A SUMMABILITY FACTOR THEOREM FOR ABSOLUTE SUMMABILITY INVOLVING ALMOST INCREASING SEQUENCES

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We obtain sufficient conditions for the series $\sum a_n \lambda_n$ to be absolutely summable of order k by a triangular matrix.

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A triangle is a lower triangular matrix with all principal diagonal entries being nonzero. Given an almost increasing sequence $\{X_n\}$ and a sequence $\{\lambda_n\}$ satisfying certain conditions, we obtain sufficient conditions for the series $\sum a_n \lambda_n$ to be absolutely summable of order $k \geq 1$ by a triangle T . As a corollary we obtain the corresponding result when T is a weighted mean matrix.

Theorem 1 of this paper is an example of a summability factor theorem. There is a large literature dealing with summability factor theorems. For example, MathSciNet lists over 500 papers dealing with this topic. For some other papers treating absolute summability factor theorems (of order $k \geq 1$) the reader may wish to consult [3, 4, 5].

Let T be a lower triangular matrix, and $\{s_n\}$ a sequence. Then

$$T_n := \sum_{\nu=0}^n t_{n\nu} s_\nu. \quad (1)$$

A series $\sum a_n$, with partial sums s_n , is said to be summable $|T|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty. \quad (2)$$

We may associate with T two lower triangular matrices $\bar{T}$ and $\hat{T}$ as follows:

$$\begin{aligned} \bar{t}_{n\nu} &= \sum_{r=\nu}^n t_{nr} \quad n, \nu = 0, 1, 2, \dots, \\ \hat{t}_{n\nu} &= \bar{t}_{n\nu} - \bar{t}_{n-1,\nu} \quad n = 1, 2, 3, \dots \end{aligned} \quad (3)$$

We may write

$$T_n = \sum_{\nu=0}^n t_{n\nu} \sum_{i=0}^{\nu} a_i \lambda_i = \sum_{i=0}^n a_i \lambda_i \sum_{\nu=i}^n t_{n\nu} = \sum_{i=0}^n \bar{t}_{ni} a_i \lambda_i. \quad (4)$$

Thus

$$\begin{aligned}
 T_n - T_{n-1} &= \sum_{i=0}^n \bar{t}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i} a_i \lambda_i \\
 &= \sum_{i=0}^n \bar{t}_{ni} a_i \lambda_i - \sum_{i=0}^n \bar{t}_{n-1,i} a_i \lambda_i \\
 &= \sum_{i=0}^n (\bar{t}_{ni} - \bar{t}_{n-1,i}) a_i \lambda_i \\
 &= \sum_{i=0}^n \hat{t}_{ni} a_i \lambda_i = \sum_{i=0}^n \hat{t}_{ni} \lambda_i (s_i - s_{i-1}) \\
 &= \sum_{i=0}^{n-1} \hat{t}_{ni} \lambda_i s_i - \sum_{i=0}^n \hat{t}_{ni} \lambda_i s_{i-1} \\
 &= \sum_{i=0}^{n-1} \hat{t}_{ni} \lambda_i s_i + \hat{t}_{nn} \lambda_n s_n - \sum_{i=0}^n \hat{t}_{ni} \lambda_i s_{i-1} \\
 &= \sum_{i=0}^{n-1} \hat{t}_{ni} \lambda_i s_i + t_{nn} \lambda_n s_n - \sum_{i=0}^{n-1} \hat{t}_{n,i+1} \lambda_{i+1} s_i \\
 &= \sum_{i=0}^n (\hat{t}_{ni} \lambda_i - \hat{t}_{n,i+1} \lambda_{i+1}) s_i + t_{nn} \lambda_n s_n.
 \end{aligned} \tag{5}$$

We may write

$$\begin{aligned}
 (\hat{t}_{ni} \lambda_i - \hat{t}_{n,i+1} \lambda_{i+1}) &= \hat{t}_{ni} \lambda_i - \hat{t}_{n,i+1} \lambda_{i+1} - t_{n,i+1} \lambda_i + t_{n,i+1} \lambda_i \\
 &= (\hat{t}_{ni} - \hat{t}_{n,i+1}) \lambda_i + t_{n,i+1} (\lambda_i - \lambda_{i+1}) \\
 &= \lambda_i \Delta_i \hat{t}_{ni} + \hat{t}_{n,i+1} \Delta \lambda_i.
 \end{aligned} \tag{6}$$

Therefore

$$\begin{aligned}
 T_n - T_{n-1} &= \sum_{i=0}^{n-1} \Delta_i \hat{t}_{ni} \lambda_i s_i + \sum_{i=0}^{n-1} \hat{t}_{n,i+1} \Delta \lambda_i s_i + t_{nn} \lambda_n s_n \\
 &= T_{n1} + T_{n2} + T_{n3}.
 \end{aligned} \tag{7}$$

A positive sequence $\{b_n\}$ is said to be almost increasing if there exists a positive increasing sequence $\{c_n\}$ and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ for each n .

THEOREM 1. *Let $\{X_n\}$ be an almost increasing sequence and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences such that*

- (i) $|\Delta \lambda_n| \leq \beta_n$,
- (ii) $\lim \beta_n = 0$,
- (iii) $\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty$,
- (iv) $|\lambda_n| X_n = O(1)$

are satisfied.

Let T be a triangle satisfying the following:

- (v) $nt_{nn} = O(1)$,
- (vi) $t_{n-1,v} \geq t_{nv}$ for $n \geq v + 1$,
- (vii) $\hat{t}_{n0} = 1$ for all n ,
- (viii) if $\sum_{n=1}^m (1/n) |s_n|^k = O(X_m)$, then the series $\sum a_n \lambda_n$ is summable $|T|_k$, $k \geq 1$.

PROOF. To prove the theorem it will be sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} |T_{nr}|^k < \infty \quad \text{for } r = 1, 2, 3. \quad (8)$$

From [2, page 86] it follows that (vi) and (vii) imply that

$$\sum_{i=0}^{n-1} |\Delta_i \hat{t}_{ni}| = O(t_{nn}). \quad (9)$$

From (iv) and the fact that $\{X_n\}$ is almost increasing, it follows that $\lambda_n = O(1)$. Using Hölder's inequality, (9), and (v),

$$\begin{aligned} \sum_{n=1}^{m+1} n^{k-1} |T_{n1}|^k &\leq \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\Delta_i \hat{t}_{ni}| |\lambda_i| |s_i| \right)^k \\ &\leq \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\Delta_i \hat{t}_{ni}| |\lambda_i|^k |s_i|^k \right) \left(\sum_{i=0}^{n-1} |\Delta_i \hat{t}_{ni}| \right)^{k-1} \\ &= O(1) \sum_{n=1}^{m+1} (nt_{nn})^{k-1} \sum_{i=0}^{n-1} |\Delta_i \hat{t}_{ni}| |\lambda_i|^k |s_i|^k \\ &= O(1) \sum_{n=1}^{m+1} (nt_{nn})^{k-1} \sum_{i=0}^{n-1} |\Delta_i \hat{t}_{ni}| |\lambda_i| |\lambda_i|^{k-1} |s_i|^k \\ &= O(1) \sum_{i=0}^m |\lambda_i| |s_i|^k \sum_{n=i+1}^{m+1} (nt_{nn})^{k-1} |\Delta_i \hat{t}_{ni}| \\ &= O(1) \sum_{i=0}^m |\lambda_i| |s_i|^k t_{ii} \\ &= O(1) \sum_{i=0}^m |\lambda_i| \left(\sum_{r=0}^i |s_r|^k t_{rr} - \sum_{r=0}^{i-1} |s_r|^k t_{rr} \right) \\ &= O(1) \left[\sum_{i=0}^m |\lambda_i| \sum_{r=0}^i |s_r|^k t_{rr} - \sum_{j=0}^{m-1} |\lambda_{j+1}| \sum_{r=0}^j |s_r|^k t_{rr} \right] \\ &= O(1) \left[\sum_{i=0}^{m-1} \Delta |\lambda_i| \sum_{r=0}^i |s_r|^k t_{rr} + |\lambda_m| \sum_{r=0}^m |s_r|^k t_{rr} \right]. \end{aligned} \quad (10)$$

If we define

$$X_i = \sum_{r=0}^i |s_r|^k t_{rr}, \quad (11)$$

then $\{X_i\}$ is an almost increasing sequence. Using (iv) and (i),

$$\sum_{n=1}^{m+1} n^{k-1} |T_{n1}|^k = O(1) \sum_{i=0}^{m-1} \beta_i X_i + O(1) = O(1), \quad (12)$$

using the result of [1, Lemma 3].

Using (i), Hölder's inequality, (ii), (v), (vi), and (vii),

$$\begin{aligned} \sum_{n=1}^{m+1} n^{k-1} |T_{n2}|^k &\leq \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=0}^{n-1} \hat{t}_{n,i+1} s_i \Delta \lambda_i \right|^k \\ &\leq \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\hat{t}_{n,i+1}| |\Delta \lambda_i| |s_i| \right)^k \\ &\leq \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\hat{t}_{n,i+1}| |s_i| \beta_i \right)^k \\ &\leq \sum_{n=1}^{m+1} n^{k-1} \sum_{i=0}^{n-1} |\hat{t}_{n,i+1}| |s_i|^k \beta_i \left(\sum_{i=0}^{n-1} |\hat{t}_{n,i+1}| \beta_i \right)^{k-1} \\ &= O(1) \sum_{n=1}^{m+1} (nt_{nn})^{k-1} \sum_{i=0}^{n-1} |\hat{t}_{n,i+1}| |s_i|^k \beta_i \\ &= O(1) \sum_{i=1}^m \beta_i |s_i|^k \sum_{n=i+1}^{m+1} |\hat{t}_{n,i+1}| \\ &= O(1) \sum_{i=1}^m \beta_i |s_i|^k = O(1) \sum_{i=1}^m i \beta_i \frac{1}{i} |s_i|^k \\ &= O(1) \sum_{i=1}^m \beta_i \left[\sum_{r=1}^i \frac{|s_r|^k}{r} - \sum_{r=1}^{i-1} \frac{|s_r|^k}{r} \right] \\ &= O(1) \left[\sum_{i=1}^m i \beta_i \sum_{r=1}^i \frac{|s_r|^k}{r} - \sum_{j=1}^{m-1} (j+1) \beta_{j+1} \sum_{r=1}^j \frac{|s_r|^k}{r} \right] \\ &= O(1) \sum_{i=1}^{m-1} \Delta(i \beta_i) \sum_{r=1}^i \frac{1}{r} |s_r|^k + O(1) m \beta_m \sum_{i=0}^m \frac{1}{i} |s_i|^k \\ &= O(1) \sum_{i=1}^{m-1} |\Delta(i \beta_i)| X_i + O(1) m \beta_m X_m \\ &= O(1) \sum_{i=0}^{m-1} i |\Delta(\beta_i)| X_i + O(1) \sum_{i=0}^{m-1} \beta_i X_i + O(1) m \beta_m X_m = O(1), \end{aligned} \quad (13)$$

again using [1, Lemma 3].

Using (v),

$$\begin{aligned}
 \sum_{n=1}^{m+1} n^{k-1} |T_{n3}|^k &\leq \sum_{n=1}^{m+1} n^{k-1} |t_{nn} \lambda_n s_n|^k \\
 &= O(1) \sum_{n=1}^m (n t_{nn})^{k-1} t_{nn} |\lambda_n|^k |s_n|^k \\
 &= O(1) \sum_{n=1}^m t_{nn} |\lambda_n|^{k-1} |\lambda_n| |s_n|^k = O(1),
 \end{aligned} \tag{14}$$

as in the proof of T_{n1} . $\square$

A weighted mean matrix, denoted by $(\overline{N}, p_n)$, is a lower triangular matrix with non-zero entries p_k/P_n , where $\{p_n\}$ is a nonnegative sequence with $p_0 > 0$ and $P_n := \sum_{i=0}^n p_i \rightarrow \infty$ as $n \rightarrow \infty$.

COROLLARY 2. *Let $\{X_n\}$ be an almost increasing sequence and let condition (viii) of Theorem 1 be satisfied. If $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy conditions (i)–(iv) of Theorem 1 and if $\{p_n\}$ is a sequence such that*

(i) $n p_n / P_n = O(1)$,

then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.

PROOF. With $T = (\overline{N}, p_n)$, condition (v) of Theorem 1 reduces to condition (i). Conditions (vi) and (vii) of Theorem 1 are automatically satisfied. $\square$

It should be noted that, in [1], an incorrect definition of absolute summability was used (see, e.g., [2]). Corollary 2 gives the correct version of Bor's theorem.

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