ON SCALAR TYPE SPECTRAL OPERATORS, INFINITE DIFFERENTIABLE AND GEVREY ULTRADIFFERENTIABLE C_0 -SEMIGROUPS

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Necessary and sufficient conditions for a scalar type spectral operator in a Banach space to be a generator of an infinite differentiable or a Gevrey ultradifferentiable C_0 -semigroup are found, the latter formulated exclusively in terms of the operator's spectrum.

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1. Introduction. Despite what was said in the final remarks to [22], the author did decide to tackle the problems of the generation of *infinite differentiable* and *Gevrey ultradifferentiable* C_0 -semigroups by a *scalar type spectral operator* in a complex Banach space. The more so as, in the former case, the task turned out to be more of a challenge than it seemed initially, the existence of a general characterization of infinite differentiable C_0 -semigroups [25] (see also [6, 26]) notwithstanding. In the latter case, such characterizations are not to be found in the plethora of the literature on the subject including such authoritative and exhaustive sources as [6, 9, 11, 15, 26, 28, 31].

In [22], the criteria of a scalar type spectral operator in a complex Banach space being a generator of a C_0 -semigroup and an analytic C_0 -semigroup were found. In the present paper, necessary and sufficient conditions for a scalar type spectral operator in a complex Banach space to be a generator of an *infinite differentiable* or a *Gevrey ultradifferentiable* C_0 -semigroup are established. The main purpose is to show that such criteria, as well as those of [22], can be formulated exclusively in terms of the operator's spectrum, without any restrictions on its *resolvent* behavior. This fact distinguishes the case of *scalar type spectral operators* and makes the aforementioned results significantly more transparent and purely qualitative.

2. Preliminaries

2.1. Scalar type spectral operators. Henceforth, unless otherwise specified, A is a *scalar type spectral operator* in a complex Banach space X with norm $\|\cdot\|$ and $E_A(\cdot)$ is its *spectral measure* (the *resolution of the density*), the operator's spectrum $\sigma(A)$ being the *support* for the latter [2, 5].

Note that, in a Hilbert space, the *scalar type spectral operators* are those similar to the *normal* ones [29].

For such operators, there has been developed an *operational calculus* for Borel measurable functions on $\mathbb{C}$ (on $\sigma(A)$) [2, 5], $F(\cdot)$ being such a function, a new *scalar type*

spectral operator

$$F(A) = \int_{\mathbb{C}} F(\lambda) dE_A(\lambda) = \int_{\sigma(A)} F(\lambda) dE_A(\lambda) \quad (2.1)$$

is defined as follows:

$$\begin{aligned} F(A)f &:= \lim_{n \rightarrow \infty} F_n(A)f, \quad f \in D(F(A)), \\ D(F(A)) &:= \left\{ f \in X \mid \lim_{n \rightarrow \infty} F_n(A)f \text{ exists} \right\} \end{aligned} \quad (2.2)$$

($D(\cdot)$ is the *domain* of an operator), where

$$F_n(\cdot) := F(\cdot) \chi_{\{\lambda \in \sigma(A) \mid |F(\lambda)| \leq n\}}(\cdot), \quad n = 1, 2, \dots, \quad (2.3)$$

($\chi_\alpha(\cdot)$ is the *characteristic function* of a set α), and

$$F_n(A) := \int_{\sigma(A)} F_n(\lambda) dE_A(\lambda), \quad n = 1, 2, \dots, \quad (2.4)$$

being the integrals of *bounded* Borel measurable functions on $\sigma(A)$, are *bounded scalar type spectral operators* on X defined in the same manner as for *normal operators* (see, e.g., [4, 27]).

The properties of the *spectral measure*, $E_A(\cdot)$, and the *operational calculus* underlying the entire subsequent argument are exhaustively delineated in [2, 5]. We just observe here that, due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded*, that is, there is an $M > 0$ such that, for any Borel set δ ,

$$\|E_A(\delta)\| \leq M, \quad (2.5)$$

see [3].

Observe that, in (2.5), the notation $\|\cdot\|$ was used to designate the norm in the space of bounded linear operators on X . We will adhere to this rather common economy of symbols in what follows, adopting the same notation for the norm in the dual space X^* as well.

Due to (2.5), for any $f \in X$ and $g^* \in X^*$ (X^* is the *dual space*), the total variation $v(f, g^*, \cdot)$ of the complex-valued measure $\langle E_A(\cdot)f, g^* \rangle$ ($\langle \cdot, \cdot \rangle$ is the *pairing* between the space X and its dual, X^*) is *bounded*. Indeed,

$$\begin{aligned} v(f, g^*, \sigma(A)) & \quad (\delta \text{ being an arbitrary Borel subset of } \sigma(A), [3]) \\ & \leq 4 \sup_{\delta \subseteq \sigma(A)} |\langle E_A(\delta)f, g^* \rangle| \leq 4 \sup_{\delta \subseteq \sigma(A)} \|E_A(\delta)\| \|f\| \|g^*\| \quad (\text{by (2.5)}) \\ & \leq 4M \|f\| \|g^*\|. \end{aligned} \quad (2.6)$$

For the reader's convenience, we reformulate here [23, Proposition 3.1], heavily relied upon in what follows, which allows to characterize the domains of the Borel measurable functions of a scalar type spectral operator in terms of positive measures (see [23] for a complete proof).

PROPOSITION 2.1 [23, Proposition 3.1]. *Let A be a scalar type spectral operator in a complex Banach space X and $F(\cdot)$ a complex-valued Borel measurable function on $\mathbb{C}$ (on $\sigma(A)$). Then $f \in D(F(A))$ if and only if*

(i) *for any $g^* \in X^*$,*

$$\int_{\sigma(A)} |F(\lambda)| d\nu(f, g^*, \lambda) < \infty, \quad (2.7)$$

(ii) $\sup_{\{g^* \in X^* : \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) : |F(\lambda)| > n\}} |F(\lambda)| d\nu(f, g^*, \lambda) \rightarrow 0$ as $n \rightarrow \infty$.

Observe that, $F(\cdot)$ being an arbitrary Borel measurable function on $\mathbb{C}$ (on $\sigma(A)$), for any $f \in D(F(A))$, $g^* \in X^*$, and arbitrary Borel sets $\delta \subseteq \sigma$,

$$\begin{aligned} & \int_{\sigma} |F(\lambda)| d\nu(f, g^*, \lambda) \quad (\text{see [3]}) \\ & \leq 4 \sup_{\delta \subseteq \sigma} \left| \int_{\delta} F(\lambda) d\langle E_A(\lambda)f, g^* \rangle \right| \\ & = 4 \sup_{\delta \subseteq \sigma} \left| \int_{\sigma} \chi_{\delta}(\lambda) F(\lambda) d\langle E_A(\lambda)f, g^* \rangle \right| \quad (\text{by the properties of the o.c.}) \\ & = 4 \sup_{\delta \subseteq \sigma} \left| \left\langle \int_{\sigma} \chi_{\delta}(\lambda) F(\lambda) dE_A(\lambda)f, g^* \right\rangle \right| \quad (\text{by the properties of the o.c.}) \\ & = 4 \sup_{\delta \subseteq \sigma} |\langle E_A(\delta)E_A(\sigma)F(A)f, g^* \rangle| \\ & \leq 4 \sup_{\delta \subseteq \sigma} \|E_A(\delta)E_A(\sigma)F(A)f\| \|g^*\| \\ & \leq 4 \sup_{\delta \subseteq \sigma} \|E_A(\delta)\| \|E_A(\sigma)F(A)f\| \|g^*\| \quad (\text{by (2.5)}) \\ & \leq 4M \|E_A(\sigma)F(A)f\| \|g^*\| \\ & \leq 4M \|E_A(\sigma)\| \|F(A)f\| \|g^*\|. \end{aligned} \quad (2.8)$$

In particular,

$$\begin{aligned} & \int_{\sigma(A)} |F(\lambda)| d\nu(f, g^*, \lambda) \quad (\text{by (2.8)}) \\ & \leq 4M \|E_A(\sigma(A))\| \|F(A)f\| \|g^*\| \\ & \quad (\text{since } E_A(\sigma(A)) = I \text{ (} I \text{ is the identity operator on } X\text{)}) \\ & \leq 4M \|F(A)f\| \|g^*\|. \end{aligned} \quad (2.9)$$

On account of compactness, the terms *spectral measure* and *operational calculus* for scalar type spectral operators, frequently referred to, will be abbreviated to *s.m.* and *o.c.*, respectively.

Observe also that, as follows directly from the results of [1, 23], if a scalar type spectral operator A generates a C_0 -semigroup $\{T(t) \mid t \geq 0\}$, the latter is of the form

$$T(t) = e^{tA}, \quad 0 \leq t < \infty. \quad (2.10)$$

2.2. The Gevrey classes of vectors. Let A be a linear operator in a Banach space X with norm $\|\cdot\|$,

$$C^\infty(A) \stackrel{\text{def}}{=} \bigcap_{n=0}^{\infty} D(A^n), \quad (2.11)$$

and $0 \leq \beta < \infty$.

The sets of vectors

$$\begin{aligned} \mathcal{E}^{\{\beta\}}(A) &\stackrel{\text{def}}{=} \{f \in C^\infty(A) \mid \exists \alpha > 0, \exists c > 0 : \|A^n f\| \leq c \alpha^n [n!]^\beta, n = 0, 1, \dots\}, \\ \mathcal{E}^{(\beta)}(A) &\stackrel{\text{def}}{=} \{f \in C^\infty(A) \mid \forall \alpha > 0 \exists c > 0 : \|A^n f\| \leq c \alpha^n [n!]^\beta, n = 0, 1, \dots\} \end{aligned} \quad (2.12)$$

are called the β th-order *Gevrey classes* of the operator A of *Roumie's* and *Beurling's types*, respectively.

For $0 \leq \beta < \beta' < \infty$,

$$\mathcal{E}^{(\beta)}(A) \subseteq \mathcal{E}^{\{\beta\}}(A) \subseteq \mathcal{E}^{(\beta')}(A) \subseteq \mathcal{E}^{\{\beta'\}}(A) \subseteq C^\infty(A). \quad (2.13)$$

In particular, $\mathcal{E}^{\{1\}}(A)$ and $\mathcal{E}^{(1)}(A)$ are the well-known classes of *analytic* and *entire vectors*, correspondingly [10, 24].

Observe that, in the definitions of the Gevrey classes, due to *Stirling's formula*, one can replace $[n!]^\beta$ by $n^{\beta n}$.

According to [17], for a *scalar type spectral operator* A in a complex Banach space X and $0 < \beta < \infty$,

$$\begin{aligned} \mathcal{E}^{\{\beta\}}(A) &\supseteq \bigcup_{t>0} D(e^{t|A|^{1/\beta}}), \\ \mathcal{E}^{(\beta)}(A) &\supseteq \bigcap_{t>0} D(e^{t|A|^{1/\beta}}), \end{aligned} \quad (2.14)$$

the inclusions becoming equalities provided that the space X is *reflexive*.

2.3. Gevrey ultradifferentiability. A smoothness higher than *infinite differentiability* ranging up to *real analyticity* and *entireness* was introduced for numerical functions by Gevrey in 1918 [7] and is naturally extrapolated to functions with values in a Banach space.

Let I be an interval of the *real axis*, $\mathbb{R}$, $C^\infty(I, X)$ the set of all X -valued functions *strongly infinite differentiable* on I , and $0 \leq \beta < \infty$.

The sets of vectors

$$\begin{aligned} \mathcal{E}^{\{\beta\}}(I, X) &\stackrel{\text{def}}{=} \left\{ g(\cdot) \in C^\infty(I, X) \mid \forall [a, b] \subseteq I \exists \alpha > 0, \exists c > 0 : \right. \\ &\quad \left. \max_{a \leq t \leq b} \|g^{(n)}(t)\| \leq c \alpha^n [n!]^\beta, n = 0, 1, \dots \right\}, \\ \mathcal{E}^{(\beta)}(I, X) &\stackrel{\text{def}}{=} \left\{ g(\cdot) \in C^\infty(I, X) \mid \forall [a, b] \subseteq I, \forall \alpha > 0 \exists c > 0 : \right. \\ &\quad \left. \max_{a \leq t \leq b} \|g^{(n)}(t)\| \leq c \alpha^n [n!]^\beta, n = 0, 1, \dots \right\} \end{aligned} \quad (2.15)$$

are the β th-order Gevrey classes of strongly ultradifferentiable functions of Roumie's and Beurling's types, respectively (see, e.g., [7, 12, 13, 14]).

Just as above, due to Stirling's formula, one can replace $[n!]^\beta$ by $n^{\beta n}$.

For $0 \leq \beta < \beta' < \infty$, the following inclusions hold:

$$\mathcal{E}^{(\beta)}(I, X) \subseteq \mathcal{E}^{(\beta)}(I, X) \subseteq \mathcal{E}^{(\beta')}(I, X) \subseteq \mathcal{E}^{(\beta')}(I, X). \quad (2.16)$$

In particular, $\mathcal{E}^{(1)}(I, X)$ is the class of all *real analytic* on I vector functions (i.e., *analytically continuable* into complex neighborhoods of the interval I) and $\mathcal{E}^{(1)}(I, X)$ is the class of all *entire* vector functions (i.e., allowing *entire* continuations) (for numerical functions, see [16]).

Note that it is well known that the Gevrey classes of functions of orders greater than one are *quasianalytic*.

3. On the strong smoothness of an orbit of a C_0 -semigroup generated by a scalar type spectral operator. Let A be a scalar type spectral operator generating a C_0 -semigroup $\{T(t) \mid t \geq 0\}$.

PROPOSITION 3.1. *Let I be a subinterval of $[0, \infty)$ and $0 < \beta < \infty$. Then the restriction of an orbit $T(\cdot)f$, $f \in X$, to I*

- (i) *belongs to $C^\infty(I, X)$ if and only if $T(t)f \in C^\infty(A)$, for any $t \in I$,*
- (ii) *belongs to $\mathcal{E}^{(\beta)}(I, X)$ (resp., $\mathcal{E}^{(\beta)}(I, X)$) if and only if $T(t)f \in \mathcal{E}^{(\beta)}(A)$ (resp., $\mathcal{E}^{(\beta)}(A)$), for any $t \in I$.*

PROOF

(i) **"ONLY IF" PART.** Assume that the restriction of an orbit $T(\cdot)f$ of the C_0 -semigroup generated by A to a subinterval I of $[0, \infty)$ belongs to $C^\infty(I, X)$.

Taking into account that $T(\cdot)f$ is a *weak solution* of the evolution equation

$$y'(t) = Ay(t) \quad (3.1)$$

on $[0, \infty)$ [1], we have, for any $g \in D(A^*)$,

$$\langle T'(t)f, g \rangle = \frac{d}{dt} \langle T(t)f, g \rangle = \langle T(t)f, A^*g \rangle, \quad t \in I. \quad (3.2)$$

Whence, by the *closedness* of the operator A ,

$$T(t)f \in D(A), \quad T'(t)f = AT(t)f, \quad \text{for any } t \in I, \quad (3.3)$$

(see [1, 8] for details).

Let $n > 1$. Then, differentiating (3.3) for an arbitrary fixed $t \in I$, we obtain

$$T''(t)f = \lim_{\Delta t \rightarrow 0} \frac{T'(t + \Delta t)f - T'(t)f}{\Delta t} = \lim_{\Delta t \rightarrow 0} A \frac{T(t + \Delta t)f - T(t)f}{\Delta t}, \quad (3.4)$$

where the increments Δt are such that $t + \Delta \in I$.

Since

$$\lim_{\Delta t \rightarrow 0} \frac{T(t + \Delta t)f - T(t)f}{\Delta t} = T'(t)f, \quad (3.5)$$

by the closedness of A , we infer that

$$T'(t)f \in D(A), \quad T''(t)f = AT'(t)f, \quad \text{for any } t \in I. \quad (3.6)$$

Thus, (3.3) and (3.6) imply

$$T(t)f \in D(A^2), \quad T''(t)f = A^2T(t)f, \quad \text{for any } t \in I. \quad (3.7)$$

Continuing inductively in this manner, we infer that for any $n = 1, 2, \dots$,

$$T(t)f \in C^\infty(A), \quad T^{(n)}(t)f = A^nT(t)f, \quad t \in I. \quad (3.8)$$

“If” PART. Let $T(\cdot)f$ be an orbit of the C_0 -semigroup generated by A such that

$$T(t)f \in C^\infty(A) \quad \text{for any } t \in I, \quad (3.9)$$

where I is a subinterval of $[0, \infty)$.

Recall that, as was noted in Section 2, the C_0 -semigroup $\{T(t) \mid t \geq 0\}$ generated by A is of the form (2.10).

The fact that

$$e^{tA}f \in C^\infty(A), \quad t \in I, \quad (3.10)$$

by [23, Proposition 3.1], implies that, for any $g^* \in X^*$,

$$\int_{\sigma(A)} |\lambda|^n e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) < \infty, \quad n = 1, 2, \dots, \quad t \in I. \quad (3.11)$$

Given a natural n and an arbitrary fixed $t \in [0, T)$, we choose a segment $[a, b] \subset [0, T)$ ($a < b$) so that t is its midpoint if $0 < t < T$, or $a = 0$ if $t = 0$. For increments Δt such that $a \leq t + \Delta t \leq b$ and any $g^* \in X^*$, we have

$$\begin{aligned} & \left| \left\langle \frac{A^{n-1}e^{t+\Delta t}f - A^{n-1}e^t f}{\Delta t} - A^n e^t f, g^* \right\rangle \right| \quad (\text{by (2.10) and the properties of the o.c.}) \\ &= \left| \left\langle \int_{\sigma(A)} \left[\frac{\lambda^{n-1}e^{(t+\Delta t)\lambda} - \lambda^{n-1}e^{t\lambda}}{\Delta t} - \lambda^n e^{t\lambda} \right] dE_A(\lambda) f, g^* \right\rangle \right| \\ & \quad (\text{by the properties of the o.c.}) \\ &\leq \left| \int_{\sigma(A)} \left[\frac{\lambda^{n-1}e^{(t+\Delta t)\lambda} - \lambda^{n-1}e^{t\lambda}}{\Delta t} - \lambda^n e^{t\lambda} \right] d \langle E_A(\lambda) f, g^* \rangle \right| \\ &\leq \int_{\sigma(A)} \left| \frac{\lambda^{n-1}e^{(t+\Delta t)\lambda} - \lambda^{n-1}e^{t\lambda}}{\Delta t} - \lambda^n e^{t\lambda} \right| d\nu(f, g^*, \lambda) \\ & \quad (\text{by the Lebesgue dominated convergence theorem}) \\ &\longrightarrow 0 \quad \text{as } \Delta t \longrightarrow 0. \end{aligned} \quad (3.12)$$

Indeed, for any $\lambda \in \sigma(A)$,

$$\begin{aligned}
 & \left| \frac{\lambda^{n-1}e^{(t+\Delta t)\lambda} - \lambda^{n-1}e^{t\lambda}}{\Delta t} - \lambda^n e^{t\lambda} \right| \\
 & \leq \left| \frac{\lambda^{n-1}e^{(t+\Delta t)\lambda} - \lambda^{n-1}e^{t\lambda}}{\Delta t} \right| + |\lambda^n e^{t\lambda}| \quad (\text{by the total change theorem}) \\
 & \leq \max_{a \leq s \leq b} |\lambda^n e^{s\lambda}| + |\lambda^n e^{t\lambda}| \leq 2|\lambda|^n \max_{a \leq s \leq b} e^{s \operatorname{Re} \lambda} \\
 & \leq 2 \begin{cases} |\lambda|^n e^{a \operatorname{Re} \lambda}, & \text{if } \operatorname{Re} \lambda < 0, \\ |\lambda|^n e^{b \operatorname{Re} \lambda}, & \text{if } \operatorname{Re} \lambda \geq 0, \end{cases} \quad (\text{by (3.11), considering that } a, b \in I) \\
 & \in \mathcal{L}^1(\sigma(A), \nu(f, g^*, \cdot)), \quad n = 1, 2, \dots, \\
 & \left| \frac{\lambda^{n-1}e^{(t+\Delta t)\lambda} - \lambda^{n-1}e^{t\lambda}}{\Delta t} - \lambda^n e^{t\lambda} \right| \rightarrow 0 \quad \text{as } \Delta t \rightarrow 0.
 \end{aligned} \tag{3.13}$$

We have shown that, for any $t \in I$ and an arbitrary $n = 1, 2, \dots$,

$$\frac{A^{n-1}e^{t+\Delta t}f - A^{n-1}e^t f}{\Delta t} \xrightarrow{w} A^n \mathcal{Y}(t) \rightarrow 0 \quad \text{as } \Delta t \rightarrow 0. \tag{3.14}$$

Thus, we have proved that, for any $g^* \in X^*$,

$$\frac{d^n}{dt^n} \langle e^{tA} f, g^* \rangle = \langle A^n e^{tA} f, g^* \rangle, \quad n = 1, 2, \dots, t \in I. \tag{3.15}$$

Now, let

$$\Delta_n := \{\lambda \in \mathbb{C} \mid |\lambda| \leq n\}. \tag{3.16}$$

We fix an arbitrary natural k and consider the sequence of functions

$$E_A(\Delta_n) A^k e^{tA} f, \quad n = 1, 2, \dots, t \in I. \tag{3.17}$$

By the properties of the o.c.,

$$\begin{aligned}
 & E_A(\Delta_n) A^k e^{tA} \\
 & = \int_{\mathbb{C}} \chi_{\Delta_n}(\lambda) \lambda^k e^{t\lambda} dE_A(\lambda) \quad (\text{where } \chi_{\Delta_n}(\cdot) \text{ is the characteristic function of the set } \Delta_n) \\
 & = \int_{\mathbb{C}} [\lambda \chi_{\Delta_n}(\lambda)]^k e^{t\lambda \chi_{\Delta_n}(\lambda)} dE_A(\lambda) \\
 & = [AE_A(\Delta_n)]^k e^{tAE_A(\Delta_n)}.
 \end{aligned} \tag{3.18}$$

Since, by the properties of the o.c., for any natural n , the operator $AE_A(\Delta_n)$ is a bounded operator on X ($\|AE_A(\Delta_n)\| \leq 4Mn$) [5], the vector function

$$E_A(\Delta_n) A^k e^{tA} f = [AE_A(\Delta_n)]^k e^{tAE_A(\Delta_n)} f, \quad n = 1, 2, \dots, t \in I, \tag{3.19}$$

is strongly continuous.

For an arbitrary segment $[a, b] \subseteq I$, we have

$$\begin{aligned}
 & \sup_{a \leq t \leq b} \|A^k e^{tA} f - E_A(\Delta_n) A^k e^{tA} f\| \quad (\text{by the properties of the o.c.}) \\
 &= \sup_{a \leq t \leq b} \left\| \int_{\{\lambda \in \sigma(A) \mid |\lambda| > n\}} \lambda^k e^{t\lambda} dE_A(\lambda) f \right\| \\
 & \quad (\text{as follows from the Hahn-Banach theorem}) \\
 &= \sup_{a \leq t \leq b} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left| \left\langle \int_{\{\lambda \in \sigma(A) \mid |\lambda| > n\}} \lambda^k e^{t\lambda} dE_A(\lambda) f, g^* \right\rangle \right| \\
 & \quad (\text{by the properties of the o.c.}) \\
 &= \sup_{a \leq t \leq b} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left| \int_{\{\lambda \in \sigma(A) \mid |\lambda| > n\}} \lambda^k e^{t\lambda} d\langle E_A(\lambda) f, g^* \rangle \right| \\
 &\leq \sup_{a \leq t \leq b} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid |\lambda| > n\}} |\lambda|^k e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 &\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sup_{a \leq t \leq b} \left[\int_{\{\lambda \in \sigma(A) \mid |\lambda| > n, \operatorname{Re} \lambda \leq 0\}} |\lambda|^k e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \right. \\
 & \quad \left. + \int_{\{\lambda \in \sigma(A) \mid |\lambda| > n, \operatorname{Re} \lambda > 0\}} |\lambda|^k e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \right] \\
 &\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid |\lambda| > n, \operatorname{Re} \lambda \leq 0\}} |\lambda|^k e^{a \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \quad + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid |\lambda| > n, \operatorname{Re} \lambda > 0\}} |\lambda|^k e^{b \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 &\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid |\lambda| > n\}} |\lambda|^k e^{a \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \quad + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid |\lambda| > n\}} |\lambda|^k e^{b \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \quad (\text{by (2.8)}) \\
 &\leq 4M \left[\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \|E_A(\{\lambda \in \sigma(A) \mid |\lambda| > n\}) A^k e^{aA} f\| \|g^*\| \right. \\
 & \quad \left. + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \|E_A(\{\lambda \in \sigma(A) \mid |\lambda| > n\}) A^k e^{bA} f\| \|g^*\| \right] \\
 &\leq 4M [\|E_A(\{\lambda \in \sigma(A) \mid |\lambda| > n\}) A^k e^{aA} f\| + \|E_A(\{\lambda \in \sigma(A) \mid |\lambda| > n\}) A^k e^{bA} f\|] \\
 & \quad (\text{by the strong continuity of the s.m.}) \\
 &\rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}
 \tag{3.20}$$

Therefore, the vector function

$$A^k e^{tA} f, \quad k = 1, 2, \dots, t \in I, \tag{3.21}$$

is strongly continuous, being the uniform limit of a sequence of strongly continuous functions on an arbitrary segment $[a, b] \subseteq I$.

We fix an arbitrary $a \in I$ and integrate (3.15) for $n = 1$ between a and an arbitrary $t \in I$. Considering the strong continuity of $Ae^{tA}f$, $t \in I$, we have

$$\langle e^{tA}f - e^{aA}f, g^* \rangle = \left\langle \int_a^t Ae^{sA}f ds, g^* \right\rangle, \quad g^* \in X^*. \quad (3.22)$$

Whence, as follows from the *Hahn-Banach theorem*,

$$e^{tA}f - e^{aA}f = \int_a^t Ae^{sA}f ds. \quad (3.23)$$

By the strong continuity of $Ae^{tA}f$, $t \in I$, we infer that

$$\frac{d}{dt}e^{tA}f = Ae^{tA}f, \quad t \in I. \quad (3.24)$$

Consequently, by (3.15) for $n = 2$,

$$\frac{d}{dt} \left\langle \frac{d}{dt}e^{tA}f, g^* \right\rangle = \langle A^2e^{tA}f, g^* \rangle, \quad t \in I. \quad (3.25)$$

Whence, analogously,

$$\frac{d^2}{dt^2}e^{tA}f = A^2e^{tA}f, \quad t \in I. \quad (3.26)$$

Continuing inductively in this manner, we infer that, for any natural n ,

$$\frac{d^n}{dt^n}e^{tA}f = A^n e^{tA}f, \quad t \in I. \quad (3.27)$$

(ii) **“ONLY IF” PART.** Assume that an orbit $T(\cdot)f$, $f \in X$, of the C_0 -semigroup $\{T(t) \mid t \geq 0\}$ generated by A restricted to a subinterval $I \subseteq [0, \infty)$ belongs to $\mathcal{E}^{\{\beta\}}(I, X)$ (resp., $\mathcal{E}^{(\beta)}(I, X)$).

This necessarily implies that $T(\cdot)f \in C^\infty(I, H)$. Whence, by (i),

$$T(t)f \in C^\infty(A), \quad T^{(n)}(t)f = A^n T(t)f, \quad n = 1, 2, \dots, \quad t \in I. \quad (3.28)$$

Furthermore, the fact that the restriction of $\mathcal{Y}(\cdot)$ to I belongs to the class $\mathcal{E}^{\{\beta\}}(I, X)$ (resp., $\mathcal{E}^{(\beta)}(I, X)$) implies that, for an arbitrary $t \in I$, a certain (any) $\alpha > 0$, and a certain $c > 0$,

$$\|A^n T(t)f\| = \|T^{(n)}(t)f\| \leq c\alpha^n [n!]^\beta, \quad n = 0, 1, \dots \quad (3.29)$$

Therefore,

$$T(t)f \in \mathcal{E}^{\{\beta\}}(A) \text{ (resp., } \mathcal{E}^{(\beta)}(A)), \quad t \in I. \quad (3.30)$$

“If” PART. Let an orbit $T(\cdot)f$, $f \in X$, of the C_0 -semigroup $\{T(t) \mid t \geq 0\}$ generated by A be such that (3.30) holds, where I is a subinterval of $[0, \infty)$.

Hence, for arbitrary $t \in I$ and some (any) $\alpha(t) > 0$, there is such a $c(t, \alpha) > 0$ that

$$\|A^n T(t)f\| \leq c(t, \alpha) \alpha(t)^n [n!]^\beta, \quad n = 0, 1, 2, \dots \quad (3.31)$$

The inclusions

$$\mathcal{E}^{(\beta)}(A) \subseteq \mathcal{E}^{\{\beta\}}(A) \subseteq C^\infty(A) \quad (3.32)$$

imply, by (i), that (3.28) holds. Recall that

$$T(t)f = e^{tA}f, \quad 0 \leq t < \infty. \quad (3.33)$$

We fix an arbitrary subsegment $[a, b] \subseteq I$. For $n = 0, 1, \dots$, we have

$$\begin{aligned} & \max_{a \leq t \leq b} \|T^{(n)}(t)f\| \\ &= \max_{a \leq t \leq b} \|A^n T(t)f\| \\ &= \max_{a \leq t \leq b} \|A^n e^{tA}f\| \quad (\text{by the properties of the o.c. and the Hahn-Banach theorem}) \\ &= \max_{a \leq t \leq b} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left| \left\langle \int_{\sigma(A)} \lambda^n e^{t\lambda} dE_A(\lambda) f, g^* \right\rangle \right| \quad (\text{by the properties of the o.c.}) \\ &\leq \max_{a \leq t \leq b} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left| \int_{\sigma(A)} \lambda^n e^{t\lambda} d\langle E_A(\lambda) f, g^* \rangle \right| \\ &\leq \max_{a \leq t \leq b} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} |\lambda|^n e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ &= \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \max_{a \leq t \leq b} \left[\int_{\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \leq 0\}} |\lambda|^n e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \right. \\ &\quad \left. + \int_{\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda > 0\}} |\lambda|^n e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \right] \\ &\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \leq 0\}} |\lambda|^n e^{a \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ &\quad + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda > 0\}} |\lambda|^n e^{b \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ &\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} |\lambda|^n e^{a \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ &\quad + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} |\lambda|^n e^{b \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \quad (\text{by (2.9)}) \\ &\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \|A^n e^{aA}f\| \|g^*\| + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \|A^n e^{bA}f\| \|g^*\| \\ &\leq 4M [\|A^n e^{aA}f\| + \|A^n e^{bA}f\|] = 4M [\|A^n \mathcal{Y}(a)\| + \|A^n \mathcal{Y}(b)\|] \quad (\text{by (3.28)}) \\ &= 4M [\|\mathcal{Y}^{(n)}(a)\| + \|\mathcal{Y}^{(n)}(b)\|]. \end{aligned} \quad (3.34)$$

Hence, in view of (3.31), we obtain

$$\begin{aligned} \max_{a \leq t \leq b} \|T^{(n)}(t)f\| \\ \leq 4M[c(a, \alpha) + c(b, \alpha)] \max[\alpha(a), \alpha(b)]^n [n!]^\beta, \quad n = 0, 1, 2, \dots, \end{aligned} \quad (3.35)$$

which implies that the restriction of $T(\cdot)f$ to the subinterval $I \subseteq [0, T)$ belongs to the Gevrey class $\mathcal{E}^{\{\beta\}}(I, X)$ (resp., $\mathcal{E}^{(\beta)}(I, X)$). $\square$

4. Infinite differentiable C_0 -semigroups. Recall that a C_0 -semigroup $\{T(t) \mid t \geq 0\}$ in a Banach space X is said to be *infinite differentiable* (a C^∞ -semigroup) if, for any $f \in X$, the orbit $T(\cdot)f$ is infinite differentiable on $(0, \infty)$ in the strong sense. Note that, due to the semigroup property $T(t+s) = T(t)T(s)$, $t, s \geq 0$, the first-order strong differentiability of an orbit on $(0, \infty)$ immediately implies its infinite strong differentiability on $(0, \infty)$.

THEOREM 4.1. *A C_0 -semigroup generated by a scalar type spectral operator A in a complex Banach space X is infinite differentiable if and only if, for an arbitrary positive b , there is a real a such that*

$$\operatorname{Re} \lambda \leq a - b \ln |\operatorname{Im} \lambda|, \quad \lambda \in \sigma(A). \quad (4.1)$$

PROOF

“ONLY IF” PART. This part immediately follows from the general criterion of the generation of infinite differentiable C_0 -semigroups [25] (see also [6, 26]).

“IF” PART. Here, unlike in [22], resorting to the general criterion, that is, proving that there is a $C > 0$ such that in the region

$$R_b := \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > a - b \ln |\operatorname{Im} \lambda|\} \subseteq \rho(A) \quad (4.2)$$

($\rho(\cdot)$ is the *resolvent set* of an operator) the estimate

$$\|R(\lambda, A)\| \leq C |\operatorname{Im} \lambda| \quad (4.3)$$

holds, brings about rather formidable difficulties. The reader could try evaluating the distance from a point $\lambda \in R_b$ to the boundary of the region R_b such an approach would inevitably entail.

Utilizing the general criterion not being an option, we are to prove directly that all the orbits of the semigroup generated by A are strongly differentiable on $(0, \infty)$.

Since A generates a C_0 -semigroup, the latter, as was shown in [22], consists of its exponentials,

$$e^{tA} = \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \quad t \geq 0, \quad (4.4)$$

and there is a real ω such that

$$\operatorname{Re} \lambda \leq \omega, \quad \lambda \in \sigma(A). \quad (4.5)$$

Without loss of generality, we can regard that

$$\operatorname{Re} \lambda \leq 0, \quad \lambda \in \sigma(A), \quad (4.6)$$

that is, we deal with a *contraction* semigroup. Indeed, otherwise, we can consider the C_0 -semigroup $T(t) := e^{-\omega t} e^{tA}$, $t \geq 0$, which, evidently, satisfies (4.1).

We need to show that, for any $f \in X$,

$$e^{tA} f \in D(A), \quad 0 < t < \infty. \quad (4.7)$$

Let $0 < t < \infty$. Since the constant b can acquire arbitrary positive values, we can set $b := 1/t$. Then, for any Borel set $\sigma \subseteq \sigma(A)$ and arbitrary $f \in X$ and $g^* \in X^*$,

$$\begin{aligned} & \int_{\sigma} |\lambda| e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ & \leq \int_{\sigma} (|\operatorname{Re} \lambda| + |\operatorname{Im} \lambda|) e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ & \quad (\text{for } \lambda \in \sigma, \operatorname{Re} \lambda \leq \min(0, a - b \ln |\operatorname{Im} \lambda|) \Rightarrow \operatorname{Re} \lambda \leq 0 \text{ and } |\operatorname{Im} \lambda| \leq e^{b^{-1}(a - \operatorname{Re} \lambda)}) \\ & \leq \int_{\sigma} (-\operatorname{Re} \lambda + e^{ab^{-1}} e^{b^{-1}(-\operatorname{Re} \lambda)}) e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \quad (\text{since } x \leq e^x, \quad 0 \leq x < \infty) \\ & \leq \int_{\sigma} (b e^{b^{-1}(-\operatorname{Re} \lambda)} + e^{ab^{-1}} e^{b^{-1}(-\operatorname{Re} \lambda)}) e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ & = \left[b + e^{ab^{-1}} \right] \int_{\sigma} e^{(t-b^{-1}) \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \quad \left(\text{by the choice } b = \frac{1}{t} \right) \\ & = \left[\frac{1}{t} + e^{at} \right] \int_{\sigma} 1 d\nu(f, g^*, \lambda) = \left[\frac{1}{t} + e^{at} \right] \nu(f, g^*, \sigma). \end{aligned} \quad (4.8)$$

This estimate, by [23, Proposition 3.1], implies (4.7).

Indeed,

(i) for any $f \in X$ and $g^* \in X^*$, we have

$$\begin{aligned} \int_{\sigma(A)} |\lambda| e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) & \leq \left[\frac{1}{t} + e^{at} \right] \nu(f, g^*, \sigma(A)) \quad (\text{by (2.6)}) \\ & \leq 4M \left[\frac{1}{t} + e^{at} \right] \|f\| \|g^*\|, \quad 0 < t < \infty, \end{aligned} \quad (4.9)$$

(ii) analogously, for any $0 < t < \infty$ and an arbitrary $f \in X$,

$$\begin{aligned}
 & \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid |\lambda| e^{t \operatorname{Re} \lambda} > n\}} |\lambda| e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \leq \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \left[\frac{1}{t} + e^{at} \right] \int_{\{\lambda \in \sigma(A) \mid |\lambda| e^{t \operatorname{Re} \lambda} > n\}} 1 d\nu(f, g^*, \lambda) \quad (\text{by (2.8)}) \\
 & \leq \left[\frac{1}{t} + e^{at} \right] \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} 4M \|E_A(\{\lambda \in \sigma(A) \mid |\lambda| e^{t \operatorname{Re} \lambda} > n\})f\| \|g^*\| \quad (4.10) \\
 & \leq 4M \left[\frac{1}{t} + e^{at} \right] \|E_A(\{\lambda \in \sigma(A) \mid |\lambda| e^{t \operatorname{Re} \lambda} > n\})f\| \\
 & \quad (\text{by the strong continuity of the s.m.}) \\
 & \rightarrow 0, \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

Thus, by [23, Proposition 3.1], (4.7) holds. $\square$

5. Gevrey ultradifferentiable C_0 -semigroups. Let $0 < \beta < \infty$. We will call a C_0 -semigroup $\{T(t) \mid t \geq 0\}$ in a Banach space X an $\mathcal{E}^{\{\beta\}}$ -semigroup (resp., an $\mathcal{E}^{(\beta)}$ -semigroup) if, for any $f \in X$, the orbit $T(\cdot)f$ belongs to the Gevrey class $\mathcal{E}^{\{\beta\}}((0, \infty), X)$ (resp., $\mathcal{E}^{(\beta)}((0, \infty), X)$). We will call a C_0 -semigroup a *Gevrey ultradifferentiable* semigroup if, for some $0 < \beta < \infty$, it is an $\mathcal{E}^{\{\beta\}}$ -semigroup or, which, due to inclusions (2.16), is the same, an $\mathcal{E}^{(\beta)}$ -semigroup.

THEOREM 5.1. *Let $1 \leq \beta < \infty$. A C_0 -semigroup generated by a scalar type spectral operator A in a complex Banach space X is an $\mathcal{E}^{\{\beta\}}$ -semigroup if and only if there are a positive b and a real a such that*

$$\operatorname{Re} \lambda \leq a - b |\operatorname{Im} \lambda|^{1/\beta}, \quad \lambda \in \sigma(A). \quad (5.1)$$

PROOF

“IF” PART. As is easily seen, the sufficiency condition of Theorem 5.1 is stronger than that of Theorem 4.1. Therefore, by Theorem 4.1, we infer that A generates an infinitely differentiable C_0 -semigroup consisting, according to [22], of its exponentials presented in (4.4). For any $f \in X$ and $n = 1, 2, \dots$,

$$\frac{d^n}{dt^n} e^{tA} f = A^n e^{tA} f, \quad 0 < t < \infty. \quad (5.2)$$

According to Proposition 3.1, we need to show that, for any $f \in X$,

$$e^{tA} f \in \mathcal{E}^{\{\beta\}}(A), \quad 0 < t < \infty. \quad (5.3)$$

In view of inclusions (2.14), it suffices to show that

$$e^{tA} f \in \bigcup_{s>0} D(e^{s|A|^{1/\beta}}), \quad 0 < t < \infty. \quad (5.4)$$

We fix an arbitrary Borel subset σ of $\sigma(A)$ and an arbitrary $t > 0$. We also set $s := t/[1 + (2/b)^\beta]^{1/\beta} > 0$ (such a peculiar choice of s will make sense later).

For any $f \in X$ and $g^* \in X^*$,

$$\begin{aligned} & \int_{\sigma} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} d\nu(f, g^*, \lambda) \\ &= \int_{\{\lambda \in \sigma \mid \operatorname{Re}\lambda \leq \min(-1, a)\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} d\nu(f, g^*, \lambda) \\ &+ \int_{\{\lambda \in \sigma \mid \min(-1, a) < \operatorname{Re}\lambda \leq a\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} d\nu(f, g^*, \lambda) < \infty. \end{aligned} \quad (5.5)$$

Indeed, the latter integral is finite due to the boundedness of the set $\{\lambda \in \sigma \mid \min(-1, a) < \operatorname{Re}\lambda \leq a\}$ (note that, for $a \leq -1$, the set is, obviously, empty), the continuity of the integrated function, and the finiteness of the positive measure $\nu(f, g^*, \cdot)$ (see (2.6)).

For the former of the above two integrals, we have

$$\begin{aligned} & \int_{\{\lambda \in \sigma \mid \operatorname{Re}\lambda \leq \min(-1, a)\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} d\nu(f, g^*, \lambda) \\ & \quad (\lambda \in \sigma, \operatorname{Re}\lambda \leq \min(-1, a) \Rightarrow \operatorname{Re}\lambda \leq -1 \text{ and } |\operatorname{Im}\lambda| \leq b^{-\beta}(a - \operatorname{Re}\lambda)^{\beta}) \quad (5.6) \\ & \leq \int_{\{\lambda \in \sigma \mid \operatorname{Re}\lambda \leq \min(-1, a)\}} e^{s[-\operatorname{Re}\lambda + b^{-\beta}(a - \operatorname{Re}\lambda)^{\beta}]^{1/\beta}} e^{t\operatorname{Re}\lambda} d\nu(f, g^*, \lambda). \end{aligned}$$

We consider separately the two possible cases $a \leq 0$ and $a > 0$.

If $a \leq 0$, then $a - \operatorname{Re}\lambda \leq -2\operatorname{Re}\lambda$ for all λ 's such that $\operatorname{Re}\lambda \leq \min(-1, a)$, and we have

$$\begin{aligned} & \int_{\{\lambda \in \sigma \mid \operatorname{Re}\lambda \leq \min(-1, a)\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} d\nu(f, g^*, \lambda) \\ & \leq \int_{\{\lambda \in \sigma \mid \operatorname{Re}\lambda \leq \min(-1, a)\}} e^{s[-\operatorname{Re}\lambda + b^{-\beta}(-2\operatorname{Re}\lambda)^{\beta}]^{1/\beta}} e^{t\operatorname{Re}\lambda} d\nu(f, g^*, \lambda) \\ & \quad (\text{since, for } 1 \leq \beta < \infty, x \leq x^{\beta}, x \geq 1) \\ & \leq \int_{\{\lambda \in \sigma \mid \operatorname{Re}\lambda \leq \min(-1, a)\}} e^{s[(-\operatorname{Re}\lambda)^{\beta} + (2/b)^{\beta}(-\operatorname{Re}\lambda)^{\beta}]^{1/\beta}} e^{t\operatorname{Re}\lambda} d\nu(f, g^*, \lambda) \\ & = \int_{\{\lambda \in \sigma \mid \operatorname{Re}\lambda \leq \min(-1, a)\}} e^{(t-s[1+(2/b)^{\beta}]^{1/\beta})\operatorname{Re}\lambda} d\nu(f, g^*, \lambda) \quad (5.7) \\ & \quad \left(\text{by the choice } s = \frac{t}{[1+(2/b)^{\beta}]^{1/\beta}} \right) \\ & = \int_{\{\lambda \in \sigma \mid \operatorname{Re}\lambda \leq \min(-1, a)\}} 1 d\nu(f, g^*, \lambda) \\ & = \nu(f, g^*, \{\lambda \in \sigma \mid \operatorname{Re}\lambda \leq \min(-1, a)\}) \quad (\text{by (2.6)}) \\ & \leq 4M\|f\|\|g^*\| < \infty. \end{aligned}$$

If $a > 0$,

$$\begin{aligned} & \int_{\{\lambda \in \sigma \mid \operatorname{Re}\lambda \leq \min(-1, a)\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} d\nu(f, g^*, \lambda) \\ &= \int_{\{\lambda \in \sigma \mid \operatorname{Re}\lambda \leq \min(-1, -a)\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} d\nu(f, g^*, \lambda) \\ &+ \int_{\{\lambda \in \sigma \mid \min(-1, -a) < \operatorname{Re}\lambda \leq -1\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda} d\nu(f, g^*, \lambda) < \infty. \end{aligned} \quad (5.8)$$

Indeed, the latter integral is finite due to the boundedness of the set $\{\lambda \in \sigma \mid \min(-a, -1) < \operatorname{Re} \lambda \leq -1\}$ (note that, for $a \geq 1$, the set is, obviously, empty), the continuity of the integrated function, and the finiteness of the positive measure $\nu(f, g^*, \cdot)$ (see (2.6)).

The former of the above two integrals is finite as well:

$$\begin{aligned}
 & \int_{\{\lambda \in \sigma \mid \operatorname{Re} \lambda \leq \min(-1, -a)\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \quad (\text{since, for } x \leq -a, a - x \leq -2x) \\
 & \leq \int_{\{\lambda \in \sigma \mid \operatorname{Re} \lambda \leq \min(-1, -a)\}} e^{s[-\operatorname{Re} \lambda + b^{-\beta}(-2 \operatorname{Re} \lambda)^\beta]^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \quad (\text{since, for } 1 \leq \beta < \infty, x \leq x^\beta, x \geq 1) \\
 & \leq \int_{\{\lambda \in \sigma \mid \operatorname{Re} \lambda \leq \min(-1, -a)\}} e^{s[(-\operatorname{Re} \lambda)^\beta + (2/b)^\beta (-\operatorname{Re} \lambda)^\beta]^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & = \int_{\{\lambda \in \sigma \mid \operatorname{Re} \lambda \leq \min(-1, -a)\}} e^{(t-s[1+(2/b)^\beta]^{1/\beta}) \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \tag{5.9} \\
 & \quad \left(\text{by the choice } s = \frac{t}{[1+(2/b)^\beta]^{1/\beta}} \right) \\
 & = \int_{\{\lambda \in \sigma \mid \operatorname{Re} \lambda \leq \min(-1, -a)\}} 1 d\nu(f, g^*, \lambda) \\
 & = \nu(f, g^*, \{\lambda \in \sigma \mid \operatorname{Re} \lambda \leq \min(-1, -a)\}) \quad (\text{by (2.6)}) \\
 & \leq 4M \|f\| \|g^*\| < \infty.
 \end{aligned}$$

Thus, we have proved that, for an arbitrary Borel subset $\sigma \subseteq \sigma(A)$, any $f \in X$, and $g^* \in X^*$,

$$\int_{\sigma} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) < \infty, \quad t > 0, \tag{5.10}$$

with $s = t/[1 + (2/b)^\beta]^{1/\beta} > 0$.

This, in particular, implies that, for any $f \in X$ and $g^* \in X^*$,

$$\int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) < \infty, \quad t > 0, \tag{5.11}$$

with $s = t/[1 + (2/b)^\beta]^{1/\beta} > 0$.

Furthermore, for any $f \in X$, $g^* \in X^*$, $t > 0$, and $s = t/[1 + (2/b)^\beta]^{1/\beta} > 0$,

$$\sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma \mid e^{2s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{5.12}$$

Indeed, as follows from the preceding argument, the specific choice of $s = t/[1 + (2/b)^\beta]^{1/\beta} > 0$ allows to partition the set $\{\lambda \in \sigma \mid e^{2s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}$ into two subsets σ_1 and σ_2 in such a way that σ_1 is bounded and

$$e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} = 1, \quad \lambda \in \sigma_2. \tag{5.13}$$

Therefore,

$$\begin{aligned}
& \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
& \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma_1 \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
& \quad + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma_2 \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
& \quad (\text{since } \sigma_1 \text{ is bounded, there is such a } C > 0 \text{ that } e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} \leq C, \lambda \in \sigma_1; \text{ by (2.8)}) \\
& \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} C4M \|E_A(\{\lambda \in \sigma_1 \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\})f\| \|g^*\| \\
& \quad + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \|E_A(\{\lambda \in \sigma_2 \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\})f\| \|g^*\| \\
& \leq 4CM \|E_A(\{\lambda \in \sigma_1 \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\})f\| \\
& \quad + 4M \|E_A(\{\lambda \in \sigma_2 \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\})f\| \quad (\text{by the strong continuity of the s.m.}) \\
& \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{5.14}$$

According to [23, Proposition 3.1], we have proved that, for any $f \in X$ and $t > 0$,

$$e^{tA} f \in D(e^{s|A|^{1/\beta}}), \tag{5.15}$$

where $s = t/[1 + (2/b)^\beta]^{1/\beta} > 0$.

Hence, for any $f \in X$,

$$e^{tA} f \in \bigcup_{s>0} D(e^{s|A|^{1/\beta}}) \subseteq \mathcal{E}^{\{\beta\}}(A), \quad 0 < t < \infty. \tag{5.16}$$

“ONLY IF” PART. We prove this part by *contrapositive*.

Assume the negation of “for some positive b and real a , $\sigma(A) \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq a - b|\operatorname{Im} \lambda|^{1/\beta}\}$,” that is, for any positive b and real a , the set $\sigma(A) \setminus \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq a - b|\operatorname{Im} \lambda|^{1/\beta}\} \neq \emptyset$. Whence it is easy to infer that, for any natural n , the set

$$\sigma(A) \setminus \left\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -\frac{1}{n} |\operatorname{Im} \lambda|^{1/\beta} \right\} \tag{5.17}$$

is *unbounded*.

Hence, we can choose a sequence of points of the complex plane $\{\lambda_n\}_{n=1}^\infty$ in the following way:

$$\begin{aligned}
& \lambda_n \in \sigma(A), \quad n = 1, 2, \dots, \\
& \operatorname{Re} \lambda_n > -\frac{1}{n} |\operatorname{Im} \lambda|^{1/\beta}, \quad n = 1, 2, \dots, \\
& \lambda_0 := 0, \quad |\lambda_n| > \max[n, |\lambda_{n-1}|], \quad n = 1, 2, \dots
\end{aligned} \tag{5.18}$$

The latter, in particular, implies that the points λ_n are *distinct*:

$$\lambda_i \neq \lambda_j, \quad i \neq j. \quad (5.19)$$

Since the set

$$\left\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > -\frac{1}{n} |\operatorname{Im} \lambda|^{1/\beta}, \quad |\lambda| > \max[n, |\lambda_{n-1}|] \right\} \quad (5.20)$$

is *open* in $\mathbb{C}$ for any $n = 1, 2, \dots$, there exists such a $\varepsilon_n > 0$ that this set contains together with the point λ_n the *open disk* centered at λ_n :

$$\Delta_n = \{ \lambda \in \mathbb{C} \mid |\lambda - \lambda_n| < \varepsilon_n \}, \quad (5.21)$$

that is, for any $\lambda \in \Delta_n$,

$$\begin{aligned} \operatorname{Re} \lambda &> -\frac{1}{n} |\operatorname{Im} \lambda|^{1/\beta}, \\ |\lambda| &> \max[n, |\lambda_{n-1}|]. \end{aligned} \quad (5.22)$$

Moreover, since the points λ_n are *distinct*, we can regard that the radii of the disks, ε_n , are chosen to be small enough so that

$$\begin{aligned} 0 < \varepsilon_n < \frac{1}{n}, \quad n = 1, 2, \dots, \\ \Delta_i \cap \Delta_j = \emptyset, \quad i \neq j \quad (\text{the disks are pairwise disjoint}). \end{aligned} \quad (5.23)$$

Note that, by the properties of the *s.m.*, the latter implies that the subspaces $E_A(\Delta_n)X$, $n = 1, 2, \dots$, are *nontrivial* since $\Delta_n \cap \sigma(A) \neq \emptyset$ and Δ_n is *open*, and

$$E_A(\Delta_i)E_A(\Delta_j) = 0, \quad i \neq j. \quad (5.24)$$

We can choose a unit vector e_n in each subspace $E_A(\Delta_n)X$ and thereby obtain a vector sequence such that

$$E_A(\Delta_i)e_j = \delta_{ij}e_i \quad (5.25)$$

(δ_{ij} is the *Kronecker delta symbol*).

The latter, in particular, implies that the vectors $\{e_1, e_2, \dots\}$ are linearly independent and that

$$d_n := \operatorname{dist}(e_n, \operatorname{span}(\{e_k \mid k \in \mathbb{N}, k \neq n\})) > 0, \quad n = 1, 2, \dots \quad (5.26)$$

Furthermore,

$$d_n \not\rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5.27)$$

Indeed, if we assume the opposite: $d_n \rightarrow 0$ as $n \rightarrow \infty$, then, for any $n = 1, 2, \dots$, we can find an $f_n \in \text{span}(\{e_k \mid k \in \mathbb{N}, k \neq n\})$ such that $\|e_n - f_n\| < d_n + 1/n$, which immediately implies that $e_n = E_A(\Delta_n)(e_n - f_n) \rightarrow 0$. Thus, such an assumption leads to a contradiction.

Therefore, there is a positive ε such that

$$d_n \geq \varepsilon, \quad n = 1, 2, \dots \quad (5.28)$$

As follows from the *Hahn-Banach theorem*, for each $n = 1, 2, \dots$, there is an $e_n^* \in X^*$ such that

$$\begin{aligned} \|e_n^*\| &= 1, \\ \langle e_i, e_j^* \rangle &= \delta_{ij} d_i. \end{aligned} \quad (5.29)$$

Let

$$g^* := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n^*. \quad (5.30)$$

Hence,

$$\begin{aligned} \langle e_n, g^* \rangle &= \frac{d_n}{n^2} \quad (\text{by (5.28)}) \\ &\geq \frac{\varepsilon}{n^2}. \end{aligned} \quad (5.31)$$

Concerning the sequence of the real parts, $\{\text{Re} \lambda_n\}_{n=1}^{\infty}$, there are two possibilities: it is either *bounded* or not. We consider separately each of them.

First, assume that the sequence $\{\text{Re} \lambda_n\}_{n=1}^{\infty}$ is *bounded*, that is, there is such an $\omega > 0$ that

$$|\text{Re} \lambda_n| \leq \omega, \quad n = 1, 2, \dots \quad (5.32)$$

Let

$$f := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n. \quad (5.33)$$

As can be easily deduced from (5.24),

$$\begin{aligned} E_A(\Delta_n)f &= \frac{1}{n^2} e_n, \quad n = 1, 2, \dots, \\ E_A\left(\bigcup_{n=1}^{\infty} \Delta_n\right)f &= f. \end{aligned} \quad (5.34)$$

Also, for $n = 1, 2, \dots$,

$$\begin{aligned} v(f, g^*, \Delta_n) &\geq |\langle E_A(\Delta_n)f, g^* \rangle| \quad (\text{by (5.34)}) \\ &= \left| \left\langle \frac{1}{n^2} e_n, g^* \right\rangle \right| \quad (\text{by (5.31)}) \\ &= \frac{d_n}{n^4} \geq \frac{\varepsilon}{n^4}. \end{aligned} \quad (5.35)$$

For an arbitrary $s > 0$, we have

$$\begin{aligned}
 & \int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \quad (\text{by (5.34)}) \\
 &= \int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re} \lambda} d\nu \left(E_A \left(\bigcup_{n=1}^{\infty} \Delta_n \right) f, g^*, \lambda \right) \quad (\text{by the properties of the o.c.}) \\
 &= \int_{\bigcup_{n=1}^{\infty} \Delta_n} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 &= \sum_{n=1}^{\infty} \int_{\Delta_n} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 &\quad (\text{for } \lambda \in \Delta_n, \text{ by (5.22), (5.32), and (5.23), } |\lambda| \geq n, \text{ and } \operatorname{Re} \lambda = \\
 &\quad \operatorname{Re} \lambda_n - (\operatorname{Re} \lambda_n - \operatorname{Re} \lambda) \geq \operatorname{Re} \lambda_n - |\lambda_n - \lambda| \geq -\omega - \varepsilon_n \geq -\omega - 1) \\
 &\geq \sum_{n=1}^{\infty} e^{sn^{1/\beta}} e^{-(\omega+1)} \nu(f, g^*, \Delta_n) \quad (\text{by (5.35)}) \\
 &\geq e^{-(\omega+1)} \sum_{n=1}^{\infty} \frac{\varepsilon e^{sn^{1/\beta}}}{n^4} = \infty.
 \end{aligned} \tag{5.36}$$

This, by [23, Proposition 3.1], implies that

$$e^A f \notin \bigcup_{t>0} D(e^{t|A|^{1/\beta}}). \tag{5.37}$$

Then, by (2.14), moreover,

$$e^A f \notin \mathcal{E}^\beta(A). \tag{5.38}$$

Hence, by Proposition 3.1, we infer that the orbit $e^{tA}f$, $t \geq 0$, does not belong to $\mathcal{E}^{\{\beta\}}((0, \infty), X)$.

Now, suppose that the sequence $\{\operatorname{Re} \lambda_n\}_{n=1}^{\infty}$ is *unbounded*. The sequence being *bounded above*, since A generates a C_0 -semigroup [11] (see also [22]), this means that there is a subsequence $\{\operatorname{Re} \lambda_{n(k)}\}_{k=1}^{\infty}$ such that

$$\operatorname{Re} \lambda_{n(k)} \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty. \tag{5.39}$$

Thus, without loss of generality, we can regard that

$$\operatorname{Re} \lambda_{n(k)} \leq -k, \quad k = 1, 2, \dots \tag{5.40}$$

Consider the vector

$$f := \sum_{k=1}^{\infty} \frac{1}{k^2} e_{n(k)}. \tag{5.41}$$

By (5.24),

$$\begin{aligned} E_A(\Delta_n(k))f &= \frac{1}{k}e_{n(k)}, \quad k = 1, 2, \dots, \\ E_A\left(\bigcup_{k=1}^{\infty} \Delta_n(k)\right)f &= f. \end{aligned} \quad (5.42)$$

For an arbitrary $s > 0$, we similarly have

$$\int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re}\lambda} dv(f, g^*, \lambda) = \sum_{k=1}^{\infty} \int_{\Delta_n(k)} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re}\lambda} dv(f, g^*, \lambda) = \infty. \quad (5.43)$$

Indeed, for all $\lambda \in \Delta_n(k)$, based on (5.23), (5.40), and (5.22), we have

$$\begin{aligned} \operatorname{Re}\lambda &= \operatorname{Re}\lambda_{n(k)} - (\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda) \leq \operatorname{Re}\lambda_{n(k)} + |\lambda_{n(k)} - \lambda| \\ &\leq \operatorname{Re}\lambda_{n(k)} + \varepsilon_{n(k)} \leq -k + 1 \leq 0, \\ &\quad -\frac{1}{n(k)}|\operatorname{Im}\lambda|^{1/\beta} < \operatorname{Re}\lambda. \end{aligned} \quad (5.44)$$

Therefore, for $\lambda \in \Delta_n(k)$,

$$-\frac{1}{n(k)}|\operatorname{Im}\lambda|^{1/\beta} < \operatorname{Re}\lambda \leq -k + 1 \leq 0. \quad (5.45)$$

Whence, for $\lambda \in \Delta_n(k)$,

$$\operatorname{Re}\lambda \leq -k + 1 \leq 0, \quad |\lambda| \geq |\operatorname{Im}\lambda| \geq [n(k)(-\operatorname{Re}\lambda)]^\beta. \quad (5.46)$$

Using these estimates, we have

$$\begin{aligned} &\int_{\Delta_n(k)} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re}\lambda} dv(f, g^*, \lambda) \\ &\geq \int_{\Delta_n(k)} e^{[sn(k)-1](-\operatorname{Re}\lambda)} dv(f, g^*, \lambda) \\ &\quad (\text{for all } k\text{'s sufficiently large so that } sn(k) - 1 > 0 \text{ and } k - 1 \geq 1) \\ &\geq e^{[sn(k)-1](k-1)} v(f, g^*, \Delta_n(k)) \quad (\text{by (5.35)}) \\ &\geq \frac{\varepsilon e^{[sn(k)-1]}}{n(k)^4} \rightarrow \infty \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (5.47)$$

Similarly, we infer that the orbit $e^{tA}f$, $t \geq 0$, does not belong to the class $\mathcal{E}^{\{\beta\}}((0, \infty), X)$.

Having analyzed all the possibilities concerning $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$, we infer that the negation of “for some positive b and real a , $\sigma(A) \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq a - b|\operatorname{Im}\lambda|^{1/\beta}\}$ ” implies that *not every orbit of the C_0 -semigroup $\{e^{tA} \mid t \geq 0\}$ belongs to the Gevrey class $\mathcal{E}^{\{\beta\}}((0, \infty), H)$, that is, $\{e^{tA} \mid t \geq 0\}$ is not an $\mathcal{E}^{\{\beta\}}$ -semigroup.*

Thus, the “only if” part has been proved by *contrapositive*. $\square$

In particular, for $\beta = 1$, we obtain a criterion of the generation of an analytic C_0 -semigroup by a *scalar type spectral operator* [22] (see also [30]).

Observe that, for $0 < \beta < 1$, all the orbits of the C_0 -semigroup $\{e^{tA} \mid t \geq 0\}$ are *entire* functions, which immediately implies that A is bounded (see [20]).

6. A concluding remark. Similar results for a *normal operator* in a complex Hilbert space are discussed in a more general context in [18, 19] (see also [20, 21]).

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