q-RIEMANN ZETA FUNCTION

TAEKYUN KIM

Received 19 July 2003

We consider the modified q-analogue of Riemann zeta function which is defined by

$$
\zeta_q(s) = \sum_{n=1}^{\infty} \left( \frac{q^n(s-1) / [n]^s}{[n]} \right),
$$

where $0 < q < 1$, $s \in C$. In this paper, we give q-Bernoulli numbers which can be viewed as interpolation of the above q-analogue of Riemann zeta function at negative integers in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers. Also, we will treat some identities of q-Bernoulli numbers using non-Archimedean q-integration.

2000 Mathematics Subject Classification: 11S80, 11B68.

1. Introduction. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will respectively denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$.

The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we normally assume $|q - 1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the notation

$$
[x] = [x : q] = \frac{1 - q^x}{1 - q} = 1 + q + q^2 + \cdots + q^{x-1}.
$$

(1.1)

Note that $\lim_{q \to 1}[x] = x$ for $x \in \mathbb{Z}_p$ in the $p$-adic case.

Let $UD(\mathbb{Z}_p)$ be denoted by the set of uniformly differentiable functions on $\mathbb{Z}_p$.

For $f \in UD(\mathbb{Z}_p)$, we start with the expression

$$
\frac{1}{[p^N]} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p)
$$

(1.2)

representing the analogue of Riemann’s sums for $f$ (cf. [4]).

The integral of $f$ on $\mathbb{Z}_p$ will be defined as the limit $(N \to \infty)$ of these sums, which exists. The $p$-adic $q$-integral of a function $f \in UD(\mathbb{Z}_p)$ is defined by (see [4])

$$
\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]} \sum_{0 \leq j < p^N} f(j) q^j.
$$

(1.3)
For $d$ that is a fixed positive integer with $(p,d) = 1$, let

$$X = X_d = \lim_{N \to \infty} \frac{\mathbb{Z}}{dp^N \mathbb{Z}}, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{0 < a < dp, \ (a,p)=1} a + dp \mathbb{Z}_p,$$

$$a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

Let $\mathbb{N}$ be the set of positive integers. For $m,k \in \mathbb{N}$, the $q$-Bernoulli polynomials, $\beta_{m}^{(-m,k)}(x,q)$, of higher order for the variable $x$ in $\mathbb{C}_p$ are defined using $p$-adic $q$-integral by (cf. [4])

$$\beta_{m}^{(-m,k)}(x,q) = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + x_2 + \cdots + x_k]^m \cdot q^{-x_1(m+1)-x_2(m+2)-\cdots-x_k(m+k)} d\mu_q(x_1) d\mu_q(x_2) \cdots d\mu_q(x_k).$$

(1.5)

Now, we define the $q$-Bernoulli numbers of higher order as follows (cf. [2, 4, 7]):

$$\beta_{m}^{(-m,k)}(0,q) = \beta_{m}^{(-m,k)}(0,0).$$

(1.6)

By (1.5), it is known that (cf. [4])

$$\beta_{m}^{(-m,k)} = \lim_{N \to \infty} \frac{1}{[p^N]^k} \sum_{x_1=0}^{p^N-1} \cdots \sum_{x_k=0}^{p^N-1} [x_1 + \cdots + x_k]^m \cdot q^{-x_1(m+1)-x_2(m+2)-\cdots-x_k(m+k)}$$

$$= \frac{1}{(1-q)^m} \sum_{i=0}^{m} \binom{m}{i} (-1)^i \frac{(i-m)(i-m-1)\cdots(i-m-k+1)}{[i-m][i-m-1]\cdots[i-m-k+1]},$$

(1.7)

where $\binom{m}{i}$ are the binomial coefficients.

Note that $\lim_{q \to 1} \beta_{m}^{(-m,k)} = B_{m}^{(k)}$, where $B_{m}^{(k)}$ are ordinary Bernoulli numbers of order $k$ (cf. [2, 3, 5, 7, 9]). By (1.5) and (1.7), it is easy to see that

$$\beta_{m}^{(-m,1)}(x,q) = \sum_{i=0}^{m} \binom{m}{i} q^{ix} \beta_{i}^{(-m,1)}(x)^{m-i}$$

$$= \frac{1}{(1-q)^m} \sum_{j=0}^{m} q^{jx} \binom{m}{j} (-1)^j \frac{j-m}{[j-m]},$$

(1.8)

We modify the $q$-analogue of Riemann zeta function which is defined in [1] as follows: for $q \in \mathbb{C}$ with $0 < q < 1$, $s \in \mathbb{C}$, define

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^{(s-1)n}}{[n]^s}.$$  

(1.9)
The numerator ensures the analytic continuation for $\Re(s) > 1$. In (1.9), we can consider the following problem.

“Are there $q$-Bernoulli numbers which can be viewed as interpolation of $\zeta_q(s)$ at negative integers in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers?”

In this paper, we give the value $\zeta_q(-m)$ for $m \in \mathbb{N}$, which is the answer of the above problem, and construct a new complex $q$-analogue of Hurwitz’s zeta function and $q$-$L$-series. Also, we will treat some interesting identities of $q$-Bernoulli numbers.

2. Some identities of $q$-Bernoulli numbers $\beta_m^{(-m,1)}$. In this section, we assume $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. By (1.5), we have

$$
\beta_n^{(-n,1)}(x,q) = \int X q^{-(n+1)t} [x + t]^n d\mu_q(t)
$$

Thus, we have

$$
\beta_n^{(-n,1)}(x,q) = [d]^{n-1} \sum_{i=0}^{d-1} q^{-ni} \int_{\mathbb{Z}_p} q^{-(n+1)dx} \left[ \frac{x + i}{d} : q^d \right]^n d\mu_q(x).
$$

(2.1)

where $d, n$ are positive integers.

If we take $x = 0$, then we have

$$
[n] \beta_m^{(-m,1)} - n[n] m \beta_m^{(-m,1)}(q^n) = \sum_{k=0}^{m-1} \binom{m}{k} [n]^k \beta_k^{(-m,1)}(q^n) \sum_{j=1}^{n-1} q^{-(m-j)k} [j]^{m-k}.
$$

(2.3)

It is easy to see that $\lim_{q \to 1} \beta_m^{(-m,1)} = B_m$, where $B_m$ are ordinary Bernoulli numbers (cf. [7]).

**Remark 2.1.** By (2.3), note that

$$
n(1-n^m)B_m = \sum_{k=0}^{m-1} \binom{m}{k} n^k B_k \sum_{j=1}^{n-1} j^{m-k}.
$$

(2.4)

Let $F_q(t)$ be the generating function of $\beta_n^{(-n,1)}$ as follows:

$$
F_q(t) = \sum_{k=0}^{\infty} \beta_k^{(-k,1)} \frac{t^k}{k!}.
$$

(2.5)

By (1.7) and (2.5), we easily see that

$$
F_q(t) = - \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} q^{-mn} [n]^{m-1} \right) \frac{t^m}{m!}.
$$

(2.6)
Through differentiating both sides with respect to $t$ in (2.5) and (2.6), and comparing coefficients, we obtain the following proposition.

**Proposition 2.2.** For $m > 0$, there exists

$$-\frac{\beta_m^{(-m,1)}}{m} = \sum_{n=1}^{\infty} q^{-nm} [n]^{m-1}. \quad (2.7)$$

Moreover, $\beta_0^{(0,1)} = (q - 1) / \log q$.

**Remark 2.3.** Note that Proposition 2.2 is a $q$-analogue of $\zeta(1 - 2m)$ for any positive integer $m$.

Let $\chi$ be a primitive Dirichlet character with conductor $f \in \mathbb{N}$.

For $m \in \mathbb{N}$, we define

$$\beta_m^{(-m,1),\chi} = \int_X q^{-(m+1)x} \chi(x) [x]^m d\mu_q(x), \quad \text{for } m \geq 0. \quad (2.8)$$

Note that

$$\beta_m^{(-m,1),\chi} = [d]^{m-1} \sum_{i=0}^{d-1} \chi(i) q^{-mi} \beta_m^{(-m,1)} \left( \frac{i}{d}, q^d \right). \quad (2.9)$$

3. $q$-analogs of zeta functions. In this section, we assume $q \in \mathbb{C}$ with $|q| < 1$. In [1], the $q$-analogue of Riemann zeta function was defined by (cf. [1])

$$\zeta_q^*(s) = \sum_{n=1}^{\infty} q^{ns} [n]^s, \quad \Re(s) > 0. \quad (3.1)$$

Now, we modify the above $q$-analogue of Riemann zeta function as follows: for $q \in \mathbb{C}$ with $0 < |q| < 1$, $s \in \mathbb{C}$, define

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^{(s-1)n}}{[n]^s}. \quad (3.2)$$

By (2.5), (2.6), and (2.7), we obtain the following proposition.

**Proposition 3.1.** For $m \in \mathbb{N}$, there exists

(i) $\zeta_q(1 - m) = -\beta_m^{(-m,1)}/m$, for $m \geq 1$;

(ii) $\zeta_q(s)$ having simple pole at $s = 1$ with residue $(q - 1) / \log q$.

By (1.7) and (1.8), we see that

$$\beta_n^{(-n,1)}(x, q) = -n \sum_{k=0}^{\infty} ([k]q^x + [x])^{n-1} q^{-n(k+x)}, \quad \text{where } 0 \leq x < 1. \quad (3.3)$$
Hence, we can define $q$-analogue of Hurwitz $\zeta$-function as follows: for $s \in \mathbb{C}$, define

$$\zeta_q(s,x) = \sum_{n=0}^{\infty} \frac{q^{(s-1)(n+x)}}{([n]q^x + [x])^s}. \quad (3.4)$$

Note that $\zeta_q(s,x)$ has an analytic continuation in $\mathbb{C}$ with only one simple pole at $s = 1$.

By (3.3) and (3.4), we have the following theorem.

**Theorem 3.2.** For any positive integer $k$, there exists

$$\zeta_q(1-k,x) = -\frac{\beta_{k}^{(-k,1)}(x,q)}{k}. \quad (3.5)$$

Let $\chi$ be Dirichlet character with conductor $d \in \mathbb{N}$. By (2.9), the generalized $q$-Bernoulli numbers with $\chi$ can be defined by

$$\beta_{m,\chi}^{(-m,1)} = [d]^{m-1} \sum_{i=0}^{d-1} \chi(i) q^{-m i} \beta_{m}^{(-m,1)} \left( \frac{i}{d}, q^d \right). \quad (3.6)$$

For $s \in \mathbb{C}$, we define

$$L_q(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n) q^{(s-1)n}}{[n]^s}. \quad (3.7)$$

It is easy to see that

$$L_q(\chi,s) = [d]^{-s} \sum_{a=1}^{d} \chi(a) q^{(s-1)a} \zeta_q^{a} \left( s, \frac{a}{d} \right). \quad (3.8)$$

By (3.6), (3.7), and (3.8), we obtain the following theorem.

**Theorem 3.3.** Let $k$ be a positive integer. Then there exists

$$L_q(1-k,\chi) = -\frac{\beta_{k}^{(-k,1)}(\chi,q)}{k}. \quad (3.9)$$

Let $a$ and $F$ be integers with $0 < a < F$. For $s \in \mathbb{C}$, we consider the functions $H_q(s,a,F)$ as follows:

$$H_q(s,a,F) = \sum_{m=a(F), m > 0} \frac{q^{m(s-1)}}{[m]^s} = [F]^{-s} \zeta_q^{F} \left( s, \frac{a}{F} \right). \quad (3.10)$$

Then we have

$$H_q(1-n,a,F) = -\frac{[F]^{n-1}}{n} \beta_{n}^{(-n,1)} \left( \frac{a}{F}, q^F \right), \quad (3.11)$$

where $n$ is any positive integer.
Therefore, we obtain the following theorem.

**Theorem 3.4.** Let a and F be integers with $0 < a < F$. For $s \in \mathbb{C}$, there exists

(i) $H_q(1 - n, a, F) = -([F]^{-1}/n)\beta^{n-1} (a/F, qF)$;
(ii) $H_q(s, a, F)$ having a simple pole at $s = 1$ with residue $(1/[F](qF - 1)/\log q)$.

In a recent paper, the $q$-analogue of Riemann zeta function was studied by Cherednik (see [1]). In [1], we can consider the $q$-Bernoulli numbers which can be viewed as an interpolation of the $q$-analogue of Riemann zeta function at negative integers. In this paper, we have shown that the $q$-analogue of zeta function interpolates $q$-Bernoulli numbers at negative integers in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers (cf. [2, 5, 7]).

**Remark 3.5.** Let $q \in \mathbb{C}$ with $|1 - q| < p^{-1/(p-1)}$. Then the $p$-adic $q$-gamma function was defined as (see [8])

$$\Gamma_{p,q}(n) = (-1)^n \prod_{1 \leq j < n, (j,p) = 1} [j].$$

For all $x \in \mathbb{Z}_p$, we have

$$\Gamma_{p,q}(x + 1) = \epsilon_{p,q}(x) \Gamma_{p,q}(x),$$

where $\epsilon_{p,q}(x) = -[x]$ for $|x|_p = 1$, and $\epsilon_{p,q}(x) = -1$ for $|x|_p < 1$, (see [8]). By (3.13), we easily see that (cf. [6])

$$\log \Gamma_{p,q}(x + 1) = \log \epsilon_{p,q}(x) + \log \Gamma_{p,q}(x).$$

By the differentiation of both sides in (3.14), we have (cf. [6])

$$\frac{\Gamma_{p,q}'(x + 1)}{\Gamma_{p,q}(x + 1)} = \frac{\Gamma_{p,q}'(x)}{\Gamma_{p,q}(x)} + \frac{\epsilon_{p,q}'(x)}{\epsilon_{p,q}(x)}.$$ (3.15)

By (3.15), we easily see that (cf. [6])

$$\frac{\Gamma_{p,q}'(x)}{\Gamma_{p,q}(x)} = \left(\sum_{j=1}^{x-1} \frac{q^j}{[j]}\right) \log q \left(\frac{1}{q-1} + \frac{\Gamma_{p,q}'(1)}{\Gamma_{p,q}(1)}\right).$$

Define

$$L_{p,q}(x) = \sum_{j=0}^{x-1} \frac{\epsilon_{p,q}'(j)}{\epsilon_{p,q}(j)}.$$ (3.17)

It is easy to check that $L_{p,q}(1) = 0$. By (3.15), we also see that

$$\frac{\Gamma_{p,q}'(x)}{\Gamma_{p,q}(x)} = L_{p,q}(x) + \frac{\Gamma_{p,q}'(1)}{\Gamma_{p,q}(1)}, \quad \text{for } x \in \mathbb{Z}_p,$$ (3.18)
where $L_{p,q}(x)$ denotes the indefinite sum of $\epsilon'_{p,q}(x)/\epsilon_{p,q}(x)$. By using (3.18) after substituting $x = 1$, we obtain $L_{p,q}(1) = 0$. The classical Euler constant was known as $\gamma = -\Gamma'(1)/\Gamma(1)$. In [8], Koblitz defined the $p$-adic $q$-Euler constant $\gamma_{p,q} = -\Gamma'_p (1)/\Gamma_p(1)$ (cf. [6, 8]). By using (3.16) and the congruence of Andrews (cf. [3]), we obtain the following congruence:

$$\frac{q-1}{\log q} \left( \frac{\Gamma'_{p,q}(p)}{\Gamma_{p,q}(p)} - \gamma_{p,q} \right) = \sum_{j=1}^{p-1} q^j \equiv \frac{p-1}{2} (q-1) \pmod{p}. \quad (3.19)$$

**Acknowledgments.** The author expresses his gratitude to the referees for their valuable suggestions and comments. This work was supported by the Korea Research Foundation Grant (KRF-2002-050-C00001).

**References**


Taekyun Kim: Institute of Science Education, Kongju National University, Kongju 314-701, Korea

E-mail address: tkim@kongju.ac.kr
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