ON THE TOPOLOGY OF D -METRIC SPACES AND GENERATION OF D -METRIC SPACES FROM METRIC SPACES

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An example of a D -metric space is given, in which D -metric convergence does not define a topology and in which a convergent sequence can have infinitely many limits. Certain methods for constructing D -metric spaces from a given metric space are developed and are used in constructing (1) an example of a D -metric space in which D -metric convergence defines a topology which is T_1 but not Hausdorff, and (2) an example of a D -metric space in which D -metric convergence defines a metrizable topology but the D -metric is not continuous even in a single variable.

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1. Introduction. Dhage [2] introduced the notion of D -metric spaces and claimed that D -metric convergence defines a Hausdorff topology and that the D -metric is (sequentially) continuous in all the three variables. Many authors (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]) have taken these claims for granted and used them in proving fixed point theorems in D -metric spaces.

In this paper, we give examples to show that in a D -metric space

- (1) D -metric convergence does not always define a topology,
- (2) even when D -metric convergence defines a topology, it need not be Hausdorff,
- (3) even when D -metric convergence defines a metrizable topology, the D -metric need not be continuous even in a single variable.

In fact, we develop certain methods for constructing D -metric spaces from a given metric space and obtain from them, as by-products, examples illustrating the last two assertions. We also introduce the notions of strong convergence, and very strong convergence in a D -metric space and study in a decisive way the mutual implications among the three notions of convergence, strong convergence, and very strong convergence.

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^+$ the set of all non-negative real numbers, $\mathbb{N}$ the set of all positive integers, and $(\mathbb{R}^+)^{3*} = \{(t_1, t_2, t_3) \in (\mathbb{R}^+)^3 : t_1 \leq t_2 + t_3, t_2 \leq t_3 + t_1, t_3 \leq t_1 + t_2\}$.

NOTE 1.1. If (X, d) is a metric space, then $(d(x, y), d(y, z), d(z, x)) \in (\mathbb{R}^+)^{3*}$ for all $x, y, z \in X$.

DEFINITION 1.2 [2]. Let X be a nonempty set. A function $\rho : X \times X \times X \rightarrow [0, \infty)$ is called a D -metric on X if

- (i) $\rho(x, y, z) = 0$ if and only if $x = y = z$ (coincidence),

- (ii) $\rho(x, y, z) = \rho(p(x, y, z))$ for all $x, y, z \in X$ and for any permutation $p(x, y, z)$ of x, y, z (symmetry),
- (iii) $\rho(x, y, z) \leq \rho(x, y, a) + \rho(x, a, z) + \rho(a, y, z)$ for all $x, y, z, a \in X$ (tetrahedral inequality).

If X is a nonempty set and ρ is D -metric on X , then the ordered pair (X, ρ) is called a D -metric space. When the D -metric ρ is understood, we say that X is a D -metric space.

DEFINITION 1.3 [2, 8]. A sequence $\{x_n\}$ in a D -metric space (X, ρ) is said to be convergent (or ρ -convergent) if there exists an element x of X with the following property: given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\rho(x_m, x_n, x) < \varepsilon$ for all $m, n \geq N$. In such a case, $\{x_n\}$ is said to converge to x and x is called a limit of $\{x_n\}$.

DEFINITION 1.4 [2, 8]. A sequence $\{x_n\}$ in a D -metric space (X, ρ) is said to be Cauchy (or ρ -Cauchy) if, given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\rho(x_m, x_n, x_p) < \varepsilon$ for all $m, n, p \geq N$.

REMARK 1.5. The definition of ρ -Cauchy sequence as given by Dhage [2] appears to be slightly different from Definition 1.4, but it is actually equivalent to it. It can be shown that in a D -metric space every convergent sequence is Cauchy.

DEFINITION 1.6 [2, 8]. A D -metric space (X, ρ) is said to be complete (or ρ -complete) if every ρ -Cauchy sequence in X is ρ -convergent in X .

NOTATION 1.7. For a subset E of a D -metric space (X, ρ) , E^c denotes $\{x \in X: \text{there is a sequence in } E \text{ which converges to } x \text{ under the } D\text{-metric } \rho\}$. For any set X , $P(X)$ denotes the power set of X , that is, the collection of all subsets of X .

We now give an example of a complete D -metric space in which D -metric convergence does not define a topology and in which there are convergent sequences with infinitely many limits.

EXAMPLE 1.8. Let $X = A \cup B \cup \{0\}$, where $A = \{1/2^n : n \in \mathbb{N}\}$ and $B = \{2^n : n \in \mathbb{N}\}$.

Define $\rho : X \times X \times X \rightarrow \mathbb{R}^+$ as follows:

- (i) $\rho(x, y, z) = 0$ if $x = y = z$,
- (ii) $\rho(x, y, z) = \min\{\max\{x, y\}, \max\{y, z\}, \max\{z, x\}\}$ if $x, y, z \in A \cup \{0\}$, 0 does not occur more than once among x, y, z , and at least two among x, y, z are distinct,
- (iii) $\rho(x, y, z) = 1$ if 0 and at least one element of B occur among x, y, z , or 0 occurs exactly twice among x, y, z ,
- (iv) $\rho(x, y, z) = \min\{x, y, z\}$ if $x, y, z \in A \cup B$ and exactly one element of B occurs exactly once among x, y, z ,
- (v) $\rho(x, y, z) = \min\{\max\{1/x, 1/y\}, \max\{1/y, 1/z\}, \max\{1/z, 1/x\}\}$, if $x, y, z \in A \cup B$ and exactly one element of A occurs exactly once among x, y, z ,
- (vi) $\rho(x, y, z) = |1/x - 1/y| + |1/y - 1/z| + |1/z - 1/x|$ if $x, y, z \in B$.

Then (X, ρ) is a complete D -metric space. But ρ -convergence does not define a topology on X .

PROOF. Clearly ρ is symmetric in all the three variables and $\rho(x, y, z) = 0$ if and only if $x = y = z$. We note that $\rho(x, y, z) \leq 1$ for all $x, y, z \in X$. Let $x, y, z, u \in X$.

CASE (i). $x = y = z$.

Then $\rho(x, y, z) = 0 \leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u)$.

CASE (ii). $x, y, z \in A \cup \{0\}$, 0 does not occur more than once among x, y, z , and at least two among x, y, z are distinct.

We may assume that $x \geq y \geq z$. If $u \in A \cup \{0\}$, then $\rho(x, y, z) = y \leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u)$, since when $u > y$, $\rho(u, y, z) = y$; when $u = y$, $\rho(x, y, z) = \rho(x, u, z)$; and when $u < y$, $\rho(x, y, u) = y$. If $u \in B$, then

$$\begin{aligned} \rho(x, y, z) &= y = \min\{x, y, u\} = \rho(x, y, u) \\ &\leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u). \end{aligned} \quad (1.1)$$

CASE (iii). 0 occurs exactly twice among x, y, z .

We may assume that $x = y = 0$. Then $z \neq 0$. If $u \in X \setminus \{0\}$, then

$$\begin{aligned} \rho(x, y, z) &= \rho(0, 0, z) = 1 = \rho(0, 0, u) = \rho(x, y, u) \\ &\leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u). \end{aligned} \quad (1.2)$$

If $u = 0$, then

$$\begin{aligned} \rho(x, y, z) &= \rho(0, 0, z) = 1 = \rho(u, y, z) \\ &\leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u). \end{aligned} \quad (1.3)$$

CASE (iv). 0 and at least one element of B occur among x, y, z .

We may assume that $x = 0$ and $y \in B$. Then

$$\begin{aligned} \rho(x, y, z) &= \rho(0, y, z) = 1 = \rho(0, y, u) = \rho(x, y, u) \\ &\leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u). \end{aligned} \quad (1.4)$$

CASE (v). $x, y, z \in A \cup B$ and exactly one element of B occurs exactly once among x, y, z .

We may assume that $x \in B$. Then $y, z \in A$. We may also assume that $y \geq z$. If $u \in B$, then

$$\begin{aligned} \rho(x, y, z) &= \min\{x, y, z\} = z = \rho(u, y, z) \\ &\leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u). \end{aligned} \quad (1.5)$$

If $u \in A \cup \{0\}$, then

$$\rho(x, y, z) = \min\{x, y, z\} = z \leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u), \quad (1.6)$$

since when $u < z$, $\rho(u, y, z) = z$; when $u = z$, $\rho(x, y, z) = \rho(x, y, u)$; and when $u > z$, $\rho(u, y, z) = \min\{u, y\} \geq z$.

CASE (vi). $x, y, z \in A \cup B$ and exactly one element of A occurs exactly once among x, y, z .

We may assume that $x \in A$. Then $y, z \in B$. We may also assume that $y \geq z$. If $u \in A$, then

$$\begin{aligned}\rho(x, y, z) &= \max\left\{\frac{1}{y}, \frac{1}{z}\right\} = \frac{1}{z} = \rho(u, y, z) \\ &\leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u).\end{aligned}\tag{1.7}$$

If $u = 0$, then

$$\begin{aligned}\rho(x, y, z) &= \max\left\{\frac{1}{y}, \frac{1}{z}\right\} = \frac{1}{z} \leq \frac{1}{2} < 1 = \rho(u, y, z) \\ &\leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u).\end{aligned}\tag{1.8}$$

If $u \in B$, then

$$\begin{aligned}\rho(x, y, z) &= \max\left\{\frac{1}{y}, \frac{1}{z}\right\} = \frac{1}{z} \\ &\leq \max\left\{\frac{1}{u}, \frac{1}{z}\right\} = \rho(x, u, z) \\ &\leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u).\end{aligned}\tag{1.9}$$

CASE (vii). $x, y, z \in B$.

We may assume that $1/x \geq 1/y \geq 1/z$. If $u = 0$, then

$$\begin{aligned}\rho(x, y, z) &= 2\left(\frac{1}{x} - \frac{1}{z}\right) < 1 = \rho(u, y, z) \\ &\leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u).\end{aligned}\tag{1.10}$$

If $u \in A$, then

$$\begin{aligned}\rho(x, y, z) &= 2\left(\frac{1}{x} - \frac{1}{z}\right) \leq \frac{1}{x} + \frac{1}{x} \\ &= \rho(x, u, z) + \rho(x, y, u) \\ &\leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u).\end{aligned}\tag{1.11}$$

If $u \in B$, then

$$\begin{aligned}\rho(x, y, z) &= \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{x}\right| \\ &\leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u),\end{aligned}\tag{1.12}$$

since $|1/x - 1/y| \leq \rho(x, y, u)$, $|1/y - 1/z| \leq \rho(u, y, z)$, and $|1/z - 1/x| \leq \rho(x, u, z)$. Thus, for all $x, y, z, u \in X$, we have $\rho(x, y, z) \leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u)$. Hence ρ is a D -metric on X .

To show that (X, ρ) is D -complete.

Let $\{x_n\}$ be a Cauchy sequence in X .

CASE 1. There exists $N \in \mathbb{N}$ such that $x_n = x_N$ for all $n \geq N$.

In this case, evidently $\{x_n\}$ converges to x_N .

CASE 2. Given $N \in \mathbb{N}$, there exist $i, j \in \mathbb{N}$ such that $i > N, j > N$, and $x_i \neq x_j$.

Then there exists $N_0 \in \mathbb{N}$ such that $x_i \neq 0$ for each $i \geq N_0$, since $\rho(0, 0, x) = 1$, for all $x \in X \setminus \{0\}$, and $\{x_n\}$ is Cauchy.

SUBCASE (i). There exists $N_1 \in \mathbb{N}$ such that $N_1 \geq N_0$ and $x_i \in A$ for all $i \geq N_1$.

CLAIM 1.9. $x_n \rightarrow 0$ as $n \rightarrow \infty$ in the usual sense.

Suppose the claim does not hold. Then there exists a positive real number ε such that $x_n \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$. Given $N \in \mathbb{N}$, we can choose $i, j, k \in \mathbb{N}$ such that $k > j > i > \max\{N, N_1\}$, $x_i \geq \varepsilon$, $x_j \geq \varepsilon$, and $x_k \neq x_j$. Then

$$\rho(x_i, x_j, x_k) = \min\{\max\{x_i, x_j\}, \max\{x_j, x_k\}, \max\{x_k, x_i\}\} \geq \varepsilon. \quad (1.13)$$

This is a contradiction since $\{x_n\}$ is Cauchy. Hence the claim.

For $m, n \geq N_1$ and $a \in A \cup \{0\}$, we have

$$\begin{aligned} \rho(a, x_m, x_n) &= \min\{\max\{a, x_m\}, \max\{x_m, x_n\}, \max\{x_n, a\}\} \\ &\leq \max\{x_m, x_n\} \rightarrow 0 \quad \text{as } m, n \rightarrow \infty. \end{aligned} \quad (1.14)$$

Hence $\{x_n\}$ converges to a for any $a \in A \cup \{0\}$. It can also be shown that $\{x_n\}$ converges to b for any $b \in B$.

SUBCASE (ii). There exists $N_2 \in \mathbb{N}$ such that $N_2 \geq N_0$ and $x_i \in B$ for all $i \geq N_2$.

CLAIM 1.10. $x_n \rightarrow +\infty$ as $n \rightarrow +\infty$.

Suppose the claim does not hold. Then there exists a positive real number M such that $x_n \leq M$ for infinitely many $n \in \mathbb{N}$. Given $N \in \mathbb{N}$, we can find $i, j, k \in \mathbb{N}$ such that $k > j > i > \max\{N, N_2\}$, $x_i \leq M$, $x_j \leq M$, and $x_j \neq x_k$. Then $\rho(x_i, x_j, x_k) \geq |1/x_j - 1/x_k| \geq 1/2x_j \geq 1/2M$. This is a contradiction since $\{x_n\}$ is Cauchy. Hence the claim.

For $m, n \geq N_2$ and $a \in A$, we have

$$\rho(a, x_m, x_n) = \max\left\{\frac{1}{x_m}, \frac{1}{x_n}\right\} \rightarrow 0 \quad \text{as } m, n \rightarrow \infty. \quad (1.15)$$

Hence $\{x_n\}$ converges to a for any $a \in A$.

SUBCASE (iii). Given $N \in \mathbb{N}$, there exist $i, j \in \mathbb{N}$ such that $i > N, j > N$, $x_i \in A$, and $x_j \in B$.

CLAIM 1.11. Any element of A occurs only finitely many times in the sequence $\{x_n\}$.

Suppose the claim does not hold. Then there exists $a_0 \in A$ such that $a_0 = x_n$ for infinitely many $n \in \mathbb{N}$. Let $N \in \mathbb{N}$. Then there exist $i, j, k \in \mathbb{N}$ such that $k > j > i > N$, $x_i = x_j = a_0$, and $x_k \in B$. Then $\rho(x_i, x_j, x_k) = \min\{x_i, x_j, x_k\} = a_0$. This is a contradiction since $\{x_n\}$ is Cauchy. Hence the claim.

CLAIM 1.12. Any element of B occurs only finitely many times in the sequence $\{x_n\}$.

Suppose the claim does not hold. Then there exists $b \in B$ such that $b = x_n$ for infinitely many $n \in \mathbb{N}$. Let $N \in \mathbb{N}$. Then there exist $i, j, k \in \mathbb{N}$ such that $k > j > i > N$, $x_i = x_j = b$, and $x_k \in A$. Then $\rho(x_i, x_j, x_k) = 1/b$. This is a contradiction since $\{x_n\}$ is Cauchy. Hence the claim.

Let $c \in A$. Let $\varepsilon > 0$. From Claim 1.11, it follows that there exists $N_3 \in \mathbb{N}$ such that $x_n < \min\{\varepsilon, c\}$ whenever $n \geq N_3$ and $x_n \in A$. From Claim 1.12, it follows that there exists $N_4 \in \mathbb{N}$ such that $x_n > 1/\varepsilon$ whenever $n \geq N_4$ and $x_n \in B$. Let $N_5 = \max\{N_0, N_3, N_4\}$. Let $m, n \in \mathbb{N}$ be such that $m \geq N_5$ and $n \geq N_5$. Then $x_n, x_m \in A \cup B$. If both $x_n, x_m \in A$, then $\rho(c, x_n, x_m) = \max\{x_n, x_m\} < \varepsilon$. If both $x_n, x_m \in B$, then $\rho(c, x_n, x_m) = \max\{1/x_n, 1/x_m\} < \varepsilon$. Suppose that one of x_n, x_m belongs to A and the other belongs to B . We may assume that $x_n \in A$ and $x_m \in B$. Then $\rho(c, x_n, x_m) = \min\{c, x_n, x_m\} = x_n < \varepsilon$. Thus $\rho(c, x_n, x_m) < \varepsilon$ for all $n, m \geq N_5$. Hence $\{x_n\}$ converges to c .

Thus, in any case, $\{x_n\}$ is convergent in X with respect to the D -metric ρ . Hence (X, ρ) is a complete D -metric space.

To show that $(B^c)^c \neq B^c$.

Let $p \in B^c$. Then there exists a sequence $\{x_n\}$ in B such that $\{x_n\}$ converges to p . Hence $\{x_n\}$ is Cauchy. If there exists $N \in \mathbb{N}$ such that $x_k = x_N$ for all $k \geq N$, then $\rho(p, x_N, x_N) = 0$ and hence $p = x_N \in B$.

Suppose that such an N does not exist. Then given $N \in \mathbb{N}$, there exist $i, j \in \mathbb{N}$ such that $i > N, j > N$, and $x_i \neq x_j$. As in Subcase (ii) of Case 2 in the proof of the completeness of ρ , it can be shown that $x_n \rightarrow +\infty$ as $n \rightarrow \infty$ and that $\{x_n\}$ converges to x for any $x \in A$.

For any $x \in B$,

$$\rho(x, x_n, x_m) = \left| \frac{1}{x} - \frac{1}{x_n} \right| + \left| \frac{1}{x_n} - \frac{1}{x_m} \right| + \left| \frac{1}{x_m} - \frac{1}{x} \right| \rightarrow \frac{2}{x} \quad \text{as } m, n \rightarrow \infty. \quad (1.16)$$

Hence $\{x_n\}$ does not converge to x for any $x \in B$. We have $\rho(0, x_n, x_m) = 1$ for all $m, n \in \mathbb{N}$. Therefore $\{x_n\}$ does not converge to 0. Hence $p \in A$. Thus $B^c \subseteq B \cup A$. Clearly $B \subseteq B^c$. $\{2^n\}$ converges to x for any x in A . Hence $A \subseteq B^c$. Therefore $A \cup B = B^c$. Since $\{1/2^n\}$ is a sequence in A and it converges to x for any $x \in X$, $(B^c)^c = X$. Since $0 \notin A \cup B$, $(B^c)^c \neq B^c$. Therefore the function $f: P(X) \rightarrow P(X)$ defined as $f(E) = E^c$ for all $E \in P(X)$ is not a closure operator. Hence ρ -convergence does not define a topology on X . $\square$

DEFINITION 1.13. Let (X, ρ) be a D -metric space and $\{x_n\}$ a sequence in X . $\{x_n\}$ is said to converge strongly to an element x of X if

- (i) $\rho(x, x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$,
- (ii) $\{\rho(y, y, x_n)\}$ converges to $\rho(y, y, x)$ for all y in X .

In such a case, x is said to be a strong limit of $\{x_n\}$.

DEFINITION 1.14. Let (X, ρ) be a D -metric space and $\{x_n\}$ a sequence in X . $\{x_n\}$ is said to converge very strongly to an element x of X if

- (i) $\rho(x, x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$,
- (ii) $\{\rho(y, z, x_n)\}$ converges to $\rho(y, z, x)$ for any elements y, z of X .

In such a case, x is said to be a very strong limit of $\{x_n\}$.

REMARK 1.15. Let $\{x_n\}$ be a sequence in a D -metric space X and $x \in X$. If $\{x_n\}$ converges very strongly to x , then $\{x_n\}$ converges strongly to x . If $\{x_n\}$ converges strongly to x , then it converges to x with respect to ρ . Examples 1.21, 1.39, and 1.40 show that the converse statements are false.

PROPOSITION 1.16. Let (X, ρ) be a D -metric space. Let $\{x_n\}$ be a sequence in X converging to an element x of X . Then $\{\rho(x, x, x_n)\}$ is convergent.

PROOF. Since $\{x_n\}$ is convergent, it is D -Cauchy. We have $\rho(x, x, x_n) \leq \rho(x_m, x, x_n) + \rho(x, x_m, x_n) + \rho(x, x, x_m)$. Hence $\rho(x, x, x_n) - \rho(x, x, x_m) \leq 2\rho(x, x_m, x_n)$. Similarly, we have $\rho(x, x, x_m) - \rho(x, x, x_n) \leq 2\rho(x, x_n, x_m)$. Hence $|\rho(x, x, x_n) - \rho(x, x, x_m)| \leq 2\rho(x, x_n, x_m)$. Since this inequality is true for all $m, n \in \mathbb{N}$ and $\{x_n\}$ converges to x under the D -metric ρ , it follows that $\{\rho(x, x, x_n)\}$ is a Cauchy sequence of real numbers and hence convergent. $\square$

REMARK 1.17. Example 1.21 shows that the hypothesis of Proposition 1.16 does not ensure that the limit of $\{\rho(x, x, x_n)\}$ is $\rho(x, x, x)$.

PROPOSITION 1.18. In a D -metric space, every strongly convergent sequence has a unique strong limit.

PROOF. Let (X, ρ) be a D -metric space and $\{x_n\}$ a strongly convergent sequence in X . Let y, z be strong limits of $\{x_n\}$. Then $\{\rho(y, y, x_n)\}$ converges to both $\rho(y, y, y)$ and $\rho(y, y, z)$. Hence $\rho(y, y, z) = \rho(y, y, y) = 0$. Hence $y = z$. $\square$

THEOREM 1.19. Let (X, d) be a metric space, $x_0 \in X$, and let A be a nonempty subset of $X \setminus \{x_0\}$. Define $\rho : A \times A \times A \rightarrow \mathbb{R}^+$ as

$$\rho(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \min \{ \max \{ d(x_0, x), d(x_0, y) \}, \\ \max \{ d(x_0, y), d(x_0, z) \}, \\ \max \{ d(x_0, z), d(x_0, x) \} \} & \text{otherwise.} \end{cases} \quad (1.17)$$

Then (A, ρ) is a complete D -metric space and ρ -convergence defines a topology τ on A . If $A \cap \{x \in X : d(x_0, x) < r_0\} = \emptyset$ for some $r_0 \in (0, \infty)$, then τ is the discrete topology on X ; otherwise $\tau = \{\emptyset\} \cup \{E \subseteq A : \{x \in A : d(x_0, x) < r\} \subseteq E \text{ for some } r \in (0, \infty)\}$ and, in particular, τ is T_1 but not Hausdorff.

Let $\{x_n\} \subseteq A$. Then $\{x_n\}$ converges to x with respect to ρ for some $x \in A$ and $x_n \neq x$ for all sufficiently large $n \Rightarrow d(x_n, x_0) \rightarrow 0$ as $n \rightarrow \infty \Rightarrow \{x_n\}$ converges to x with respect to ρ for each x in A . If A has at least two elements, there does not exist a sequence in A which is strongly convergent with respect to ρ .

PROOF. Clearly ρ is symmetric in all the three variables and $\rho(x, y, z) = 0$ if and only if $x = y = z$. Let $x, y, z, u \in A$. We may assume that $d(x_0, x) \geq d(x_0, y) \geq d(x_0, z)$. Irrespective of whether $d(x_0, u) < d(x_0, y)$ or $d(x_0, u) \geq d(x_0, y)$, we have

$$\rho(x, y, z) = d(x_0, y) \leq \rho(x, y, u). \quad (1.18)$$

Hence $\rho(x, y, z) \leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u)$ for all $x, y, z, u \in A$. Hence ρ is a D -metric on A . Let $\{x_n\}$ be a sequence in A and $x \in A$. If $x_n \neq x$, we have

$$\begin{aligned} \rho(x, x_n, x_n) &= \min \{ \max \{ d(x_0, x), d(x_0, x_n) \}, \\ &\quad \max \{ d(x_0, x_n), d(x_0, x_n) \}, \\ &\quad \max \{ d(x_0, x_n), d(x_0, x) \} \} \\ &\geq d(x_0, x_n). \end{aligned} \quad (1.19)$$

That is, $\rho(x, x_n, x_n) \geq d(x_0, x_n)$ if $x_n \neq x$. Hence $d(x_0, x_n) \rightarrow 0$ as $n \rightarrow \infty$ if $\{x_n\}$ converges to x with respect to ρ for some $x \in A$ and $x_n \neq x$ for all sufficiently large n . We have

$$\begin{aligned} \rho(x_n, x_m, x) &\leq \min \{ \max \{ d(x_0, x_n), d(x_0, x_m) \}, \\ &\quad \max \{ d(x_0, x_m), d(x_0, x) \}, \\ &\quad \max \{ d(x_0, x), d(x_0, x_n) \} \} \\ &\leq \max \{ d(x_0, x_n), d(x_0, x_m) \} \quad \forall n, m \in \mathbb{N}. \end{aligned} \quad (1.20)$$

Thus $\{x_n\}$ converges to x with respect to ρ for each x in A if $d(x_0, x_n) \rightarrow 0$ as $n \rightarrow \infty$.

If $x_m \neq x_n$, then we have

$$\begin{aligned} \rho(x_n, x_m, x_n) &= \min \{ \max \{ d(x_0, x_n), d(x_0, x_m) \}, \\ &\quad \max \{ d(x_0, x_m), d(x_0, x_n) \}, \\ &\quad \max \{ d(x_0, x_n), d(x_0, x_n) \} \} \\ &\geq d(x_0, x_n). \end{aligned} \quad (1.21)$$

Hence $d(x_0, x_n) \rightarrow 0$ as $n \rightarrow \infty$ if $\{x_n\}$ is ρ -Cauchy and there does not exist an $N \in \mathbb{N}$ such that $x_n = x_N$ for all $n > N$. If there exists $N \in \mathbb{N}$ such that $x_n = x_N$ for all $n > N$, then, evidently, $\{x_n\}$ converges to x_N with respect to ρ .

If such an N does not exist and $\{x_n\}$ is ρ -Cauchy, then $d(x_0, x_n) \rightarrow 0$ as $n \rightarrow \infty$, and hence $\{x_n\}$ converges to x with respect to ρ for any x in A . Hence every ρ -Cauchy sequence in A is convergent with respect to ρ . Therefore (A, ρ) is D -complete. If $x_n \neq x$, we have

$$\begin{aligned} \rho(x, x, x_n) &= \min \{ \max \{ d(x_0, x), d(x_0, x) \}, \\ &\quad \max \{ d(x_0, x), d(x_0, x_n) \}, \\ &\quad \max \{ d(x_0, x_n), d(x_0, x) \} \} \\ &\geq d(x_0, x). \end{aligned} \quad (1.22)$$

Since $x \in A \subseteq X \setminus \{x_0\}$, $d(x_0, x) > 0$. Hence $\{\rho(x, x, x_n)\}$ does not converge to 0 if $x_n \neq x$ for infinitely many n . Consequently, $\{x_n\}$ is not strongly ρ -convergent if A has at least two elements. Let E be a subset of A . Clearly, $E \subseteq E^c$.

CASE 1. $E \cap \{x \in X : d(x, x_0) < r\} = \emptyset$ for some $r \in (0, \infty)$.

Let $z \in E^c$. Then there exists a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to z with respect to ρ . Suppose that $x_n \neq z$ for each n . Then $d(x_0, x_n) \rightarrow 0$ as $n \rightarrow \infty$. Since $E \cap \{x \in X : d(x, x_0) < r\} = \emptyset$, we have $d(x, x_0) \geq r$ for all $x \in E$. Hence $d(x_n, x_0) \geq r$ for all $n \in \mathbb{N}$. Therefore $\{d(x_n, x_0)\}$ does not converge to 0. Thus we arrived at a contradiction. Consequently, $x_n = z$ for some $n \in \mathbb{N}$. Hence $z \in E$. Therefore $E^c \subseteq E$. Thus $E = E^c$.

CASE 2. [Case 1](#) is false.

Then $E \cap \{x \in X : d(x, x_0) < 1/n\} \neq \emptyset$ for each $n \in \mathbb{N}$. Hence there exists a sequence $\{u_n\}$ in E such that $d(u_n, x_0) < 1/n$ for all $n \in \mathbb{N}$. Therefore $d(u_n, x_0) \rightarrow 0$ as $n \rightarrow \infty$. Hence $\{u_n\}$ converges to x with respect to ρ for each x in A . Hence $E^c = A$.

CASE (I). $A \cap \{x \in X : d(x, x_0) < r_0\} = \emptyset$ for some $r_0 \in (0, \infty)$.

In this case, for any subset E of A , we have $E^c = E$, and hence $(E^c)^c = E^c = E$. Thus the function f defined on the power set $P(A)$ of A as $f(E) = E^c$ for all $E \in P(A)$ is a closure operator. Therefore ρ -convergence defines a topology τ on A in which every subset of A is closed. Hence $\tau = \{E : E \subseteq A\}$. Consequently, τ is the discrete topology on X .

CASE (II). $A \cap \{x \in X : d(x, x_0) < r\} \neq \emptyset$ for any $r \in (0, \infty)$.

In this case, for a subset E of X , we have $E^c = E$ or A according to whether [Case 1](#) or [Case 2](#) holds. Hence $(E^c)^c = E^c$ for all $E \in P(A)$. Therefore the function f defined on $P(A)$ as $f(E) = E^c$ for all $E \in P(A)$ is a closure operator. Thus ρ -convergence defines a topology τ on A with respect to which a subset B of A is closed if and only if $B = E^c$ for some $E \in P(A)$. Hence

$$\begin{aligned} \tau &= \{A \setminus E^c : E \in P(A)\} \\ &= \{\emptyset\} \cup \{E \in P(A) : \{x \in A : d(x_0, x) < r\} \subseteq E \text{ for some } r \in (0, \infty)\}. \end{aligned} \quad (1.23)$$

If U_1, U_2 are nonempty open sets in τ , then $U_1 \cap U_2 \neq \emptyset$ since there exist $r_1, r_2 \in (0, \infty)$ such that

$$\{x \in A : d(x_0, x) < r_i\} \subseteq U_i, \quad i = 1, 2. \quad (1.24)$$

Hence τ is not Hausdorff. Let p, q be distinct elements of A . Since $x_0 \notin A$, $d(p, x_0)$ and $d(q, x_0)$ are positive real numbers. Let $0 < r < \min\{d(p, x_0), d(q, x_0)\}$. Let $V_0 = \{x \in A : d(x, x_0) < r\}$. Then $V_0 \cup \{p\}$ is a τ -open subset of A containing p but not q , and $V_0 \cup \{q\}$ is a τ -open subset of A containing q but not p . Hence the topology τ is T_1 . $\square$

EXAMPLE 1.20. Let $X = \mathbb{R}$ with the usual metric, $x_0 = 0$, and $A = [1, 2]$. Then the function ρ defined in [Theorem 1.19](#) on $A \times A \times A$ reduces to the following:

$$\rho(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \min\{\max\{x, y\}, \max\{y, z\}, \max\{z, x\}\} & \text{otherwise.} \end{cases} \quad (1.25)$$

From [Theorem 1.19](#) it follows that (A, ρ) is a complete D -metric space and that ρ -convergence defines a topology τ on A , which is the discrete topology on A .

EXAMPLE 1.21. Let $X = \mathbb{R}$ with the usual metric, $x_0 = 0$, and $A = \{1/n : n \in \mathbb{N}\}$. Then the function ρ defined in [Theorem 1.19](#) on $A \times A \times A$ has the same form as that given in [Example 1.20](#). From [Theorem 1.19](#) it follows that (A, ρ) is a complete D -metric space, any sequence in A which converges to zero in the usual sense converges to x with respect to ρ for each x in A , and that ρ -convergence defines a topology τ on A with respect to which nonempty subset E of A is open if and only if E contains $\{1/n : n \in \mathbb{N} \text{ and } n \geq N\}$ for some $N \in \mathbb{N}$. Further, τ is T_1 but not Hausdorff. Let $x_n = 1/n$ for all $n \in \mathbb{N}$ and $x_0 = 1/2$. Then $\{x_n\}$ converges to $1/2$ under the D -metric ρ . We have $\rho(x_0, x_0, x_n) = \rho(1/2, 1/2, 1/n) = 1/2$ for all $n \in \mathbb{N} \setminus \{2\}$. Hence $\{\rho(x_0, x_0, x_n)\}$ does not converge to $0 = \rho(x_0, x_0, x_0)$. We note that $\{1/n\}$ does not converge strongly even though it converges to every element of X .

THEOREM 1.22. Let (X, d) be a metric space, $x_0 \in X$, and let $\{x_n\}$ be a convergent sequence in $X \setminus \{x_0\}$ with limit x_0 , A a proper subset of $X \setminus \{x_0\}$ containing $\{x_n\}$, and B a subset of $X \setminus \{x_0\}$ which contains A properly. Define $\rho : B \times B \times B \rightarrow \mathbb{R}^+$ as

$$\rho(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \min \left\{ \begin{array}{l} \max \{d(x_0, x), d(x_0, y)\}, \\ \max \{d(x_0, y), d(x_0, z)\}, \\ \max \{d(x_0, z), d(x_0, x)\} \end{array} \right\} & \text{otherwise.} \end{cases} \quad (1.26)$$

Let ρ_0 denote the restriction of ρ to $A \times A \times A$. Then (B, ρ) and (A, ρ_0) are complete D -metric spaces, $A \subseteq B$, but $\{x \in B : \text{there is a sequence } \{y_n\} \text{ in } A \text{ which converges to } x \text{ with respect to } \rho\} = B \neq A$.

PROOF. The proof follows from [Theorem 1.19](#). □

REMARK 1.23. If (X, d) is a metric space, $Y \subseteq X$, d_0 is the restriction of d to $Y \times Y$, and (Y, d_0) is complete, then $\{x \in X : \text{there is a sequence } \{y_n\} \text{ in } Y \text{ which converges to } x\} = Y$. [Theorem 1.22](#) shows that an analogous result does not hold in D -metric spaces.

THEOREM 1.24. Suppose that $\Phi : (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ is

- (i) symmetric in all the three variables,
- (ii) $\Phi(t_1, t_2, t_3) = 0$ if and only if $t_1 = t_2 = t_3 = 0$,
- (iii) $\Phi(t_1, t_2, t_3) \leq \Phi(t_1, t'_2, t'_3) + \Phi(t'_1, t_2, t'_3) + \Phi(t''_1, t'_2, t_3)$ whenever $(t_1, t_2, t_3), (t_1, t'_2, t'_3), (t'_1, t_2, t'_3), (t''_1, t'_2, t_3) \in (\mathbb{R}^+)^{3*}$ and $t_i \leq t'_i + t''_i$ for all $i = 1, 2, 3$.

Let d be a metric on X and let $\rho : X \times X \times X \rightarrow \mathbb{R}^+$ be defined as

$$\rho(x, y, z) = \Phi(d(x, y), d(y, z), d(z, x)). \quad (1.27)$$

Then ρ is a D -metric on X . If Φ is continuous at $(0, 0, 0)$, then

- (1) any d -Cauchy sequence in X is ρ -Cauchy,
- (2) $\{x_n\} \subseteq X$, $x \in X$, and $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty \Rightarrow \{x_n\}$ converges to x with respect to the D -metric ρ .

Suppose that Φ has the following property:

- (iv) given $\varepsilon > 0$, there exists $\delta > 0$ such that $t < \varepsilon$ whenever $t \in \mathbb{R}^+$ and $\Phi(0, t, t) < \delta$.

Then

- (1) any ρ -Cauchy sequence is d -Cauchy,
- (2) $\{x_n\} (\subseteq X)$ converges to $x \in X$ with respect to $\rho \Rightarrow d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.

If Φ is continuous at $(0, 0, 0)$, $\{\Phi(0, t_n, t_n)\}$ converges to $\Phi(0, t, t)$ whenever $t \in \mathbb{R}^+$, and $\{t_n\}$ is a sequence in $\mathbb{R}^+$ converging to t , then $\{x_n\} \subseteq X$, $x \in X$, and $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty \Rightarrow \{x_n\}$ converges strongly to x with respect to the D -metric ρ .

If Φ is continuous at $(0, 0, 0)$ and is continuous in any two variables, then $\{x_n\} \subseteq X$, $x \in X$, and $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty \Rightarrow \{x_n\}$ converges very strongly to x with respect to the D -metric ρ .

PROOF. We prove that ρ is a D -metric on X . Since Φ is symmetric in all the three variables, so is ρ . From property (ii) of Φ , it follows that $\rho(x, y, z) = 0$ if and only if $x = y = z$.

Let $x, y, z, u \in X$. From property (iii) of Φ , we have

$$\begin{aligned} \rho(x, y, z) &= \Phi(d(x, y), d(y, z), d(z, x)) \\ &\leq \Phi(d(x, y), d(y, u), d(u, x)) \\ &\quad + \Phi(d(u, y), d(y, z), d(z, u)) \\ &\quad + \Phi(d(x, u), d(u, z), d(z, x)) \end{aligned} \quad (1.28)$$

since $d(x, y) \leq d(u, y) + d(x, u)$, $d(y, z) \leq d(y, u) + d(u, z)$, and $d(z, x) \leq d(u, x) + d(z, u)$. Hence $\rho(x, y, z) \leq \rho(x, y, u) + \rho(u, y, z) + \rho(x, u, z)$ for all $x, y, z, u \in X$. Hence ρ is a D -metric on X .

Suppose that Φ is continuous at $(0, 0, 0)$.

- (1) Let $\{x_n\}$ be a d -Cauchy sequence in X . Then $d(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$. We have

$$\begin{aligned} \rho(x_n, x_m, x_k) &= \Phi(d(x_n, x_m), d(x_m, x_k), d(x_k, x_n)) \\ &\rightarrow \Phi(0, 0, 0) = 0 \quad \text{as } n, m, k \rightarrow \infty \quad (\text{since } \Phi \text{ is continuous at } (0, 0, 0)). \end{aligned} \quad (1.29)$$

Hence $\{x_n\}$ is ρ -Cauchy in X .

- (2) Let $\{x_n\} \subseteq X$ and let $x \in X$ be such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. We have

$$\begin{aligned} \rho(x, x_n, x_m) &= \Phi(d(x, x_n), d(x_n, x_m), d(x_m, x)) \\ &\rightarrow \Phi(0, 0, 0) = 0 \quad \text{as } n, m \rightarrow \infty \end{aligned} \quad (1.30)$$

(since every d -convergent sequence is d -Cauchy and Φ is continuous at $(0, 0, 0)$).

Hence $\rho(x, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$. Hence $\{x_n\}$ converges to x with respect to ρ . Suppose that Φ has property (iv).

- (1) Let $\{x_n\}$ be a ρ -Cauchy sequence in X . Let ε be a positive real number. Then there exists $\delta > 0$ such that $t < \varepsilon$ whenever $t \in \mathbb{R}^+$ and $\Phi(0, t, t) < \delta$. Since $\rho(x_n, x_m, x_n) \rightarrow 0$ as $n, m \rightarrow \infty$, there exists $N \in \mathbb{N}$ such that $\rho(x_n, x_m, x_n) < \delta$ for all $n, m \geq N$. That is, $\Phi(d(x_n, x_m), d(x_m, x_n), d(x_n, x_n)) < \delta$ for all $n, m \geq N$. In other words, $\Phi(0, d(x_n, x_m), d(x_n, x_m)) < \delta$ for all $n, m \geq N$ (since Φ is symmetric). Hence $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. Therefore $\{x_n\}$ is d -Cauchy.

(2) Let $\{x_n\} \subseteq X$ converge to $x \in X$ with respect to the D -metric ρ . Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $t < \varepsilon$ whenever $t \in \mathbb{R}^+$ and $\Phi(0, t, t) < \delta$. Since $\rho(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, there exists $N \in \mathbb{N}$ such that $\rho(x_n, x_n, x) < \delta$ for all $n \geq N$. That is, $\Phi(d(x_n, x_n), d(x_n, x), d(x, x_n)) < \delta$ for all $n \geq N$. That is $\Phi(0, d(x_n, x), d(x_n, x)) < \delta$ for all $n \geq N$. Hence $d(x_n, x) < \varepsilon$ for all $n \geq N$. Therefore $\{x_n\}$ converges to x with respect to the metric d .

Suppose that Φ is continuous at $(0, 0, 0)$ and $\{\Phi(0, t_n, t_n)\}$ converges to $\Phi(0, t, t)$ whenever $t \in \mathbb{R}^+$ and $\{t_n\}$ is a sequence in $\mathbb{R}^+$ converging to t . Let $\{x_n\} \subseteq X$ and let $x \in X$ be such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. Let $y \in X$. Then $d(x_n, y) \rightarrow d(x, y)$ as $n \rightarrow \infty$. Hence $\{\Phi(0, d(x_n, y), d(x_n, y))\} \rightarrow \Phi(0, d(x, y), d(x, y))$ as $n \rightarrow \infty$. That is, $\rho(y, y, x_n) \rightarrow \rho(y, y, x)$ as $n \rightarrow \infty$. Since Φ is continuous at $(0, 0, 0)$, from what we have already proved, it follows that $\{x_n\}$ converges to x with respect to ρ . Hence $\{x_n\}$ converges strongly to x with respect to the D -metric ρ . Suppose that Φ is continuous at $(0, 0, 0)$ and is continuous in any two variables. Let $\{x_n\} \subseteq X$ and let $x \in X$ be such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. Let $y, z \in X$. Then $\{d(z, x_n)\}$ and $\{d(x_n, y)\}$ converge to $d(z, x)$ and $d(x, y)$, respectively. Since Φ is continuous in any two variables, it follows that $\{\Phi(d(y, z), d(z, x_n), d(x_n, y))\}$ converges to $\Phi(d(y, z), d(z, x), d(x, y))$, that is, $\{\rho(y, z, x_n)\}$ converges to $\rho(y, z, x)$. Since Φ is continuous at $(0, 0, 0)$, $\{x_n\}$ converges to x with respect to ρ . Hence $\{x_n\}$ converges very strongly to x with respect to ρ . $\square$

COROLLARY 1.25. *Let (X, d) be a metric space and let $\Phi: (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ be continuous at $(0, 0, 0)$, and have properties (i), (ii), (iii), and (iv) specified in [Theorem 1.24](#). Let ρ be defined as in [Theorem 1.24](#). Then ρ is a D -metric on X , a sequence in X is D -Cauchy if and only if it is ρ -Cauchy, and a sequence $\{x_n\}$ in X converges with respect to d to an element x of X if and only if $\{x_n\}$ converges to x with respect to ρ . In particular, ρ -convergence defines a topology on X which coincides with the metric topology induced by the metric d on X , and X is complete with respect to the metric d if and only if it is complete with respect to the D -metric ρ . Further, the following statements are true.*

(1) *If $\{\Phi(0, t_n, t_n)\}$ converges to $\Phi(0, t, t)$ whenever $t \in \mathbb{R}^+$ and $\{t_n\}$ is a sequence in $\mathbb{R}^+$ converging to t , $\{x_n\} \subseteq X$, and $x \in X$, then $\{x_n\}$ converges to x with respect to ρ if and only if $\{x_n\}$ converges strongly to x with respect to ρ .*

(2) *If Φ is continuous in any two variables, $\{x_n\} \subseteq X$, and $x \in X$, then $\{x_n\}$ converges to x with respect to ρ if and only if $\{x_n\}$ converges very strongly to x with respect to ρ .*

(3) *If Φ is continuous on $(\mathbb{R}^+)^{3*}$, then ρ is sequentially continuous in all the three variables, that is, $\{\rho(u_n, v_n, w_n)\}$ converges to $\rho(u, v, w)$ whenever $u, v, w \in X$ and $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are sequences in X converging to u , v , and w , respectively with respect to ρ .*

NOTE 1.26. [Corollary 1.25](#) is useful in generating a number of D -metrics from a given metric on a set.

We now prove a number of propositions which show that the class of functions $\Phi: (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$, which are continuous at $(0, t, t)$ for all $t \in \mathbb{R}^+$ and which satisfy properties (i), (ii), (iii), and (iv) specified in [Theorem 1.24](#), is very rich.

LEMMA 1.27. *Let $p \in [1, \infty)$. Then $(a+b)^p \geq a^p + b^p$ for all $a, b \in \mathbb{R}^+$.*

PROOF. Define $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ as $f(t) = (1+t)^p - 1 - t^p$ for all $t \in \mathbb{R}^+$. Then $f'(t) = p(1+t)^{p-1} - pt^{p-1} = p[(1+t)^{p-1} - t^{p-1}]$. Since $1+t \geq t$ for all $t \in \mathbb{R}^+$ and $p-1 \geq 0$, we have $(1+t)^{p-1} \geq t^{p-1}$ for all $t \in \mathbb{R}^+$.

Hence $f'(t) \geq 0$ for all $t \in \mathbb{R}^+$. Therefore f is monotonically increasing on $\mathbb{R}^+$. We have $f(0) = 0$. Hence $f(t) \geq f(0)$ for all $t \in \mathbb{R}^+$. Therefore

$$(1+t)^p \geq 1+t^p \quad \forall t \in \mathbb{R}^+. \quad (1.31)$$

Let $a, b \in \mathbb{R}^+$. We may assume that $a \geq b$. If $a = 0$, then $b = 0$ and $(a+b)^p = 0 = a^p + b^p$. Suppose that $a > 0$. Then, from what we have already proved above, we have

$$\left(1 + \frac{b}{a}\right)^p \geq 1 + \left(\frac{b}{a}\right)^p, \quad (1.32)$$

that is,

$$\left(\frac{a+b}{a}\right)^p \geq 1 + \frac{b^p}{a^p}. \quad (1.33)$$

Hence $(a+b)^p \geq a^p + b^p$. □

COROLLARY 1.28. *Let $p \in [1, \infty)$. Then $(a+b+c)^p \geq a^p + b^p + c^p$ for all $a, b, c \in \mathbb{R}^+$.*

PROOF. Let $a, b, c \in \mathbb{R}^+$. Then, from [Lemma 1.27](#), we have

$$(a+b+c)^p = [(a+b)+c]^p \geq (a+b)^p + c^p \geq a^p + b^p + c^p. \quad (1.34)$$

□

PROPOSITION 1.29. *Suppose that $\Psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is monotonically increasing and $\Psi(t) = 0$ if and only if $t = 0$. Let $p \in [1, \infty)$. Define $\Phi: (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ as*

$$\Phi(t_1, t_2, t_3) = [[\Psi(t_1)]^p + [\Psi(t_2)]^p + [\Psi(t_3)]^p]^{1/p} \quad \forall (t_1, t_2, t_3) \in (\mathbb{R}^+)^{3*}. \quad (1.35)$$

Then Φ has properties (i), (ii), (iii), and (iv) specified in [Theorem 1.24](#). If Ψ is continuous at 0, then Φ is continuous at $(0, 0, 0)$, and if Ψ is continuous on $\mathbb{R}^+$, then Φ is continuous on $(\mathbb{R}^+)^{3}$.*

PROOF. Clearly Φ is symmetric in all the three variables:

$$\begin{aligned} \Phi(t_1, t_2, t_3) = 0 &\iff [[\Psi(t_1)]^p + [\Psi(t_2)]^p + [\Psi(t_3)]^p]^{1/p} = 0 \\ &\iff [\Psi(t_i)]^p = 0 \quad \forall i \\ &\iff \Psi(t_i) = 0 \quad \forall i \\ &\iff t_i = 0 \quad \forall i. \end{aligned} \quad (1.36)$$

Let $t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3, t'_3, t''_3 \in \mathbb{R}^+$. Let

$$\begin{aligned} a &= [[\Psi(t_1)]^p + [\Psi(t'_2)]^p + [\Psi(t'_3)]^p]^{1/p}, \\ b &= [[\Psi(t'_1)]^p + [\Psi(t_2)]^p + [\Psi(t'_3)]^p]^{1/p}, \\ c &= [[\Psi(t'_1)]^p + [\Psi(t'_2)]^p + [\Psi(t_3)]^p]^{1/p}. \end{aligned} \quad (1.37)$$

We have

$$\begin{aligned} (a+b+c)^p &\geq a^p + b^p + c^p \\ &= [[\Psi(t_1)] + [\Psi(t'_2)] + [\Psi(t'_3)]] \\ &\quad + [[\Psi(t'_1)]^p + [\Psi(t_2)]^p + [\Psi(t'_3)]^p] \\ &\quad + [[\Psi(t'_1)]^p + [\Psi(t'_2)]^p + [\Psi(t_3)]^p] \\ &\geq [\Psi(t_1)]^p + [\Psi(t_2)]^p + [\Psi(t_3)]^p. \end{aligned} \quad (1.38)$$

Hence $a + b + c \geq [[\Psi(t_1)]^p + [\Psi(t_2)]^p + [\Psi(t_3)]^p]^{1/p}$. Therefore Φ has property (iii). We have

$$\begin{aligned} \Phi(0, t, t) &= [[\Psi(0)]^p + [\Psi(t)]^p + [\Psi(t)]^p]^{1/p} \\ &= [2[\Psi(t)]^p]^{1/p} \\ &= 2^{1/p}\Psi(t). \end{aligned} \quad (1.39)$$

Let ε be a positive real number. Choose $\delta = 2^{1/p}\Psi(\varepsilon)$. Then $\delta > 0$ since $\Psi(t) = 0$ implies $t = 0$.

$$\begin{aligned} \Phi(0, t, t) < \delta &\Rightarrow 2^{1/p}\Psi(t) < 2^{1/p}\Psi(\varepsilon) \\ &\Rightarrow \Psi(t) < \Psi(\varepsilon) \\ &\Rightarrow t < \varepsilon \quad (\text{since } \Psi \text{ is monotonically increasing}). \end{aligned} \quad (1.40)$$

Hence Φ has property (iv). □

COROLLARY 1.30. Let $p \in [1, \infty)$. Then the function $\Phi : (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ defined as $\Phi(t_1, t_2, t_3) = [t_1^p + t_2^p + t_3^p]^{1/p}$ for all $(t_1, t_2, t_3) \in (\mathbb{R}^+)^{3*}$ is continuous on $(\mathbb{R}^+)^{3*}$ and has properties (i), (ii), (iii), and (iv) specified in [Theorem 1.24](#).

PROOF. The proof follows from [Proposition 1.29](#) by taking $\Psi(t) = t$ for all $t \in \mathbb{R}^+$. □

PROPOSITION 1.31. Suppose that $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is monotonically increasing and $\Psi(t) = 0$ if and only if $t = 0$. Define $\Phi : (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ as

$$\Phi(t_1, t_2, t_3) = \max\{\Psi(t_1), \Psi(t_2), \Psi(t_3)\} \quad \forall (t_1, t_2, t_3) \in (\mathbb{R}^+)^{3*}. \quad (1.41)$$

Then Φ has properties (i), (ii), (iii), and (iv) specified in [Theorem 1.24](#). If Ψ is continuous at 0, then Φ is continuous at $(0, 0, 0)$, and if Ψ is continuous on $\mathbb{R}^+$, then Φ is continuous on $(\mathbb{R}^+)^{3*}$.

PROOF. Clearly Φ is symmetric in all the three variables.

$$\begin{aligned}\Phi(t_1, t_2, t_3) = 0 &\iff \max\{\Psi(t_1), \Psi(t_2), \Psi(t_3)\} = 0 \\ &\iff \Psi(t_i) = 0 \quad \forall i \\ &\iff t_i = 0 \quad \forall i.\end{aligned}\tag{1.42}$$

Hence Φ has property (ii).

Let $t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3, t'_3, t''_3 \in \mathbb{R}^+$. Then

$$\begin{aligned}\max\{\Psi(t_1), \Psi(t_2), \Psi(t_3)\} &\leq \max\{\Psi(t_1), \Psi(t'_2), \Psi(t'_3)\} \\ &\quad + \max\{\Psi(t'_1), \Psi(t_2), \Psi(t''_3)\} \\ &\quad + \max\{\Psi(t''_1), \Psi(t''_2), \Psi(t_3)\}.\end{aligned}\tag{1.43}$$

Hence Φ has property (iii).

Let ε be a positive real number. Choose $\delta = \Psi(\varepsilon)$. Then $\delta > 0$ since $\Psi(t) = 0$ implies $t = 0$.

$$\begin{aligned}\Phi(0, t, t) < \delta &\implies \max\{\Psi(0), \Psi(t), \Psi(t)\} < \Psi(\varepsilon) \\ &\implies \Psi(t) < \Psi(\varepsilon) \\ &\implies t < \varepsilon \quad (\text{since } \Psi \text{ is monotonically increasing}).\end{aligned}\tag{1.44}$$

Hence Φ has property (iv). □

COROLLARY 1.32. The function $\Phi : (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ defined as $\Phi(t_1, t_2, t_3) = \max\{t_1, t_2, t_3\}$ for all $(t_1, t_2, t_3) \in (\mathbb{R}^+)^{3*}$ is continuous on $(\mathbb{R}^+)^{3*}$ and has properties (i), (ii), (iii), and (iv) specified in [Theorem 1.24](#).

PROOF. The proof follows from [Proposition 1.31](#) by taking $\Psi(t) = t$ for all $t \in \mathbb{R}^+$. □

PROPOSITION 1.33. Suppose that $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is monotonically increasing, $\Psi(s+t) \leq \Psi(s) + \Psi(t)$ for all $s, t \in \mathbb{R}^+$, and $\Psi(t) = 0$ if and only if $t = 0$. Define $\Phi : (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ as

$$\begin{aligned}\Phi(t_1, t_2, t_3) &= \min\{\max\{\Psi(t_1), \Psi(t_2)\}, \max\{\Psi(t_2), \Psi(t_3)\}, \\ &\quad \max\{\Psi(t_3), \Psi(t_1)\}\} \quad \forall (t_1, t_2, t_3) \in (\mathbb{R}^+)^{3*}.\end{aligned}\tag{1.45}$$

Then Φ has properties (i), (ii), (iii), and (iv) specified in [Theorem 1.24](#). If Ψ is continuous at 0, then Φ is continuous at $(0, 0, 0)$, and if Ψ is continuous on $\mathbb{R}^+$, then Φ is continuous on $(\mathbb{R}^+)^{3*}$.

PROOF. Clearly Φ is symmetric in all the three variables.

$$\begin{aligned}
 \Phi(t_1, t_2, t_3) = 0 &\iff \min \{ \max \{ \Psi(t_1), \Psi(t_2) \}, \max \{ \Psi(t_2), \Psi(t_3) \}, \\
 &\quad \max \{ \Psi(t_3), \Psi(t_1) \} \} = 0 \\
 &\iff \max \{ \Psi(t_1), \Psi(t_2) \} = 0 \\
 &\quad \text{or } \max \{ \Psi(t_2), \Psi(t_3) \} = 0 \\
 &\quad \text{or } \max \{ \Psi(t_3), \Psi(t_1) \} = 0 \\
 &\iff t_1 = t_2 = 0 \quad \text{or} \quad t_2 = t_3 = 0 \quad \text{or} \quad t_3 = t_1 = 0 \\
 &\iff t_1 = t_2 = t_3 = 0 \quad (\text{since } (t_1, t_2, t_3) \in (\mathbb{R}^+)^{3*}).
 \end{aligned} \tag{1.46}$$

Hence Φ has property (ii).

Let $t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3, t'_3, t''_3 \in \mathbb{R}^+$ be such that $t_i \leq t'_i + t''_i$ for all $i = 1, 2, 3$. We have

$$\begin{aligned}
 &\max \{ \Psi(t_1), \Psi(t'_2) \} + \max \{ \Psi(t'_1), \Psi(t''_2) \} \\
 &\geq \max \{ \Psi(t_1), \Psi(t'_2 + t''_2) \} \quad (\text{since } \Psi(s+t) \leq \Psi(s) + \Psi(t)) \\
 &\geq \max \{ \Psi(t_1), \Psi(t_2) \} \quad (\text{since } \Psi \text{ is monotonically increasing} \\
 &\quad \text{and } t_2 \leq t'_2 + t''_2).
 \end{aligned} \tag{1.47}$$

Clearly, we have

$$\begin{aligned}
 &\max \{ \Psi(t_1), \Psi(t'_2) \} + \max \{ \Psi(t'_2), \Psi(t_3) \} \geq \max \{ \Psi(t_1), \Psi(t_3) \}, \\
 &\max \{ \Psi(t_1), \Psi(t'_2) \} + \max \{ \Psi(t_3), \Psi(t'_1) \} \geq \max \{ \Psi(t_1), \Psi(t_3) \}.
 \end{aligned} \tag{1.48}$$

Hence

$$\begin{aligned}
 &\max \{ \Psi(t_1), \Psi(t'_2) \} + \min \{ \max \{ \Psi(t'_1), \Psi(t'_2) \}, \max \{ \Psi(t'_2), \Psi(t_3) \}, \\
 &\quad \max \{ \Psi(t_3), \Psi(t'_1) \} \} \\
 &\geq \min \{ \max \{ \Psi(t_1), \Psi(t_2) \}, \max \{ \Psi(t_1), \Psi(t_3) \} \}.
 \end{aligned} \tag{1.49}$$

Similarly,

$$\begin{aligned}
 & \max \{ \Psi(t'_3), \Psi(t_1) \} + \min \{ \max \{ \Psi(t'_1), \Psi(t_2) \}, \max \{ \Psi(t_2), \Psi(t'_3) \}, \max \{ \Psi(t'_3), \Psi(t'_1) \} \} \\
 & \geq \min \{ \max \{ \Psi(t_1), \Psi(t_2) \}, \max \{ \Psi(t_1), \Psi(t_3) \} \}, \\
 & \max \{ \Psi(t'_1), \Psi(t_2) \} + \min \{ \max \{ \Psi(t'_1), \Psi(t'_2) \}, \max \{ \Psi(t'_2), \Psi(t_3) \}, \max \{ \Psi(t_3), \Psi(t'_1) \} \} \\
 & \geq \min \{ \max \{ \Psi(t_1), \Psi(t_2) \}, \max \{ \Psi(t_2), \Psi(t_3) \} \}, \\
 & \max \{ \Psi(t_2), \Psi(t'_3) \} + \min \{ \max \{ \Psi(t_1), \Psi(t'_2) \}, \max \{ \Psi(t'_2), \Psi(t'_3) \}, \max \{ \Psi(t'_3), \Psi(t_1) \} \} \\
 & \geq \min \{ \max \{ \Psi(t_1), \Psi(t_2) \}, \max \{ \Psi(t_2), \Psi(t_3) \} \}, \\
 & \max \{ \Psi(t'_2), \Psi(t_3) \} + \min \{ \max \{ \Psi(t_1), \Psi(t'_2) \}, \max \{ \Psi(t'_2), \Psi(t'_3) \}, \max \{ \Psi(t'_3), \Psi(t_1) \} \} \\
 & \geq \min \{ \max \{ \Psi(t_1), \Psi(t_3) \}, \max \{ \Psi(t_2), \Psi(t_3) \} \}, \\
 & \max \{ \Psi(t_3), \Psi(t'_1) \} + \min \{ \max \{ \Psi(t'_1), \Psi(t_2) \}, \max \{ \Psi(t_2), \Psi(t'_3) \}, \max \{ \Psi(t'_3), \Psi(t'_1) \} \} \\
 & \geq \min \{ \max \{ \Psi(t_1), \Psi(t_3) \}, \max \{ \Psi(t_2), \Psi(t_3) \} \}.
 \end{aligned}
 \tag{1.50}$$

We have

$$\begin{aligned}
 & \max \{ \Psi(t'_2), \Psi(t'_3) \} + \max \{ \Psi(t'_3), \Psi(t'_1) \} + \max \{ \Psi(t'_1), \Psi(t'_2) \} \\
 & \geq \max \{ \max \{ \Psi(t'_3), \Psi(t'_1) \} + \max \{ \Psi(t'_1), \Psi(t'_2) \}, \\
 & \quad \max \{ \Psi(t'_2), \Psi(t'_3) \} + \max \{ \Psi(t'_1), \Psi(t'_2) \}, \\
 & \quad \max \{ \Psi(t'_2), \Psi(t'_3) \} + \max \{ \Psi(t'_3), \Psi(t'_1) \} \} \\
 & \geq \max \{ \Psi(t'_1) + \Psi(t'_1), \Psi(t'_2) + \Psi(t'_2), \Psi(t'_3) + \Psi(t'_3) \} \\
 & \geq \max \{ \Psi(t'_1 + t'_1), \Psi(t'_2 + t'_2), \Psi(t'_3 + t'_3) \} \quad (\Theta \Psi(s + t) \leq \Psi(s) + \Psi(t) \quad \forall s, t \in \mathbb{R}^+) \\
 & \geq \max \{ \Psi(t_1), \Psi(t_2), \Psi(t_3) \} \quad (\text{since } \Psi \text{ is monotonically increasing} \\
 & \quad \text{and } t_i \leq t'_i + t''_i \quad \forall i = 1, 2, 3).
 \end{aligned}
 \tag{1.51}$$

Hence

$$\begin{aligned}
 & \min \{ \max \{ \Psi(t_1), \Psi(t_2) \}, \max \{ \Psi(t_2), \Psi(t_3) \}, \max \{ \Psi(t_3), \Psi(t_1) \} \} \\
 & \leq \min \{ \max \{ \Psi(t_1), \Psi(t'_2) \}, \max \{ \Psi(t'_2), \Psi(t'_3) \}, \max \{ \Psi(t'_3), \Psi(t_1) \} \} \\
 & \quad + \min \{ \max \{ \Psi(t'_1), \Psi(t_2) \}, \max \{ \Psi(t_2), \Psi(t'_3) \}, \max \{ \Psi(t'_3), \Psi(t'_1) \} \} \\
 & \quad + \min \{ \max \{ \Psi(t'_1), \Psi(t'_2) \}, \max \{ \Psi(t'_2), \Psi(t_3) \}, \max \{ \Psi(t_3), \Psi(t'_1) \} \}.
 \end{aligned}
 \tag{1.52}$$

Therefore $\Phi(t_1, t_2, t_3) \leq \Phi(t_1, t'_2, t'_3) + \Phi(t'_1, t_2, t'_3) + \Phi(t'_1, t'_2, t_3)$ whenever $(t_1, t_2, t_3), (t_1, t'_2, t'_3), (t'_1, t_2, t'_3), (t'_1, t'_2, t_3) \in (\mathbb{R}^+)^{3*}$ and $t_i \leq t'_i + t''_i$ for all $i = 1, 2, 3$. Thus Φ has property (iii).

Let ε be a positive real number. Choose $\delta = \Psi(\varepsilon)$. Then $\delta > 0$ since $\Psi(t) = 0$ implies $t = 0$.

$$\begin{aligned}
 \Phi(0, t, t) < \delta &\Rightarrow \min \{ \max \{ \Psi(0), \Psi(t) \}, \max \{ \Psi(t), \Psi(t) \}, \\
 &\quad \max \{ \Psi(t), \Psi(0) \} \} < \Psi(\varepsilon) \\
 &\Rightarrow \min \{ \Psi(t), \Psi(t), \Psi(t) \} < \Psi(\varepsilon) \\
 &\Rightarrow \Psi(t) < \Psi(\varepsilon) \\
 &\Rightarrow t < \varepsilon \quad (\text{since } \Psi \text{ is monotonically increasing}).
 \end{aligned} \tag{1.53}$$

Hence Φ has property (iv). $\square$

COROLLARY 1.34. *The function $\Phi : (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ defined as $\Phi(t_1, t_2, t_3) = \min \{ \max \{ t_1, t_2 \}, \max \{ t_2, t_3 \}, \max \{ t_3, t_1 \} \}$ for all $(t_1, t_2, t_3) \in (\mathbb{R}^+)^{3*}$ is continuous on $(\mathbb{R}^+)^{3*}$ and has properties (i), (ii), (iii), and (iv) specified in [Theorem 1.24](#).*

PROOF. The proof follows from [Proposition 1.33](#) by taking $\Psi(t) = t$ for all $t \in \mathbb{R}^+$. $\square$

PROPOSITION 1.35. *The function $\Phi : (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ defined as*

$$\Phi(t_1, t_2, t_3) = \begin{cases} t_1 + t_2 + t_3 & \text{if } \min \{ |t_1 - t_2|, |t_2 - t_3|, |t_3 - t_1| \} \leq 1, \\ 1 + t_1 + t_2 + t_3 & \text{otherwise,} \end{cases} \tag{1.54}$$

has properties (i), (ii), (iii), and (iv) specified in [Theorem 1.24](#) and is continuous at $(t_1, t_2, t_3) \in (\mathbb{R}^+)^{3}$ if $\min \{ |t_1 - t_2|, |t_2 - t_3|, |t_3 - t_1| \} \neq 1$.*

PROOF. Clearly Φ has properties (i) and (ii).

Let $(t_1, t_2, t_3), (t_1, t'_2, t'_3), (t'_1, t_2, t''_3), (t''_1, t'_2, t_3) \in (\mathbb{R}^+)^{3*}$ be such that $t_i \leq t'_i + t''_i$ for all $i = 1, 2, 3$.

CASE (i). $\min \{ |t_1 - t_2|, |t_2 - t_3|, |t_3 - t_1| \} \leq 1$.

Then

$$\begin{aligned}
 \Phi(t_1, t_2, t_3) &= t_1 + t_2 + t_3 \\
 &\leq (t_1 + t'_2 + t'_3) + (t'_1 + t_2 + t''_3) + (t''_1 + t'_2 + t_3) \\
 &\leq \Phi(t_1, t'_2, t'_3) + \Phi(t'_1, t_2, t''_3) + \Phi(t''_1, t'_2, t_3).
 \end{aligned} \tag{1.55}$$

CASE (ii). $\min \{ |t_1 - t_2|, |t_2 - t_3|, |t_3 - t_1| \} > 1$.

Then $\max \{ t_1, t_2, t_3 \} > 1$, hence

$$\begin{aligned}
 \Phi(t_1, t_2, t_3) &= 1 + t_1 + t_2 + t_3 \\
 &< 2(t_1 + t_2 + t_3) \\
 &\leq (t_1 + t'_2 + t'_3) + (t'_1 + t_2 + t''_3) + (t''_1 + t'_2 + t_3) \\
 &\quad (\text{since } t_i \leq t'_i + t''_i \quad \forall i = 1, 2, 3). \\
 &\leq \Phi(t_1, t'_2, t'_3) + \Phi(t'_1, t_2, t''_3) + \Phi(t''_1, t'_2, t_3).
 \end{aligned} \tag{1.56}$$

Thus, in either case, $\Phi(t_1, t_2, t_3) \leq \Phi(t_1, t'_2, t'_3) + \Phi(t'_1, t_2, t''_3) + \Phi(t''_1, t'_2, t_3)$. Hence Φ has property (iii).

Let ε be a positive real number. Let $\delta = \varepsilon$. We have

$$\begin{aligned}\Phi(0, t, t) < \delta &\implies 0 + t + t < \varepsilon \\ &\implies 2t < \varepsilon \\ &\implies t < \varepsilon.\end{aligned}\tag{1.57}$$

Hence Φ has property (iv). □

PROPOSITION 1.36. *The function $\Phi: (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ defined as*

$$\Phi(t_1, t_2, t_3) = \begin{cases} t_1 + t_2 + t_3 & \text{if } \min\{t_1, t_2, t_3\} \leq 1, \\ 1 + t_1 + t_2 + t_3 & \text{otherwise,} \end{cases}\tag{1.58}$$

has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24 and is continuous at $(t_1, t_2, t_3) \in (\mathbb{R}^+)^{3}$ if $\min\{t_1, t_2, t_3\} \neq 1$.*

PROOF. The proof is similar to that of Proposition 1.35. □

PROPOSITION 1.37. *Let $\Phi: (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ be defined as*

$$\Phi(t_1, t_2, t_3) = \min\{t_1 + t_2, t_2 + t_3, t_3 + t_1\} \quad \forall (t_1, t_2, t_3) \in (\mathbb{R}^+)^{3*}.\tag{1.59}$$

Then Φ is continuous on $(\mathbb{R}^+)^{3}$ and has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24.*

PROOF. Let $t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3, t'_3, t''_3 \in \mathbb{R}^+$ be such that $t_i \leq t'_i + t''_i$ for all $i = 1, 2, 3$. We have

$$\begin{aligned}(t_1 + t'_2) + (t''_1 + t''_2) &\geq t_1 + t'_2 + t''_2 \geq t_1 + t_2, & (t_1 + t'_2) + (t''_2 + t_3) &\geq t_1 + t_3, \\ (t_1 + t'_2) + (t_3 + t'_1) &\geq t_1 + t_3.\end{aligned}\tag{1.60}$$

Hence $(t_1 + t'_2) + \min\{(t''_1 + t''_2), (t''_2 + t_3), (t_3 + t'_1)\} \geq \min\{t_1 + t_2, t_1 + t_3\}$. Similarly, we have

$$\begin{aligned}(t'_3 + t_1) + \min\{(t'_1 + t_2), (t_2 + t'_3), (t'_3 + t'_1)\} &\geq \min\{t_1 + t_2, t_1 + t_3\}, \\ (t'_1 + t_2) + \min\{(t''_1 + t''_2), (t''_2 + t_3), (t_3 + t'_1)\} &\geq \min\{t_1 + t_2, t_2 + t_3\}, \\ (t_2 + t'_3) + \min\{(t_1 + t'_2), (t'_2 + t'_3), (t'_3 + t_1)\} &\geq \min\{t_1 + t_2, t_2 + t_3\}, \\ (t''_2 + t_3) + \min\{(t_1 + t'_2), (t'_2 + t'_3), (t'_3 + t_1)\} &\geq \min\{t_1 + t_2, t_2 + t_3\}, \\ (t_3 + t'_1) + \min\{(t'_1 + t_2), (t_2 + t'_3), (t'_3 + t'_1)\} &\geq \min\{t_2 + t_3, t_3 + t_1\}.\end{aligned}\tag{1.61}$$

We have

$$(t'_2 + t'_3) + (t'_3 + t'_1) + (t'_1 + t'_2) = (t'_1 + t'_1) + (t'_2 + t'_2) + (t'_3 + t'_3) \geq t_1 + t_2 + t_3.\tag{1.62}$$

Hence

$$\begin{aligned}\min\{t_1 + t_2, t_2 + t_3, t_3 + t_1\} &\leq \min\{(t_1 + t'_2), (t'_2 + t'_3), (t'_3 + t_1)\} \\ &\quad + \min\{(t'_1 + t_2), (t_2 + t'_3), (t'_3 + t'_1)\} \\ &\quad + \min\{(t''_1 + t''_2), (t''_2 + t_3), (t_3 + t'_1)\}.\end{aligned}\tag{1.63}$$

Hence $\Phi(t_1, t_2, t_3) \leq \Phi(t_1, t'_2, t'_3) + \Phi(t'_1, t_2, t''_3) + \Phi(t''_1, t'_2, t_3)$ whenever $(t_1, t_2, t_3), (t_1, t'_2, t'_3), (t'_1, t_2, t''_3), (t''_1, t'_2, t_3) \in (\mathbb{R}^+)^{3*}$ and $t_i \leq t'_i + t''_i$ for all $i = 1, 2, 3$. Hence Φ has property (iii) specified in [Theorem 1.24](#). $\square$

REMARK 1.38. Let (X, ρ) be a D -metric space and $\{x_n\}$ a sequence in X . If $\{x_n\}$ converges to an element, say $x \in X$, then $\{\rho(y, z, x_n)\}$ need not converge to $\rho(y, z, x)$ for all $y, z \in X$. The following examples show that it is so, even when every convergent sequence is strongly convergent and has a unique limit and ρ -convergence defines a topology on X which is a metric topology. While in the first example we show the existence of a convergent sequence $\{x_n\}$ with limit, say x , and elements y, z of X such that $\{\rho(y, z, x_n)\}$ is convergent but not to $\rho(y, z, x)$, in the second example we show the existence of a convergent sequence $\{x_n\}$ with limit, say x , and elements y, z of X such that $\{\rho(y, z, x_n)\}$ is not convergent.

EXAMPLE 1.39. Define a function $\Phi: (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ as

$$\Phi(t_1, t_2, t_3) = \begin{cases} t_1 + t_2 + t_3 & \text{if } \min\{|t_1 - t_2|, |t_2 - t_3|, |t_3 - t_1|\} \leq 1, \\ 1 + t_1 + t_2 + t_3 & \text{otherwise.} \end{cases} \quad (1.64)$$

From [Proposition 1.35](#), we know that Φ has properties (i), (ii), (iii), and (iv) specified in [Theorem 1.24](#), and that Φ is continuous at (t_1, t_2, t_3) if $\min\{|t_1 - t_2|, |t_2 - t_3|, |t_3 - t_1|\} \neq 1$. Define $\rho: \mathbb{R}^3 \rightarrow \mathbb{R}^+$ as $\rho(x, y, z) = \Phi(|x - y|, |y - z|, |z - x|)$ for all $x, y, z \in \mathbb{R}$. Then, from [Corollary 1.25](#), it follows that ρ is a D -metric on $\mathbb{R}$, $(\mathbb{R}, \rho)$ is ρ -complete, and ρ -convergence defines a topology on $\mathbb{R}$ which is nothing but the usual topology on $\mathbb{R}$. Further, if $\{u_n\} \subseteq \mathbb{R}$ and $u \in \mathbb{R}$, then $|u_n - u| \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\{u_n\}$ converges to u with respect to ρ if and only if $\{u_n\}$ converges to u strongly with respect to ρ . Hence every ρ -convergent sequence has a unique limit. Let $x_n = 1 + 1/n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a sequence in $\mathbb{R}$. Clearly $\{x_n\}$ converges to 1 in the usual sense. Hence $\{x_n\}$ converges to 1 with respect to ρ . Let $x = 1$, $y = 3$, and $z = 6$. Then we have

$$\begin{aligned} \rho(y, z, x) &= \rho(3, 6, 1) = \Phi(|3 - 6|, |6 - 1|, |1 - 3|) = \Phi(3, 5, 2) \\ &= 10 \quad (\text{since } \min\{|3 - 5|, |5 - 2|, |2 - 3|\} = \min\{2, 3, 1\} = 1), \\ \rho(y, z, x_n) &= \rho\left(3, 6, 1 + \frac{1}{n}\right) = \Phi\left(|3 - 6|, \left|6 - 1 - \frac{1}{n}\right|, \left|1 + \frac{1}{n} - 3\right|\right) \\ &= \Phi\left(3, 5 - \frac{1}{n}, 2 - \frac{1}{n}\right) \\ &= 11 - \frac{2}{n} \quad \forall n \geq 2 \quad \left(\text{since } \min\left\{\left|3 - 5 + \frac{1}{n}\right|, |5 - 2|, \right.\right. \\ &\quad \left.\left|\left(2 - \frac{1}{n}\right) - 3\right|\right\} = \min\left\{3, 1 + \frac{1}{n}, 2 - \frac{1}{n}\right\} > 1\right) \\ &\rightarrow 11 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (1.65)$$

Hence $\{\rho(y, z, x_n)\}$ does not converge to $\rho(y, z, x)$.

EXAMPLE 1.40. Define a function $\Phi : (\mathbb{R}^+)^{3*} \rightarrow \mathbb{R}^+$ as

$$\Phi(t_1, t_2, t_3) = \begin{cases} t_1 + t_2 + t_3 & \text{if } \min\{t_1, t_2, t_3\} \leq 1, \\ 1 + t_1 + t_2 + t_3 & \text{otherwise.} \end{cases} \quad (1.66)$$

From [Proposition 1.36](#) we know that Φ has properties (i), (ii), (iii), and (iv) specified in [Theorem 1.24](#), and that Φ is continuous at $(t_1, t_2, t_3) \in (\mathbb{R}^+)^{3*}$ if and only if $\min\{t_1, t_2, t_3\} \neq 1$. Define $\rho : \mathbb{R}^3 \rightarrow \mathbb{R}^+$ as $\rho(x, y, z) = \Phi(|x - y|, |y - z|, |z - x|)$ for all $x, y, z \in \mathbb{R}$. Then, from [Corollary 1.25](#), it follows that ρ is a D -metric on $\mathbb{R}$, $(\mathbb{R}, \rho)$ is D -complete, and ρ -convergence defines a topology on $\mathbb{R}$ which is nothing but the usual topology on $\mathbb{R}$. Further, if $\{u_n\} \subseteq \mathbb{R}$ and $u \in \mathbb{R}$, then $|u_n - u| \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\{u_n\}$ converges to u with respect to ρ if and only if $\{u_n\}$ converges to u strongly with respect to ρ . Hence every ρ -convergent sequence has a unique limit.

For any $y \in \mathbb{R}$,

$$\begin{aligned} \rho\left(y, y+2, y+3-\frac{1}{n}\right) &= \Phi\left(2, 1-\frac{1}{n}, 3-\frac{1}{n}\right) \\ &= 6 - \frac{2}{n} \rightarrow 6 \quad \text{as } n \rightarrow \infty, \\ \rho\left(y, y+2, y+3+\frac{1}{n}\right) &= \Phi\left(2, 1+\frac{1}{n}, 3+\frac{1}{n}\right) \\ &= 1 + 6 + \frac{2}{n} \\ &\rightarrow 7 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (1.67)$$

The sequences $\{y+3-1/n\}$ and $\{y+3+1/n\}$ both converge to $y+3$. Let

$$x_n = \begin{cases} y+3-\frac{1}{n} & \text{if } n \text{ is odd,} \\ y+3+\frac{1}{n} & \text{if } n \text{ is even.} \end{cases} \quad (1.68)$$

Then $\{x_n\}$ converges to $y+3$, but $\{\rho(y, y+2, x_n)\}$ does not converge.

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