

RANK AND k -NULLITY OF CONTACT MANIFOLDS

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We prove that the dimension of the 1-nullity distribution $N(1)$ on a closed Sasakian manifold M of rank l is at least equal to $2l - 1$ provided that M has an isolated closed characteristic. The result is then used to provide some examples of K -contact manifolds which are not Sasakian. On a closed, $2n + 1$ -dimensional Sasakian manifold of positive bisectional curvature, we show that either the dimension of $N(1)$ is less than or equal to $n + 1$ or $N(1)$ is the entire tangent bundle TM . In the latter case, the Sasakian manifold M is isometric to a quotient of the Euclidean sphere under a finite group of isometries. We also point out some interactions between k -nullity, Weinstein conjecture, and minimal unit vector fields.

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1. Introduction. Contact, non-Sasakian manifolds whose characteristic vector field lies in the k -nullity distribution have been fully classified by Boeckx [7]. One of the main goals of the present paper is to describe the leaves of the 1-nullity distribution and the topology of the Sasakian manifolds using the notion of “rank” of a K -contact manifold. After collecting some preliminaries on contact metric geometry in [Section 2](#), we define the rank of a closed K -contact manifold in [Section 3](#).

In [Section 4](#), we define the k -nullity distribution of a Riemannian manifold and prove [Theorem 4.3](#).

Relying on a construction of Yamazaki [25], we use [Theorem 4.3](#) in [Section 5](#), where we exhibit examples of five-dimensional manifolds whose K -contact structures are not Sasakian.

[Section 6](#) deals with Sasakian manifolds with positive bisectional curvature. Using variational calculus techniques, we prove [Theorem 6.2](#).

A conjecture of Weinstein asserts that any compact contact manifold should have at least one closed characteristic. In [Section 7](#), we point out how this conjecture holds true in the case, where the characteristic vector field belongs to the k -nullity distribution, and the contact metric manifold carries a nonsingular Killing vector field.

We conclude our paper by an observation relating k -nullity and the existence of minimal unit vector fields in [Section 8](#). It is shown here that if the characteristic vector field belongs to the k -nullity distribution, then one can deform the contact metric in such a way that the same characteristic vector field becomes a critical point of the volume functional which is defined on the space of unit vector fields.

2. Preliminaries. A *contact form* on a $2n + 1$ -dimensional manifold M is a 1-form α such that $\alpha \wedge (d\alpha)^n$ is a volume form on M . There is always a unique vector field

Z , the characteristic vector field of α , which is determined by the equations $\alpha(Z) = 1$ and $d\alpha(Z, X) = 0$ for arbitrary X . The distribution $D_p = \{V \in T_pM : \alpha(V) = 0\}$ is called the contact distribution of α . Clearly, D is a symplectic vector bundle with symplectic form $d\alpha$.

On a contact manifold (M, α, Z) , there is also a nonunique Riemannian metric g and a partial complex operator J adapted to α in the sense that the identities

$$2g(X, JY) = d\alpha(X, Y), \quad J^2X = -X + \alpha(X)Z, \tag{2.1}$$

hold for any vector fields X, Y on M . We have adopted the convention for exterior derivative so that

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]). \tag{2.2}$$

The tensors α, Z, J , and g are called contact metric structure tensors and the manifold M with such a structure will be called a *contact metric manifold* [6]. We will use the notation (M, α, Z, J, g) to denote a contact metric manifold M with specified structure tensors. Assuming that (M, g) is a complete Riemannian manifold, let $\psi_t, t \in \mathbb{R}$, denote the 1-parameter group of diffeomorphism generated by Z . The group ψ_t preserves the contact form α , that is, $\psi_t^* \alpha = \alpha$. If ψ_t is also a 1-parameter group of isometries of g , then the contact metric manifold is called a *K-contact manifold*. By ∇ we will denote the Levi-Civita covariant derivative operator of g . On a *K-contact manifold*, one has the identity

$$\nabla_X Z = -JX \tag{2.3}$$

valid for any tangent vector X . On a general contact metric manifold, the identity

$$\nabla_X Z = -JX - JhX \tag{2.4}$$

is satisfied, where $hX = (1/2)L_Z JX$. If the identity

$$(\nabla_X J)Y = g(X, Y)Z - \alpha(Y)X \tag{2.5}$$

is satisfied for any vector fields X and Y on M , then the contact metric structure (M, α, Z, J, g) is called a *Sasakian structure*. A submanifold N in a contact manifold (M, α, Z, J, g) is said to be invariant if Z is tangent to N and JX is tangent to N whenever X is. An invariant submanifold is a contact submanifold.

3. Rank of *K-contact manifolds.* On a compact *K-contact metric manifold* (M, α, Z, g, J) , the closure of the 1-parameter group ψ_t in the isometry group of (M, g) is a torus group T^l for some nonzero integer l . A *K-contact manifold* with the action of such a torus T^l is said to be of *rank l* [24]. The *K-contact manifolds* of rank 1 are those whose 1-parameter group ψ_t is periodic, that is, the integral curves of Z are all circles. It is shown in [15] or [25] that the rank l of a *K-contact manifold* is at most equal to $n + 1$ if the manifold is $2n + 1$ dimensional. Also, from [14], a closed *K-contact manifold* of dimension $2n + 1$ carries at least $n + 1$ closed characteristics, that is, $n + 1$ closed orbits

of the flow ψ_t . Each one of these closed characteristics is a 1-dimensional orbit of the action of a circle subgroup of the torus T^l , where l is the rank of the K -contact manifold.

4. k -nullity distribution. For a real number k , the k -nullity distribution of a Riemannian manifold (M, g) is the subbundle $N(k)$ defined at each point $p \in M$ by

$$N_p(k) = \{H \in T_pM \mid R(X, Y)H = k(g(Y, H)X - g(X, H)Y); \forall X, Y \in T_pM\}, \tag{4.1}$$

where R denotes the Riemann curvature tensor given by the formula

$$R(X, Y)H = \nabla_X \nabla_Y H - \nabla_Y \nabla_X H - \nabla_{[X, Y]}H, \tag{4.2}$$

for arbitrary vector fields X, Y , and H on M . If H lies in $N(k)$, then the sectional curvatures of all plane sections containing H are equal to k .

The interaction between rank and dimension of 1-nullity distribution of Sasakian manifolds can be described thanks to the following simple observation.

PROPOSITION 4.1. *The k -nullity distribution of a Riemannian manifold (M, g) is left invariant by any isometry of (M, g) .*

PROOF. If $H \in N(k)$ and φ is an isometry of (M, g) , then, for any vector fields X, Y on M , one has

$$\begin{aligned} R(\varphi_*X, \varphi_*Y)\varphi_*H &= \varphi_*R(X, Y)H \\ &= \varphi_*(k(g(Y, H)X - g(X, H)Y)) \\ &= k[g(\varphi_*Y, \varphi_*H)\varphi_*X - g(\varphi_*X, \varphi_*H)\varphi_*Y]. \end{aligned} \tag{4.3}$$

Since φ_* is an automorphism of the tangent bundle of M , the above identity shows that $\varphi_*H \in N(k)$. □

By R_k we denote the tensor field defined for arbitrary vector fields X, Y, H by

$$R_k(X, Y)H = R(X, Y)H - k\{g(Y, H)X - g(X, H)Y\}. \tag{4.4}$$

R_k satisfies similar identities as the curvature tensor R , mainly,

- (i) $g(R_k(X, Y)H, V) = -g(R_k(X, Y)V, H)$,
- (ii) $g(R_k(X, Y)H, V) = g(R_k(X, H)Y, V)$,
- (iii) $\nabla_X R_k(Y, H)V + \nabla_Y R_k(H, X)V + \nabla_H R_k(X, Y)V = 0$.

Now, let X, Y, V be any tangent vectors at $p \in M$. Extend X, Y and V into local vector fields such that at p one has $\nabla X = 0 = \nabla Y = \nabla V$. Let H, W be two vector fields in the nullity distribution of R_k , that is,

$$R_k(X, Y)H = 0 = R_k(X, Y)W, \tag{4.5}$$

for any X, Y on M . Using identity (iii), one obtains

$$\begin{aligned}
 0 &= g(\nabla_H R_k(X, Y)V + \nabla_X R_k(Y, H)V + \nabla_Y R_k(H, X)V, W) \\
 &= g\left(\nabla_H(R_k(X, Y)V) + \nabla_X(R_k(Y, H)V) + \nabla_Y(R_k(H, X)V) \right. \\
 &\quad \left. - R_k(Y, \nabla_X H)V - R_k(\nabla_Y H, X)V + \text{Others}, W\right) \\
 &= Zg(R_k(X, Y)V, W) - g(R_k(X, Y)V, \nabla_H W) + Xg(R_k(Y, H)V, W) \\
 &\quad - g(R_k(Y, H)V, \nabla_X W) + Yg(R_k(H, X)V, W) - g(R_k(H, X)V, \nabla_Y W) \\
 &\quad - g(R_k(Y, \nabla_X H)V, W) - g(R_k(\nabla_Y H, X)V, W) + g(\text{Others}, W).
 \end{aligned}
 \tag{4.6}$$

“Others” stands for terms vanishing at p . Applying identities (i) and (ii), and evaluating at p , we obtain

$$0 = g(R_k(X, Y)\nabla_H W, V),
 \tag{4.7}$$

for arbitrary X, Y , and V . This means that $\nabla_H W$ also belongs to the k -nullity distribution whenever H and W do. The above argument proves that $N(k)$ is an integrable subbundle with totally geodesic leaves of constant curvature k [20]. Hence, if $k > 0$ and $\dim N(k) > 1$, then each leaf of $N(k)$ is a compact manifold [13, Corollary 19.5].

On a contact metric $2n + 1$ -dimensional manifold M , $n > 1$, Blair and Koufogiorgos showed that if the characteristic vector field Z lies in $N(k)$, then $k \leq 1$. If $k < 1$ and $k \neq 0$, then the dimension of $N(k)$ is equal to 1 [1]. The corresponding result for $n = 1$ is due to Sharma [19]. If $k = 0$, then M is locally $E^{n+1} \times \mathbf{S}^n(4)$ and Z is tangent to the Euclidean factor giving that the dimension of $N(0)$ is equal to $n + 1$ [5]. If $k = 1$, the contact metric structure is Sasakian and we wish to investigate the dimension of $N(1)$ on a Sasakian manifold. Contact, non-Sasakian manifolds whose characteristic vector field lies in the k -nullity distribution have been fully classified by Boeckx in [7]. First, we will describe the leaves of the 1-nullity distribution on a Sasakian manifold.

PROPOSITION 4.2. *Let (M, α, Z, J, g) be a closed Sasakian manifold. If the dimension of $N(1)$ is bigger than 1, then each leaf of $N(1)$ is a closed Sasakian submanifold which is isometric to a quotient of a Euclidean sphere under a finite group of isometries.*

PROOF. Let N be such a leaf of $N(1)$. Since the leaf is a totally geodesic submanifold and Z is tangent to it, one has that $JX = -\nabla_X Z$ is tangent to the leaf for any X tangent to it. So, N is an invariant contact submanifold of the Sasakian manifold M and therefore it is also Sasakian. Since N is complete of constant curvature 1, it is isometric to a quotient of a Euclidean sphere under a finite group of Euclidean isometries [23]. \square

To simplify notations, we will denote the dimension of $N(k)$ by $\dim N(k)$. As a consequence of Proposition 4.2 and the work in [14], we obtain the following theorem.

THEOREM 4.3. *Let M be a closed Sasakian $2n + 1$ -dimensional manifold of rank l , with structure tensors α, Z, J , and g . The following hold.*

- (1) *If $\dim N(1) > 1$ and M has an isolated closed characteristic, then $\dim N(1) \geq 2l - 1$. In particular, if $l = n + 1$, then $\dim N(1) = 2n + 1$ and M is isometric to the quotient of a Euclidean $2n + 1$ -sphere under a finite group of Euclidean isometries.*

- (2) If M has a finite number of closed characteristics, then again $\dim N(1) = 2n + 1$, and M is isometric to the quotient of a Euclidean $2n + 1$ -sphere under a finite group of Euclidean isometries.

PROOF. Under the hypothesis, one has $l \geq 2$. There is then a torus T^l acting on M by isometric strict contact diffeomorphisms. Let Z_1, \dots, Z_l be a basis of periodic Killing vector fields for the Lie algebra of T^l . Any isolated closed characteristic of α is a common orbit of all the Z_i and Z (see [14]). Let N be a leaf of $N(1)$ containing an isolated closed characteristic, then, since by Proposition 4.1, each Z_i preserves the foliation by leaves of $N(1)$, one sees that each of Z_i preserves the leaf N . Therefore, each Z_i is tangent to N . It follows that $\dim N(1) \geq 2l - 1$ since JZ_i is also tangent to N for each i and at most $l - 1$ of the JZ_i 's can be linearly independent.

In the case $l = n + 1$, $N(1)$ has only one leaf, the manifold M itself. In the case M has a finite number S of closed characteristics, then $N(1)$ cannot have more than S leaves because, being a closed Sasakian manifold, each leaf of $N(1)$ must contain at least one closed characteristic. It follows again that there is only one leaf which must be the manifold M itself. In any of the above two cases, M is a closed manifold of constant curvature 1 and it is well known that any compact, constant curvature-1 manifold is isometric to a quotient of a Euclidean sphere under a finite group of Euclidean isometries. \square

5. K -Contact, non-Sasakian manifolds. In dimension 3, a K -contact manifold is automatically Sasakian, not so in higher dimensions. We will provide 5-dimensional examples documenting the existence of K -contact structures which are not Sasakian. One well-known way of obtaining K -contact structures which are not Sasakian is as follows. Let S be a closed manifold admitting a symplectic form but not Kähler form. Examples of such manifolds may be found for instance in [21] or [12]. Let ω be a symplectic form on S whose cohomology class $[\omega]$ lies in $H^2(S, \mathbb{Z})$, and let $\pi : E \rightarrow S$ be the Boothby-Wang fibration associated with ω [8]. If g and J are a metric and an almost complex operator adapted to ω , then E carries a K -contact structure whose tensors (α, Z, J^*, g^*) are naturally derived from (ω, J, g) . The contact form α is just the connection 1-form of the S^1 -bundle, $d\alpha = \pi^*\omega$, $\pi_*J^* = J\pi_*$, and $g^* = \pi^*g + \alpha \otimes \alpha$. The characteristic vector field Z is, up to a sign, the unit tangent vector field along the fibers of π . That the above K -contact manifold E is not Sasakian follows from the well-known result of Hatakeyama which states that a regular contact manifold with structure tensors (α, Z, J, g) is Sasakian if and only if the space of orbits of Z is a Kähler manifold with projected tensors [10]. As a consequence of Theorem 4.3, we derive other examples of K -contact structures which are not Sasakian. These come as simply connected, 5-dimensional K -contact manifolds of maximum rank 3.

In [25], closed simply connected K -contact manifolds of dimension 5 and rank 3 have been classified. Let M denote $\mathbb{S}^2 \times \mathbb{S}^3$ and N denote the nontrivial oriented \mathbb{S}^3 bundle over \mathbb{S}^2 . Let r be an integer, $r > 3$. In [25], Yamazaki showed that the connected sum $Q = \#_{r-3}M$ of $r - 3$ copies of M carries a K -contact structure of rank 3 with exactly r closed characteristics. Also, he showed that the connected sum $W = N\#_{r-4}M$ of N with $r - 4$ copies of M carries a K -contact structure of rank 3 with exactly r closed

characteristics. None of the manifolds Q and W above is homeomorphic to S^5 , therefore, As an immediate consequence of [Theorem 4.3](#) in this note, none of the K -contact structures on Q and W is Sasakian.

6. Sasakian manifolds of positive bisectonal curvature. On compact Sasakian manifolds, one has the following lemma due to Binh and Tamássy [[4](#)].

LEMMA 6.1. *Let (M, α, Z, J, g) be a closed $2n + 1$ -dimensional Sasakian manifold and $N \subset M$, a $2r + 1$ -dimensional invariant submanifold. Let $\gamma(t)$ be a normal geodesic issuing from $\gamma(0) = x \in N$ in a direction perpendicular to N . Then, there exist orthonormal vectors $E_i \in T_x N, i = 1, 2, \dots, r$, such that their parallel translated $E_i(t)$ along $\gamma(t)$ completed with $JE_i(t)$ form a vector system which is orthonormal and parallel along $\gamma(t)$.*

PROOF. Let E_1, \dots, E_r be an orthonormal system of vectors in $T_x N$ such that $Z, E_1, J e_1, \dots, E_r, J E_r$ is an orthonormal system of vectors tangent to N at x . We translate E_i parallel along $\gamma(t)$ to obtain $E_i(t), E_i(0) = E_i$. We claim that $JE_i(t)$ is also parallel along $\gamma(t)$. Indeed, denoting $\dot{\gamma}$ by V and applying identity [\(2.5\)](#), one obtains

$$\nabla_V (JE_i(t)) = (\nabla_V J)E_i(t) = -g(Z, E_i(t))V. \tag{6.1}$$

Hence, $JE_i(t)$ will be parallel along γ if and only if $f_i(t) = g(Z, E_i(t)) = 0$ for all t . We will show that $f_i(t)$ satisfies the linear differential equations

$$\begin{aligned} f_i''(t) &= -f_i(t), \\ f_i'(t) &= Vg(E_i, Z) = -g(E_i(t), JV), \\ f_i''(t) &= -Vg(E_i(t), JV) = -g(E_i(t), Z) = -f_i(t). \end{aligned} \tag{6.2}$$

Moreover, $f_i(0) = g(E_i, Z) = 0$ and $f_i'(0) = -g(E_i, JV) = 0$, because N is invariant and so is the normal bundle of N in M . The initial value problem

$$f_i'' + f_i = 0, \quad f_i(0) = 0, \quad f_i'(0) = 0 \tag{6.3}$$

has the unique solution $f_i(t) = 0$ for all t . □

Given two unit tangent vectors X and Y such that $\alpha(X) = 0 = \alpha(Y)$ on a contact $2n + 1$ -dimensional manifold (M, α, Z, J, g) , the bisectonal curvature $H(X \wedge Y)$ of the plane spanned by X and Y is defined by

$$H(X \wedge Y) = g(R(X, Y)Y, X) + g(R(X, JY)JY, X). \tag{6.4}$$

The J -sectional curvature is by definition the sectional curvature of a plane spanned by X and JX .

For closed, Sasakian manifolds of constant J -sectional curvature, Sharma [[19](#)] has shown that the dimension of $N(1)$ is either 1 or $2n + 1$. Assuming only that the manifold has positive bisectonal curvature, we prove the following result.

THEOREM 6.2. *Let (M, α, Z, J, g) be a closed $2n + 1$ -dimensional Sasakian manifold of positive bisectonal curvature. Then $\dim N(1) \leq n + 1$ or $\dim N(1) = 2n + 1$.*

PROOF. Suppose that N_1 and N_2 are two distinct $2l - 1$ -dimensional leaves of $N(1)$, where $2l - 1 > 1$. Denoting by T the distance between N_1 and N_2 , there exists a minimal geodesic $c(t)$, $t \in [0, T]$, from N_1 to N_2 such that $c(0) \in N_1$, $c(T) \in N_2$, and $c(t)$ is the shortest such curve. Let $V(t)$ be the unit tangent vector to the geodesic $c(t)$. Then $V(0)$ is orthogonal to N_1 and $V(T)$ is orthogonal to N_2 . Let $E_i, JE_i, i = 1, 2, \dots, l - 1$, be an orthonormal basis for the contact distribution at $c(0) \in N_1$ (recall N_1 is a contact submanifold). Let $E_i(t)$ denote the parallel translation of E_i from $c(0)$ to $c(t)$. Then $E_i(t), JE_i(t), i = 1, 2, \dots, l - 1$, is a parallel orthonormal frame field along $c(t)$ as was shown in [Lemma 6.1](#) (see also [\[4\]](#)). Suppose now that $2l - 1 > n + 1$. Then the span of $E_i, JE_i, i = 1, 2, \dots, l - 1$, has dimension $2l - 2$ which is bigger than $2n - (2l - 2)$, the fiber dimension of the normal bundle of N_2 . Consequently, one can find a unit vector $F \in T_{c(0)}N_1$ which is a linear combination of the $E_i(0), JE_i(0), i = 1, 2, \dots, l - 1$, such that its parallel translated $F(T) \in T_{c(T)}N_2$. Since N_1 and N_2 are invariant contact submanifolds, one has also $JF(T) \in T_{c(T)}N_2$. The vector fields $F(t)$ and $JF(t)$ along $c(t)$ provide variations $c_s(t)$ of the geodesic $c(t)$ with endpoints in N_1 and N_2 . Let $V_s(t)$ denote the tangent vector to the curves in such a variation. Then the arclength functional $L(s)$ is given by

$$L(s) = \int_0^T \|V_s(t)\| dt. \tag{6.5}$$

One has $L'(0) = 0$ because $c(t)$ is a minimal geodesic. Furthermore, by Singe formula for the second variation [\[9\]](#), one has

$$L''_F(0) = \sigma_{N_2}(F, F)(T) - \sigma_{N_1}(F, F)(0) - \int_0^T g(R(F, V)V, F)(t) dt, \tag{6.6}$$

where σ_{N_i} is the second fundamental form of the submanifold N_i and $g(R(F, V)V, F)$ is the sectional curvature of the plane spanned by F and V . Similarly,

$$L''_{JF}(0) = \sigma_{N_2}(JF, JF)(T) - \sigma_{N_1}(JF, JF)(0) - \int_0^T g(R(JF, V)V, JF)(t) dt. \tag{6.7}$$

Adding the two second variations and recalling that $N_i, i = 1, 2$, is totally geodesic and the bisectional sectional curvature $H(V \wedge F)$ is positive by assumption, one has that $\sigma_{N_1}(F, F)(0) = 0 = \sigma_{N_2}(F, F)(T)$ and

$$L''_F(0) + L''_{JF}(0) = - \int_0^T H(V \wedge F)(t) dt < 0. \tag{6.8}$$

Therefore, at least one of the second variations at $c(t)$ is strictly negative, contradicting the minimality of the geodesic $c(t)$. Thus, we have established that if $\dim N(1) > n + 1$, then $N(1)$ has only one leaf, which has to be the manifold M itself. \square

7. Weinstein conjecture. Let (M, α, Z, J, g) be a closed contact manifold. Weinstein conjecture [\[22\]](#) asserts that the characteristic vector field Z of α should have at least one closed orbit. If Z belongs to the 1-nullity distribution $N(1)$, then (M, α, Z, J, g) is Sasakian, and Weinstein conjecture has been settled in that case [\[15\]](#). In the non-Sasakian case, one has the following.

THEOREM 7.1. *Let (M, α, Z, J, g) be a closed contact metric structure such that Z belongs to the k -nullity distribution $N(k)$, $0 < k < 1$. Suppose that there is a nonsingular, Killing vector field on (M, g) . Then Z has at least two closed characteristics.*

PROOF. Since $0 < k < 1$, $\dim N(k) = 1$. Let C be a nonsingular Killing vector field on M ; we may assume C to be a periodic vector field. Since C preserves the k -nullity distribution $N(k)$, one has

$$[C, Z] = fZ \tag{7.1}$$

for some smooth function f on M . But also, using identity (2.4), one obtains the following:

$$\alpha([C, Z]) = g(Z, [C, Z]) = -g(Z, JC + JhC) - g(Z, \nabla_Z C). \tag{7.2}$$

But since C is Killing, $g(Z, \nabla_Z C) = -g(\nabla_Z C, Z)$, thus $g(\nabla_Z C, Z) = 0$ and

$$f = \alpha([C, Z]) = g(\nabla_Z C, Z) = 0, \tag{7.3}$$

from which follows the identity

$$[C, Z] = 0. \tag{7.4}$$

Moreover, for an arbitrary vector field X on M ,

$$\begin{aligned} L_C \alpha(X) &= C\alpha(X) - \alpha([C, X]) \\ &= Cg(Z, X) - g(Z, [C, X]) \\ &= g([C, Z], X) \\ &= 0. \end{aligned} \tag{7.5}$$

Therefore, one has

$$L_C \alpha = 0. \tag{7.6}$$

As in [3], one defines a smooth function S on M by

$$S = i_C \alpha. \tag{7.7}$$

S is a basic function relative to Z , indeed,

$$dS(Z) = i_Z di_C \alpha = i_Z (L_C \alpha - i_C d\alpha) = 0. \tag{7.8}$$

The differential of S is given by

$$dS = di_C \alpha = L_C \alpha - i_C d\alpha = -i_C d\alpha. \tag{7.9}$$

A point $p \in M$ is a critical point of S if and only if $C(p)$ is proportional to $Z(p)$. Moreover, since S is basic relative to Z and $[C, Z] = 0$, any C orbit containing a critical point p is itself a critical manifold and coincides with the Z orbit containing p . Thus, it is a closed orbit of Z . Since S must have at least two critical points on two distinct Z orbits, we conclude that Z must have at least two closed orbits. □

8. Minimal unit vector fields. Given a contact metric manifold (M, α, Z, J, g) , one can look at Z as an embedding

$$Z : M \rightarrow T^1M, \quad (8.1)$$

where T^1M is the unit tangent sphere bundle endowed with the Sasaki metric. One can then ask when is Z a minimal unit vector field, that is, when is Z a critical point for the volume functional defined on the space of unit vector fields on M .

The manifold T^1M is also equipped with a natural contact metric structure whose characteristic vector field generates the well-known geodesic flow of M [17]. If $Z(M) \subset T^1M$ is a contact metric submanifold, then it is a minimal unit vector field [11]. Some examples of minimal unit vector fields whose images in T^1M are also contact submanifolds have been presented in [16]. Here, we prove the following.

THEOREM 8.1. *Let (M, α, Z, J, g) be a closed contact metric manifold with Z belonging to the k -nullity distribution, $k < 1$. Let g_l denote the deformation of g given by*

$$g_l = lg + (1 - l)\alpha \otimes \alpha, \quad (8.2)$$

with

$$l = \frac{1}{\sqrt{2-k}} \quad (8.3)$$

and let T^1M be endowed with the Sasaki metric induced by g_l . Then $Z : M \rightarrow T^1M$ is a minimal unit vector field and $Z(M) \subset T^1M$ is a contact submanifold.

PROOF. Since Z belongs to the k -nullity distribution, the operator h^2 restricted to the contact distribution acts as multiplication by $\lambda^2 = (1 - k)$ [2]. The condition on l then implies that $\lambda^2 = 1/l^2 - 1$, and by [18, Theorem 4.1], Z is a minimal unit vector field whose image $Z(M) \subset T^1M$ is a contact submanifold. \square

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