tag{3.26}$$

implies, together with (3.19) and (3.25), that

$$\lim_{n\to\infty}\|S^n x_n - x_n\| = 0 = \lim_{n\to\infty}\|T^n x_n - x_n\|. \tag{3.27}$$

Lemma 3.1 now reveals that

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0 = \lim_{n\to\infty}\|Tx_n - x_n\|, \tag{3.28}$$

which is as desired. □

Theorem 3.3. *Let E be a uniformly convex Banach space satisfying Opial's condition and let C, S, T, and $\{x_n\}$ be as taken in Lemma 3.2. If $F(S)\cap F(T)\neq\phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let $p\in F(S)\cap F(T)$. Then, as proved in Lemma 3.2, $\lim_{n\to\infty}\|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(S)\cap F(T)$. To prove this, let w_1 and w_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 3.2, $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and $I-S$ is demiclosed with respect to zero by Lemma 2.7; therefore, we obtain $Sw_1 = w_1$. Similarly, $Tw_1 = w_1$. Again, in the same way, we can prove that $w_2\in F(S)\cap F(T)$. Next, we prove the uniqueness. For this, suppose that $w_1\neq w_2$; then by Opial's condition,

$$\begin{aligned}\lim_{n\to\infty}\|x_n - w_1\| &= \lim_{n_i\to\infty}\|x_{n_i} - w_1\| < \lim_{n_i\to\infty}\|x_{n_i} - w_2\| \\ &= \lim_{n\to\infty}\|x_n - w_2\| = \lim_{n_j\to\infty}\|x_{n_j} - w_2\| \\ &< \lim_{n_j\to\infty}\|x_{n_j} - w_1\| = \lim_{n\to\infty}\|x_n - w_1\|,\end{aligned} \tag{3.29}$$

a contradiction. Hence the proof is over. □

Remark 3.4. If we take $u_n = v_n = 0$ for all $n\in\mathbb{N}$, the above theorem reduces to [3, Theorem 1] of Khan and Takahashi. Moreover, [6, Theorem 2.1] of Schu becomes a special case of the above theorem when $u_n = v_n = 0$ as well as $T = I$, the identity mapping.

Finally, we approximate common fixed points by the following strong convergence theorem.

Theorem 3.5. *Let E be a uniformly convex Banach space and C its bounded closed convex subset. Let S, T, and $\{x_n\}$ be as taken in Lemma 3.2. If $F(S)\cap F(T)\neq\phi$ and either S or T is completely continuous, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. Assume that $T: C\to C$ is completely continuous. Since $\{x_n\}$ is a bounded sequence and T is completely continuous, therefore $\{Tx_n\}$ must have a convergent subsequence $\{Tx_{n_i}\}$. Hence by (3.28), $\{x_n\}$ must have a subsequence $\{x_{n_i}\}$ such that $x_{n_i}\to q$ (say) in C as $n_i\to\infty$. Now continuity of S and T gives that $Sx_{n_i}\to Sq$ and $Tx_{n_i}\to Tq$ as $n_i\to\infty$. Then, again by (3.28), $\|Sq - q\| = 0 = \|Tq - q\|$. This yields that $q\in F(S)\cap F(T)$ so that $\{x_{n_i}\}$ converges strongly to q in $F(S)\cap F(T)$. As proved in

Lemma 3.2, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(S)\cap F(T)$; therefore, $\{x_n\}$ must itself converge to $q \in F(S)\cap F(T)$. Hence the proof. □

Remark 3.6. If we put $T = I$, $v_n = 0$ in the above theorem, then [2, Theorem 1] of Huang is obtained. When we take $S = T$ in the above theorem, then [2, Theorem 2] of Huang follows except when $b_n = 0$. Since a self-mapping with compact domain is completely continuous, therefore [3, Theorem 2] of Khan and Takahashi can also be obtained by putting $u_n = v_n = 0$. It is also worth mentioning that the results presented in this paper are for two mappings while the results in Huang [2] are for one mapping only. Meanwhile, calculations in this paper are made much simpler as compared to Huang [2].

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