SOME THEOREMS ON THE EXPLICIT EVALUATION OF RAMANUJAN’S THETA-FUNCTIONS

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Bruce C. Berndt et al. and Soon-Yi Kang have proved many of Ramanujan’s formulas for the explicit evaluation of the Rogers-Ramanujan continued fraction and theta-functions in terms of Weber-Ramanujan class invariants. In this note, we give alternative proofs of some of these identities of theta-functions recorded by Ramanujan in his notebooks and deduce some formulas for the explicit evaluation of his theta-functions in terms of Weber-Ramanujan class invariants.

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1. Introduction. Ramanujan’s general theta-function \( f(a, b) \) is given by

\[
f(a, b) = \sum_{k = -\infty}^{\infty} \frac{a^{k(k+1)/2}b^{k(k-1)/2}}{a^k + b^k},
\]

where \( |ab| < 1 \). If we set \( a = q^{2iz}, b = q^{-2iz}, \) and \( q = e^{\pi i\tau} \), where \( z \) is complex and \( \text{Im}(\tau) > 0 \), then \( f(a, b) = \vartheta_3(z, \tau) \), where \( \vartheta_3(z, \tau) \) denotes one of the classical theta-functions in its standard notation [9, page 464]. After Ramanujan, we define the following special types of his theta-function.

If \( |q| < 1 \), then

\[
\begin{align*}
\phi(q) &:= f(q, q) = 1 + 2 \sum_{k=1}^{\infty} q^{2k}, \\
\psi(q) &:= f(q, q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2}, \\
f(-q) &:= f(-q, -q^2) = \sum_{k=0}^{\infty} (-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty} (-1)^k q^{k(3k+1)/2}, \\
\chi(q) &:= (-q; q^2)_\infty,
\end{align*}
\]

where \( (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k) \). The function \( \chi(q) \) is only for notational purposes. Also, note that \( f(-q) = q^{-1/24} \eta(z) \), where \( q = e^{2\pi iz} \) and \( \eta \) denotes the Dedekind eta-function. Much of Ramanujan’s discoveries about theta-functions can be found in Chapters 16–21 of the organized pages of his second notebook [8]. Proofs and other references of all
the identities can be found in [1]. However, in the unorganized pages of his notebooks [8], Ramanujan recorded many other beautiful identities. Proofs of these identities can be found in [2, 3]. In Section 2, we prove some of these identities by using some other identities of theta-functions. Berndt [2, 3] proved these identities via parameterization.

At scattered places in his notebooks [8], Ramanujan recorded several values of his theta-function \( \phi(q) \). Proofs of all the values claimed by Ramanujan can be found in [3, Chapter 35]. Berndt and Chan [4] also verified all of Ramanujan’s nonelementary values of \( \phi(e^{-\pi n}) \) and found three new values for \( n = 13, 27, \) and 63. Kang [6] also calculated some quotients of theta-functions \( \phi \) and \( \psi \). In Section 3, we give some theorems for the explicit evaluation of the quotients of theta-functions \( \phi, \psi, \) and \( f \), by combining Weber-Ramanujan class invariants with the identities proved in Section 2 and some other identities of theta-functions. Some of these evaluations can be used to find explicit values of the famous Rogers-Ramanujan continued fraction \( R(q) \) defined by

\[
R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \ldots,
\]

where \( |q| < 1 \).

We end this introduction by defining Weber-Ramanujan class invariants \( G_n \) and \( g_n \). For \( q = \exp(-\pi \sqrt{n}) \), where \( n \) is a positive rational number, the Weber-Ramanujan class invariants \( G_n \) and \( g_n \) are defined by

\[
G_n := 2^{-1/4} q^{-1/24} \chi(q), \quad (1.7)
\]

\[
g_n := 2^{-1/4} q^{-1/24} \chi(-q). \quad (1.8)
\]

2. Theta-function identities. The following identity was recorded by Ramanujan on page 295 of his first notebook [8]. Berndt [3, page 366] proved this by using parameterization. Here we give an alternative proof.

**Theorem 2.1.** If \( \phi(q), \psi(q), \) and \( \chi(q) \) are defined by (1.2), (1.3), and (1.5), respectively, then

\[
\psi^2(-q) + 5q \psi^2(-q^5) = \frac{\phi^2(q)}{\chi(q) \chi(q^5)}. \quad (2.1)
\]

**Proof.** From [1, Entry 9(vii), page 258, and Entry 10(v), page 262], we find that

\[
\psi^2(q) - q \psi^2(q^5) = \frac{\phi(-q^5) f(-q^5)}{\chi(-q)}. \quad (2.2)
\]

From [1, Entry 24(iii), page 39], we note that

\[
f(q) = \frac{\phi(q)}{\chi(q)}. \quad (2.3)
\]
From (2.2) and (2.3), we deduce that
\[
\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q^5)}{\chi(-q)\chi(-q^5)}. \tag{2.4}
\]

Now, we recall from [1, Entry 9(iii), page 258] that
\[
\phi^2(q) - \phi^2(q^5) = 4q\chi(q)f(-q^5)f(-q^{20}). \tag{2.5}
\]
Replacing \(q\) by \(-q\) in (2.5), we deduce that
\[
\phi^2(-q^5) = \phi^2(-q) + 4q\chi(-q)f(q^5)f(-q^{20}). \tag{2.6}
\]
Employing (2.6) in (2.4), we find that
\[
\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4q \frac{f(q^5)f(-q^{20})}{\chi(-q^5)}. \tag{2.7}
\]
Again, by [1, Entry 24(iii), page 39], we find that
\[
f(-q^4) = \psi(q^2)\chi(-q^2). \tag{2.8}
\]
Using (2.8) in (2.7), we obtain
\[
\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4q \frac{f(q^5)\psi(q^{10})\chi(q^{10})}{\chi(-q^5)}. \tag{2.9}
\]
Now, by [1, Entry 24(iv), page 39], we note that
\[
\chi(q)\chi(-q) = \chi(-q^2). \tag{2.10}
\]
Thus, from (2.9), we obtain
\[
\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4q f(q^5)\psi(q^{10})\chi(q^5). \tag{2.11}
\]
From [1, Entry 25(iv), page 40], we note that
\[
\phi(q)\psi(q^2) = \psi^2(q). \tag{2.12}
\]
Employing (2.3) and (2.12), with \(q\) replaced by \(q^5\), we conclude from (2.11) that
\[
\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4q\psi^2(q^5). \tag{2.13}
\]
Replacing \(q\) by \(-q\) in (2.13), we complete the theorem. \(\Box\)

The next theorem was recorded by Ramanujan on page 4 of his second notebook [8]. Berndt [2, page 202] proved this theorem by parameterization. Here we give an alternative proof by using some identities of theta-functions.
Theorem 2.2. With \( \psi(q) \) and \( \chi(q) \) defined in (1.3) and (1.5), respectively,
\[
\frac{\chi^3(q)}{\chi(q^3)} = 1 + 3q \frac{\psi(-q^9)}{\psi(-q)},
\]
(2.14)
\[
\frac{\chi^5(q)}{\chi(q^5)} = 1 + 5q \frac{\psi^2(-q^5)}{\psi^2(-q)}.
\]
(2.15)

Proof of (2.14). From [1, Chapter 16, Corollary (ii) of Entry 31, page 49], we find that
\[
\psi(q) - q\psi(q^9) = f(q^3, q^6).
\]
(2.16)
Using the Jacobi triple product identity, Berndt [1, page 350] proved that
\[
f(q, q^2) = \phi(-q^3) \chi(-q).
\]
(2.17)
Replacing \( q \) by \( q^3 \) in (2.17) and then using the resultant identity in (2.16), we find that
\[
\psi(q) - q\psi(q^9) = \frac{\phi(-q^9)}{\chi(-q^3)}.
\]
(2.18)
Now, from [1, Corollary (i) of Entry 31, page 49 and Example (v), page 51], we find that
\[
\phi(-q^9) = \phi(-q) + 2q\psi(q^9)\chi(-q^3).
\]
(2.19)
Invoking (2.19) in (2.18), we deduce that
\[
\psi(q) - 3q\psi(q^9) = \frac{\phi(-q)}{\chi(-q^3)}.
\]
(2.20)
Thus,
\[
1 - 3q \frac{\psi(q^9)}{\psi(q)} = \frac{\phi(-q)}{\chi(-q^3)\psi(q)}.
\]
(2.21)
Now, from [1, Entry 24(iii), page 39], we note that
\[
\chi(q) = \frac{3}{\psi(-q)} \sqrt{\phi(q)}.
\]
(2.22)
Replacing \( q \) by \(-q\) in (2.21) and then using (2.22), we complete the proof of (2.14).

Proof of (2.15). From Theorem 2.1, we find that
\[
1 + 5q \frac{\psi^2(-q^5)}{\psi^2(-q)} = \frac{\phi^2(q)}{\chi(q)\chi(q^5)\psi^2(-q)}.
\]
(2.23)
Employing (2.22) in (2.23), we arrive at (2.15), which completes the proof.
3. Explicit evaluations of theta-functions

**Theorem 3.1.** If $\psi(q)$, $G_n$, and $g_n$ are defined by (1.3), (1.7), and (1.8), respectively, then

\[
e^{-\pi \sqrt{n}} \frac{\psi(-e^{-9\pi \sqrt{n}})}{\psi(-e^{-\pi})} = \frac{1}{3} \left( \sqrt{2} \frac{G_n^3}{G_{9n}} - 1 \right),
\]

(3.1)

\[
e^{-\pi \sqrt{n}} \frac{\psi(e^{-9\pi \sqrt{n}})}{\psi(e^{-\pi})} = \frac{1}{3} \left( 1 - \sqrt{2} \frac{g_n^3}{g_{9n}} \right).
\]

(3.2)

**Proof.** From (2.14) and the definition of $G_n$ from (1.7), we easily arrive at (3.1). To prove (3.2), we replace $q$ by $-q$ in (2.14) and then use the definition of $g_n$ from (1.8).

Since $G_{9n}$ and $g_{9n}$ can be calculated from the respective values of $G_n$ and $g_n$ [5], from the theorem above, we see that the quotients of theta-functions on the left-hand sides can be evaluated if the corresponding values of $G_n$ and $g_n$ are known. We give a few examples below.

**Corollary 3.2.**

\[
e^{-\pi} \frac{\psi(-e^{-9\pi})}{\psi(-e^{-\pi})} = \frac{3 \sqrt{2} \left( \sqrt{3} - 1 \right) - 1}{3}.
\]

(3.3)

**Proof.** Putting $n = 1$ in (3.1), we find that

\[
e^{-\pi} \frac{\psi(-e^{-9\pi})}{\psi(-e^{-\pi})} = \frac{1}{3} \left( \sqrt{2} \frac{G_1^3}{G_9} - 1 \right).
\]

(3.4)

From [3, page 189],

\[
G_1 = 1, \quad G_9 = \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right)^{1/3}.
\]

(3.5)

Employing (3.5) in (3.4) and then simplifying, we complete the proof.

From [1, Entry 11(ii), page 123], we find that

\[
\psi(-e^{-\pi}) = \phi(e^{-\pi}) 2^{-3/4} e^{\pi/8}.
\]

(3.6)

Since

\[
\phi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)}
\]

(3.7)

is classical [9], (3.3) and (3.6) provide an explicit evaluation for $\psi(-e^{-9\pi})$.

**Corollary 3.3.**

\[
e^{-\pi \sqrt{5}/3} \frac{\psi(-e^{-3\pi \sqrt{5}/3})}{\psi(-e^{-\pi \sqrt{5}/3})} = \frac{(3 + \sqrt{5})(\sqrt{5} - \sqrt{3}) - 2}{6}.
\]

(3.8)
Proof. Putting \( n = 5/9 \) in (3.1), we obtain

\[
e^{-\pi \sqrt{5/3}} \frac{\psi(-e^{-3\pi \sqrt{5}/9})}{\psi(-e^{-\pi \sqrt{5/3}/3})} = \frac{1}{3} \left( \sqrt{2} \frac{G_{5/9}^3}{G_5} - 1 \right). \tag{3.9}\]

Now, from [3, pages 189 and 345], we note that

\[
G_5 = \left( \frac{1 + \sqrt{5}}{2} \right)^{1/4}, \quad G_{5/9} = \left( \sqrt{3} + 2 \right)^{1/4} \left( \frac{\sqrt{5} - \sqrt{3}}{\sqrt{2}} \right)^{1/3}. \tag{3.10}\]

Employing (3.10) in (3.9) and then simplifying, we arrive at (3.8).

\[\square\]

Corollary 3.4.

\[
e^{-\pi \sqrt{2}} \frac{\psi(e^{-9\pi \sqrt{2}/18})}{\psi(e^{-\pi \sqrt{2}/18})} = \frac{1}{3} \left( 1 - \sqrt{2} g_{18}^3 \right). \tag{3.11}\]

Proof. Putting \( n = 2 \) in (3.2), we find that

\[
e^{-\pi \sqrt{2}} \frac{\psi(e^{-9\pi \sqrt{2}/18})}{\psi(e^{-\pi \sqrt{2}/18})} = \frac{1}{3} \left( 1 - \sqrt{2} \frac{g_5}{g_{25}} \right). \tag{3.12}\]

From [3, page 200], we note that

\[
g_2 = 1, \quad g_{18} = \left( \sqrt{2} + \sqrt{2} \right)^{1/3}. \tag{3.13}\]

Using (3.13) in (3.12), we easily arrive at (3.11).

\[\square\]

Theorem 3.5. With \( \psi(q) \), \( G_n \), and \( g_n \) defined in (1.3), (1.7), and (1.8), respectively,

\[
e^{-\pi \sqrt{n}} \frac{\psi^2(-e^{-5\pi \sqrt{n}/25})}{\psi^2(-e^{-\pi \sqrt{n}/25})} = \frac{1}{5} \left( 2 \frac{G_n^5}{G_{25n}} - 1 \right), \tag{3.14}\]

\[
e^{-\pi \sqrt{n}} \frac{\psi^2(-e^{-5\pi \sqrt{n}/25})}{\psi^2(-e^{-\pi \sqrt{n}/25})} = \frac{1}{5} \left( 1 - 2 \frac{g_n^5}{g_{25n}} \right). \tag{3.15}\]

Proof. From (2.15) and the definition of \( G_n \) from (1.7), we easily arrive at (3.14). Replacing \( q \) by \(-q\) in (2.15) and then using the definition of \( g_n \) from (1.8), we arrive at (3.15).

\[\square\]

If the class invariants are known, then we can explicitly find the values of the quotients of the left-hand-side expressions of the theorem. Next we give some examples.

Corollary 3.6 [6].

\[
e^{-n} \frac{\psi^2(-e^{-5\pi})}{\psi^2(-e^{-\pi})} = \frac{1}{5\sqrt{5} + 10}. \tag{3.16}\]

Proof. Putting \( n = 1 \) in (3.14), we find that

\[
e^{-n} \frac{\psi^2(-e^{-5\pi})}{\psi^2(-e^{-\pi})} = \frac{1}{5} \left( 2 \frac{G_1^5}{G_{25}} - 1 \right). \tag{3.17}\]
From [3, page 189],

\[ G_1 = 1, \quad G_{25} = \frac{1 + \sqrt{5}}{2}. \] (3.18)

Employing (3.18) in (3.17) and then simplifying, we complete the proof. 

**Corollary 3.7.**

\[ e^{-\pi / \sqrt{5}} \psi^2 (-e^{-\sqrt{5} \pi}) \psi^2 (-e^{-\pi / \sqrt{5}}) = \frac{1}{\sqrt{5}}. \] (3.19)

**Proof.** We put \( n = 1/5 \) in (3.14) to obtain

\[ e^{-\pi / \sqrt{5}} \psi^2 (-e^{-\sqrt{5} \pi}) \psi^2 (-e^{-\pi / \sqrt{5}}) = \frac{1}{5} (2G_5^4 - 1). \] (3.20)

Since, from [3, page 189],

\[ G_5 = \left( \frac{1 + \sqrt{5}}{2} \right)^{1/4}, \] (3.21)

we can easily complete the proof by (3.20). 

**Corollary 3.8.**

\[ e^{-\pi \sqrt{3/5}} \psi^2 (-e^{-\pi \sqrt{3/5}}) = \frac{3 - \sqrt{5}}{5 + \sqrt{5}}. \] (3.22)

**Proof.** Putting \( n = 3/5 \) in (3.14), we obtain

\[ e^{-\pi \sqrt{3/5}} \psi^2 (-e^{-\pi \sqrt{3/5}}) = \frac{1}{5} \left( 2 \frac{G_{3/5}^5}{G_5} - 1 \right). \] (3.23)

Now, from [3, page 341], we note that

\[ G_{15} = 2^{-1/12} (1 + \sqrt{5})^{1/3}, \quad G_{3/5} = 2^{-1/12} (\sqrt{5} - 1)^{1/3}. \] (3.24)

Employing (3.24) in (3.23) and then simplifying, we arrive at (3.22).

**Corollary 3.9.**

\[ e^{-\pi \sqrt{2}} \psi^2 (e^{-5 \pi \sqrt{2}}) \psi^2 (e^{-\pi \sqrt{2}}) = \frac{1}{5} \left( 1 - \frac{2}{a} \right), \] (3.25)

where

\[ a = g_{50} = \frac{1}{3} \left( 1 + \frac{5 + \sqrt{5}}{4} \right)^{1/3} \left( \frac{3}{\sqrt{1 + 7 \sqrt{5} + 6 \sqrt{6}} + \frac{3}{\sqrt{1 + 7 \sqrt{5} - 6 \sqrt{6}}} \right). \] (3.26)
Proof. We put $n = 2$ in (3.15) to obtain
\[ e^{-\pi \sqrt{2}} \frac{\psi^2(e^{-5\pi \sqrt{2}})}{\psi^2(e^{-\pi \sqrt{2}})} = \frac{1}{5} \left( 1 - 2 \frac{g_5^5}{g_{50}} \right). \] (3.27)

From [3, page 201],
\[ g_{50} = \frac{1}{3} \left( 1 + \left( \frac{5 + \sqrt{5}}{4} \right)^{1/3} \left( \frac{1}{2} \sqrt{5} + 6 \sqrt{6} + \frac{3}{2} \sqrt{5} - 6 \sqrt{6} \right) \right). \] (3.28)

Employing (3.13) and (3.28) in (3.27), we complete the proof.

Since for $q = e^{-\pi \sqrt{3}}$, $n$ positive rational, the explicit formulas for $\phi^2(q^5)/\phi^2(q)$, $\phi(q^9)/\phi(q)$, and $\phi^4(q^3)/\phi^4(q)$ are known [3, page 339, (8.11); page 334, (5.7); page 330, (4.5), respectively], namely,
\[ \frac{\phi^2(e^{-5\pi \sqrt{3}})}{\phi^2(e^{-\pi \sqrt{3}})} = \frac{1}{3} \left( 1 + 2 \frac{G_{25n}}{G_{n}} \right), \] (3.29)
\[ \frac{\phi(e^{-9\pi \sqrt{3}})}{\phi(e^{-3\pi \sqrt{3}})} = \frac{1}{3} \left( 1 + \sqrt{2} \frac{G_{9n}}{G_{n}} \right), \] (3.30)
\[ \frac{\phi^4(e^{-3\pi \sqrt{3}})}{\phi^4(e^{-\pi \sqrt{3}})} = \frac{1}{9} \left( 1 + 2 \sqrt{2} \frac{G_{9n}}{G_{n}} \right), \] (3.31)
we now derive some identities by which the corresponding values of the quotients $\psi^2(-q^5)/\psi^2(-q)$, $\psi(-q^9)/\psi(-q)$, and $\psi^4(-q^3)/\psi^4(-q)$ can be found.

**Theorem 3.10** [7]. If $\phi(q)$ and $\psi(q)$ are defined by (1.2) and (1.3), respectively, then
\[ q \frac{\psi^2(-q^5)}{\psi^2(-q)} = \frac{1 - \phi^2(q^5)/\phi^2(q)}{(5 \phi^2(q^5)/\phi^2(q)) - 1}. \] (3.32)

Proof. We replace $q$ by $-q$ in (2.4) and then divide the resulting identity by (2.1) to obtain
\[ \frac{\phi^2(q^5)}{\phi^2(q)} = \frac{\psi^2(-q) + q \psi^2(-q^5)}{\psi^2(-q) + 5q \psi^2(-q^5)}. \] (3.33)

This is indeed equivalent to (3.32).

**Theorem 3.11.** With $\phi(q)$ and $\psi(q)$ defined in (1.2) and (1.3), respectively,
\[ q \frac{\psi(-q^9)}{\psi(-q)} = \frac{1 - \phi(q^9)/\phi(q)}{(3 \phi(q^9)/\phi(q)) - 1}. \] (3.34)

Proof. Replace $q$ by $-q$ in (2.18) and (2.20) and then, dividing the first resulting identity by the second, we find that
\[ \frac{\phi(q)}{\phi(q^9)} = \frac{\psi(-q) + q \psi(-q^9)}{\psi(-q) + 3q \psi(-q^9)}. \] (3.35)

It is now easy to see that (3.35) and (3.34) are equivalent.
Theorem 3.12. With $\phi(q)$ and $\psi(q)$ defined in (1.2) and (1.3), respectively,

$$1 + 9q \frac{\psi^4(-q^3)}{\psi^4(-q)} = \frac{8}{(9\phi^4(q^3)/\phi^4(q)) - 1}.$$  \hspace{1cm} (3.36)

Proof. From Theorem 3.11, we note that

$$1 + 3q \frac{\psi(-q^3)}{\psi(-q)} = \frac{2}{(3\phi(q^3)/\phi(q)) - 1}.$$ \hspace{1cm} (3.37)

From the third equality of [1, Entry 1(ii), page 345] and the second equality of [1, Entry 1(iii), page 345], we note that

$$1 + 3q \frac{\psi(-q^3)}{\psi(-q)} = \left(1 + 9q \frac{\psi^4(-q^3)}{\psi^4(-q)}\right)^{1/3},$$

$$3\frac{\phi(q^3)}{\phi(q)} - 1 = \left(9\frac{\phi^4(q^3)}{\phi^4(q)} - 1\right)^{1/3},$$ \hspace{1cm} (3.38)

respectively. Employing (3.38) in (3.37) and then cubing the resultant identity, we complete the proof. \hfill \square

Corollary 3.13.

$$e^{-\pi} \frac{\psi^4(-e^{-3\pi})}{\psi^4(-e^{-\pi})} = \frac{2 - \sqrt{3}}{3\sqrt{3}}.$$ \hspace{1cm} (3.39)

Proof. It is known from [3, page 327] (or can be found easily from (3.31)) that

$$\frac{\phi^4(e^{-3\pi})}{\phi^4(e^{-\pi})} = \frac{1}{6\sqrt{3} - 9}.$$ \hspace{1cm} (3.40)

The proof of the corollary now follows immediately by putting $q = e^{-\pi}$ in Theorem 3.12 and then using (3.40). \hfill \square

Now, from [1, Entries 24(ii) and 24(iv), page 39], we note that

$$f^3(q) = \phi^2(q)\psi(-q),$$

$$f^3(-q^2) = \phi(q)\psi^2(-q).$$ \hspace{1cm} (3.41)

From (3.41), we find the following quotients of $f$ in terms of $\phi$ and $\psi$:

$$F_1(q) := \frac{f^6(q)}{q^3f^6(-q^5)} = \frac{\psi^2(-q)}{q\psi^2(-q^5)} \times \frac{\phi^4(q)}{\phi^4(q^5)},$$

$$F_2(q) := \frac{f^6(-q^2)}{q^2f^6(-q^{10})} = \frac{\phi^2(q)}{\phi^2(q^2)} \times \frac{\psi^4(-q)}{q^2\psi^4(-q^5)}.$$ \hspace{1cm} (3.42)
The values of $F_1(q)$ and $F_2(q)$ can be determined explicitly for $q = e^{-\pi\sqrt{n}}$ by employing Theorem 3.5 and (3.29). We give a couple of examples below.

**Corollary 3.14.**

\[
F_1(e^{-\pi\sqrt{5}}) = 5\sqrt{5}, \\
F_2(e^{-\pi\sqrt{5}}) = 5\sqrt{5}.
\]  

**Proof.** As in Corollary 3.7, by putting $n = 1/5$ in (3.29), it can be easily seen that

\[
\frac{\phi^2(e^{-\sqrt{5}\pi})}{\phi^2(e^{-\pi\sqrt{5}})} = \frac{1}{\sqrt{5}}.
\]  

Putting $q = e^{-\pi\sqrt{5}}$ in (3.42) and then employing (3.44) and Corollary 3.7, we complete the proof.

**Corollary 3.15.**

\[
F_1(e^{-\pi\sqrt{3/5}}) = \frac{5(5 + \sqrt{5})}{2}, \\
F_2(e^{-\pi\sqrt{3/5}}) = \frac{5(25 + 11\sqrt{5})}{2}.
\]  

**Proof.** As in Corollary 3.8, by putting $n = 3/5$ in (3.29), it can be easily seen that

\[
\frac{\phi^2(e^{-\sqrt{15}\pi})}{\phi^2(e^{-\pi\sqrt{3/5}})} = \frac{2}{5 - \sqrt{5}}.
\]  

Putting $q = e^{-\pi\sqrt{3/5}}$ in (3.42) and then employing (3.46) and Corollary 3.8, we complete the proof.

Now, for the explicit evaluation of $R(q)$ defined in (1.6), we note from [6] that

\[
\frac{1}{R_5(q^2)} - 11 - R_5(q^2) = \frac{f^6(-q^2)}{q^2 f^6(-q^{10})}, \\
\frac{1}{S_5(q)} + 11 - S_5(q) = \frac{f^6(q)}{q f^6(q^5)},
\]  

where $S(q) = -R(-q)$.

From (3.47) and (3.42), we see that to find the explicit values of $R(q^2)$ and $S(q)$, for $q = e^{-\pi\sqrt{n}}$, it is enough to find $F_1(q)$ and $F_2(q)$. See [6].

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**References**

SOME THEOREMS ON THE EXPLICIT EVALUATION …


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