ON SIMULTANEOUS APPROXIMATION FOR SOME MODIFIED
BERNSTEIN-TYPE OPERATORS

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We study the simultaneous approximation for a certain variant of Bernstein-type operators. 2000 Mathematics Subject Classification: 41A25, 41A30.

1. Introduction. To approximate Lebesgue integrable functions on the interval $I \equiv [0,1]$, the modified Bernstein operators are defined by

$$M_{n,\alpha,\beta}(f,x) = (n - \alpha + 1) \sum_{k=\beta}^{n-\alpha+\beta} p_{n,k}(x) \int_0^1 p_{n-\alpha,k-\beta}(t) f(t) dt$$

$$+ \sum_{k \in I_n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

(1.1)

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

(1.2)

for $n \geq \alpha$, where $\alpha, \beta$ are integers satisfying $\alpha \geq \beta \geq 0$ and $I_n \subseteq \{0,1,2,\ldots,n\}$ is a certain index set. For $\alpha = \beta = 0$, $I_n = \{0\}$, this definition reduces to the Bernstein-Durrmeyer operators, which were first studied by Derriennic [3]. Also if $\alpha = \beta = 1$, $I_n = \{0\}$, we obtain the recently introduced sequence of Gupta and Maheshwari [4], that is, $M_{n,1,1}(f,x) \equiv P_n(f,x)$ which is defined as

$$P_n(f,x) = \int_0^1 W_n(x,t) f(t) dt$$

$$= n \sum_{k=1}^{n} p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t) f(t) dt + (1-x)^n f(0), \quad x \in I \equiv [0,1],$$

(1.3)

where $W_n(x,t) = n \sum_{k=1}^{n} p_{n,k}(x) p_{n-1,k-1}(t) + (1-x)^n \delta(t)$, $\delta(t)$ being a Dirac Delta function.

In [4] Gupta and Maheshwari have estimated the rate of convergence for functions of bounded variation for the operators $P_n$, $n \in \mathbb{N}$. The approximation properties for different values of $\alpha, \beta$ were studied by several researchers. Recently Abel [1] obtained the complete asymptotic expansion for the Bernstein-Durrmeyer operators ($\alpha = \beta = 0, I_n = \{0\}$) in a concise form in simultaneous approximation. The operators $M_{n,\alpha,\beta}(f,x)$ are
linear positive operators but their approximation properties are different with different values of \( \alpha \) and \( \beta \). In the present paper, we study the pointwise convergence and asymptotic formula in simultaneous approximation for the operators \( M_{n,1}(f,x) \equiv P_n(f,x) \). In the end we give a remark that similar results can be obtained for different values of \( \alpha \) and \( \beta \), for example, we mention the asymptotic formula for another particular case, namely, \( M_{n,0}(f,x) \equiv B_n(f,x) \). Our main theorems can be stated as follows.

**Theorem 1.1.** Let \( f \in C[0,1] \) and let \( f^{(r)} \) exist at a point \( x \in (0,1) \), then

\[
P_n^{(r)}(f,x) = f^{(r)}(t) + o(1) \quad \text{as } n \to \infty.
\]  

**Theorem 1.2.** Let \( f \in C[0,1] \). If \( f^{(r+2)} \) exists at a point \( x \in (0,1) \), then

\[
\lim_{n \to \infty} n \left[ P_n^{(r)}(f,x) - f^{(r)}(x) \right] = x(1-x)f^{(r+2)}(x) + \left[ r - x(1+2r) \right] f^{(r+1)}(x) - r^2 f^{(r)}(x).
\]  

2. Auxiliary results. In this section, we mention some results which are necessary to prove the main theorem.

**Lemma 2.1.** For \( m \in \mathbb{N}_0 \) (the set of nonnegative integers), if the following definition holds:

\[
P_n((t-x)^m,x) = \mu_n,m(x) = \frac{n}{(n+1)} \sum_{k=1}^{n} p_{n,k}(x) \int_{0}^{1} p_{n-1,k-1}(t)(t-x)^m dt + (-x)^m(1-x)^n,
\]

then

\[
\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = \frac{-x}{(n+1)}, \quad \mu_{n,2}(x) = \frac{x(1-x)(2n+1) - (1-3x)x}{(n+1)(n+2)}
\]  

and for \( m \geq 1 \) there holds the recurrence relation

\[
[n+m+1] \mu_{n,m+1}(x) = x(1-x) \left[ \mu_{n,m}^{(1)}(x) + 2m \mu_{n,m-1}(x) \right] + [m(1-2x)-x] \mu_{n,m}(x).
\]

**Proof.** The values of \( \mu_{n,0}(x) \) and \( \mu_{n,1}(x) \) can easily follow from the definition. We prove the recurrence relation as follows:

\[
x(1-x)\mu_{n,m}^{(1)}(x) = n \sum_{k=1}^{n} x(1-x)p_{n,k}^{(1)}(x) \int_{0}^{1} p_{n-1,k-1}(t)(t-x)^m dt + mn \sum_{k=1}^{n} x(1-x)p_{n,k}(x) \int_{0}^{1} p_{n-1,k-1}(t)(t-x)^{m-1} dt + \{n(-x)^m(1-x)^{n-1} + m(-x)^{m-1}(1-x)^n\}x(1-x).
\]
Now using the identity $x(1 - x)p_{n,k}^{(1)}(x) = (k - nx)p_{n,k}(x)$, we obtain

\[
x(1 - x) \left[ \mu_{n,m}(x) + m \mu_{n,m-1}(x) \right]
= n \sum_{k=1}^{n} (k - nx)p_{n,k}(x) \int_{0}^{1} p_{n-1,k-1}(t)(t-x)^m \, dt + n(-x)^{m+1}(1-x)^n
= n \sum_{k=1}^{n} p_{n,k}(x) \int_{0}^{1} \left[ k - 1 - (n-1)t + (n-1)(t-x) + (1-x) \right] p_{n-1,k-1}(t)(t-x)^m \, dt
+ n(-x)^{m+1}(1-x)^n
= n \sum_{k=1}^{n} p_{n,k}(x) \int_{0}^{1} t(1-t)p_{n-1,k-1}^{(1)}(t)(t-x)^m \, dt
+ (n-1)\mu_{n,m+1}(x) + (1-x)\mu_{n,m}(x) - (-x)^{m}(1-x)^n
= n \sum_{k=1}^{n} p_{n,k}(x) \int_{0}^{1} \left[ (1-2x)(t-x) + (t-x)^2 + x(1-x) \right]
\times p_{n-1,k-1}^{(1)}(t)(t-x)^m \, dt
+ (n-1)\mu_{n,m+1}(x) + (1-x)\mu_{n,m}(x) - (-x)^{m}(1-x)^n
= -(m+1)(1-2x)\left[ \mu_{n,m}(x) - (-x)^{m}(1-x)^n \right]
+ (m+2)\left[ \mu_{n,m+1}(x) - (-x)^{m+1}(1-x)^n \right]
- x(1-x)m\left[ \mu_{n,m-1}(x) - (-x)^{m-1}(1-x)^n \right] + (n-1)\mu_{n,m+1}(x)
+ (1-x)\mu_{n,m}(x) - (-x)^{m}(1-x)^n
= [(1-x) - (m+1)(1-2x)]\mu_{n,m}(x) + (n + m + 1)\mu_{n,m+1}(x)
- mx(1-x)\mu_{n,m-1}(x).
\]

This completes the proof of the recurrence relation. \qed

The value of $\mu_{n,2}(x)$ can be easily obtained from the above recurrence relation.

**Remark 1.** For each fixed $x \in [0, 1]$, it follows from the above lemma that

\[
P_n(\psi_x^s, x) = O\left( n^{-\left[\frac{s+1}{2}\right]} \right), \quad n \to \infty,
\]

where $\psi_x = t - x$. 

Lemma 2.2. For \( m \in \mathbb{N} \cup \{0\} \), if the \( m \)th-order moment is defined as

\[
U_{n,m}(x) = \sum_{k=0}^{n} p_{n,k}(x) \left( \frac{v}{n} - x \right)^m,
\]

then \( U_{n,0}(x) = 1 \), \( U_{n,1}(x) = 0 \), and

\[
nU_{n,m+1}(x) = x(1-x) [U_{n,m}^{(1)}(x) + mU_{n,m-1}(x)].
\]

Consequently,

\[
U_{n,m}(x) = \mathcal{O}\left(n^{-(m+1)/2}\right).
\]

Lemma 2.3 [5]. There exist the polynomials \( Q_{i,j,r}(x) \) independent of \( n \) and \( v \) such that

\[
[x(1-x)]^r D^r [p_{n,k}(x)] = \sum_{\substack{2i+j\leq r \\ i,j \leq 0}} n^i(k-nx)^j Q_{i,j,r}(x) p_{n,k}(x), \quad D = \frac{d}{dx}.
\]

3. Proofs of theorems

Proof of Theorem 1.1. By Taylor’s expansion of \( f \), we have

\[
f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} (t-x)^i + \varepsilon(t,x)(t-x)^r,
\]

where \( \varepsilon(t,x) \to 0 \) as \( t \to \infty \).

Hence

\[
P_n^{(r)}(f, x) = \int_{0}^{1} W_n^{(r)}(t, x) f(t) dt
= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_{0}^{1} W_n^{(r)}(t, x) (t-x)^i dt + \int_{0}^{1} W_n^{(r)}(t, x) \varepsilon(t,x)(t-x)^r dt
= R_1 + R_2.
\]

First to estimate \( R_1 \), using binomial expansion of \( (t-x)^m \) and Lemma 2.1, we have

\[
R_1 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{v=0}^{i} \binom{i}{v} (-x)^{i-v} \frac{\partial^v}{\partial x^v} \int_{0}^{1} W_n(t,x) t^v dt
= \frac{f^{(r)}(x)}{r!} \frac{\partial^r}{\partial x^r} \int_{0}^{1} W_n(t,x) t^r dt = f^{(r)}(x) + o(1), \quad n \to \infty.
\]
Next using Lemma 2.3 we obtain
\[
|R_2| \leq n \sum_{2i+j \leq r \atop i,j \geq 0} n^i \frac{|Q_{i,j,r}(x)|}{|x(1-x)|^r} \sum_{k=1}^{n} |k-nx|^j p_{n,k}(x) \\
\times \int_0^1 p_{n-1,k-1}(t) |\varepsilon(t,x)| |t-x|^r \, dt + \frac{n!}{(n-r)!} (1-x)^{n-r} |\varepsilon(0,x)| x^r
\]
\[= R_3 + R_4. \tag{3.4}\]

Since \(\varepsilon(t,x) \to 0\) as \(t \to x\) for a given \(\varepsilon > 0\), there exists a \(\delta > 0\) such that \(|\varepsilon(t,x)| < \varepsilon\) whenever \(0 < |t-x| < \delta\). Thus for some \(M_1 > 0\), we can write
\[
R_3 \leq nM_1 \sum_{2i+j \leq r \atop i,j \geq 0} n^i \sum_{k=1}^{n} p_{n,k}(x) |k-nx|^j \\
\times \left\{ \varepsilon \int_{|t-x| < \delta} p_{n-1,k-1}(t) |t-x|^r \\
+ \int_{|t-x| \geq \delta} p_{n-1,k-1}(t) M_2 |t-x|^r \, dt \right\} = R_5 + R_6, \tag{3.5}\]

where
\[
M_1 = \sup_{2i+j \leq r \atop i,j \geq 0} \frac{|Q_{i,j,r}(x)|}{|x(1-x)|^r}. \tag{3.6}\]

and \(M_2\) is independent of \(t\). Applying the Schwarz inequality for integration and summation respectively, we obtain
\[
R_5 \leq \varepsilon \cdot M_1 n \sum_{2i+j \leq r \atop i,j \geq 0} n^i \sum_{k=1}^{n} p_{n,k}(x) |k-nx|^j \left( \int_0^1 p_{n-1,k-1}(t) \, dt \right)^{1/2} \\
\times \left( \int_0^1 p_{n-1,k-1}(t) |t-x|^{2r} \, dt \right)^{1/2} \\
\leq \varepsilon \cdot M_1 n \sum_{2i+j \leq r \atop i,j \geq 0} n^i \sum_{k=1}^{n} p_{n,k}(x) \left( \sum_{k=1}^{n} p_{n,k}(x) (k-nx)^{2j} \right)^{1/2} \\
\times \left( \sum_{k=1}^{n} p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t) |t-x|^{2r} \, dt \right)^{1/2}. \tag{3.7}\]

Using Lemmas 2.2 and 2.1, we get
\[
R_5 \leq \varepsilon \cdot M_1 \sum_{2i+j \leq r \atop i,j \geq 0} n^i O(n^{j/2}) O(n^{-r/2}) = O(1). \tag{3.8}\]
Again using the Schwarz inequality and Lemmas 2.2 and 2.1, we get

\[ R_6 \leq nM_2 \sum_{2i+j \leq r \atop i,j \geq 0} n^i \sum_{k=1}^{n} p_{n,k}(x) |k-nx|^j \int_{|t-x| \geq \delta} p_{n-1,k-1}(t) |t-x|^{r'} dt \]

\[ \leq nM_2 \sum_{2i+j \leq r \atop i,j \geq 0} n^i \sum_{k=1}^{n} p_{n,k}(x) |k-nx|^j \left( \int_{|t-x| \geq \delta} p_{n-1,k-1}(t) dt \right)^{1/2} \]

\[ \times \left( \int_{|t-x| \geq \delta} p_{n-1,k-1}(t)(t-x)^{2r} dt \right)^{1/2} \]

\[ \leq nM_2 \sum_{2i+j \leq r \atop i,j \geq 0} n^i \left( \sum_{k=1}^{n} p_{n,k}(x)(k-nx)^{2j} \right)^{1/2} \]

\[ \times \left( \sum_{k=1}^{n} p_{n,k}(x) \int_{0}^{1} p_{n-1,k-1}(t)(t-x)^{2r} dt \right)^{1/2} \]

\[ = \sum_{2i+j \leq r \atop i,j \geq 0} n^i O(n^{j/2}) O(n^{-r/2}) = O(n^{j-r/2}) = O(1). \]

Thus, due to arbitrariness of \( \epsilon > 0 \), it follows that \( R_3 = o(1) \). Also \( R_4 \to 0 \) as \( n \to \infty \) and hence \( R_2 = o(1) \). Collecting the estimates of \( R_1 \) and \( R_2 \), we get the required result.

**Proof of Theorem 1.2.** Using Taylor's expansion of \( f \), we have

\[ f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} (t-x)^i + \epsilon(t,x)(t-x)^{r+2}, \] (3.10)

where \( \epsilon(t,x) \to 0 \) as \( t \to x \). Applying Lemma 2.1, we have

\[ n[P_n^{(r)}(f(t),x) - f^{(r)}(x)] = n \left[ \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \int_{0}^{1} W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x) \right] \]

\[ + \left[ n \int_{0}^{1} W_n^{(r)}(t,x) \epsilon(t,x)(t-x)^{r+2} dt \right] = E_1 + E_2, \]

\[ E_1 = n \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \int_{0}^{1} W_n^{(r)}(t,x)t^j dt - n f^{(r)}(x) \]

\[ = \frac{f^{(r)}(x)}{r!} n[P_n^{(r)}(t',x) - (r!)] + \frac{f^{(r+1)}(x)}{(r+1)!} \]

\[ \times n[(r+1)(-x)P_n^{(r)}(t',x) + P_n^{(r)}(t'^{r+1},x)] \]

\[ + \frac{f^{(r+2)}(x)}{(r+2)!} n \left[ \frac{(r+2)(r+1)}{2} x^2 P_n^{(r)}(t',x) \right. \]

\[ + (r+2)(-x)P_n^{(r)}(t'^{r+1},x) + P_n^{(r)}(t'^{r+2},x) \].
It is easily verified from Lemma 2.1 that for each \( x \in (0, 1) \),

\[
P_n(t^v, x) = \frac{(n!)^2}{(n-v)!(n+v)!} x^v + t^{v(v-1)} \frac{(n!)^2}{(n-v+1)!(n+v)!} x^{v-1} + O(n^{-2}). \quad (3.12)
\]

Therefore

\[
E_1 = n f^{(r)}(x) \left[ \frac{(n!)^2}{(n-r)!(n+r)!} - 1 \right] 
+ n f^{(r+1)}(x) \left[ \frac{(n!)^2}{(n-r)!(n+r)!} \right] 
+ \frac{(n!)^2}{(n-r)!(n+r+1)!(r+1)!} x 
+ r(r+1) \frac{(n!)^2}{(n-r)!(n+r+1)!(r+1)!} \left[ \frac{(n!)^2}{(n-r)!(n+r+1)!} \right] 
+ \left[ \frac{(n!)^2}{(n-r-2)!(n+r+2)!} \right] \frac{(n!)^2}{(n-r-2)!(n+r+2)!} \frac{(r+2)!}{2} x^2 
+ \frac{(n!)^2}{(n-r-1)!(n+r+2)!} (r+1) x + O(n^{-2}). \quad (3.13)
\]

In order to complete the proof of the theorem, it is sufficient to show that \( \{x(1+x)\}^{r} E_2 \to 0 \) as \( n \to \infty \), which can easily be proved along the lines of the proof of Theorem 1.1 and by using Lemmas 2.1, 2.2, and 2.3.

**Remark 2.** Just like the operators in (1.3), very recently Abel and Gupta [2] considered the following operators:

\[
B_n(f, x) = (n+1) \sum_{k=1}^{n} p_{n,k}(x) \int_{0}^{1} p_{n,k-1}(t) f(t) dt + (1-x)^n f(0), \quad x \in I \equiv [0, 1], \quad (3.14)
\]

where \( p_{n,k}(x) \) is as defined by (1.3). These operators are \( M_{n,0,1}(f,x) = B_n(f,x) \).
For these operators, we can easily verify the following: for \( m \in \mathbb{N}_0 \) (the set of nonnegative integers), if we define

\[
B_n \left( (t-x)^m, x \right) = \phi_{n,m}(x) = (n+1) \sum_{k=1}^{n} p_{n,k}(x) \int_{0}^{1} p_{n,k-1}(t)(t-x)^m \, dt + (-x)^m (1-x)^n,
\]

then for \( m \geq 1 \) there holds the recurrence relation

\[
[n + m + 2] \phi_{n,m+1}(x) = x(1-x) \left[ \phi_{n,m}^{(1)}(x) + 2m \phi_{n,m-1}(x) \right] + [m(1-2x) - 2x] \phi_{n,m}(x).
\]

Also, it is easily verified that

\[
B_n \left( t^v, x \right) = \frac{n!(n+1)!}{(n-v)!(n+v+1)!} x^v + \frac{n!(n+1)!}{(n-v+1)!(n+v+1)!} x^{v-1} + O(n^{-2}).
\]

Thus we have the following asymptotic formula for the operators \( B_n \).

**Theorem 3.1.** Let \( f \in C[0,1] \). If \( f^{(r+2)} \) exists at a point \( x \in (0,1) \), then

\[
\lim_{n \to \infty} n \left[ B_n^{(r)}(f,x) - f^{(r)}(x) \right] = x(1-x)f^{(r+2)}(x) + \left[ r - 2x(1+r) \right] f^{(r+1)}(x) - r(r+1)f^{(r)}(x).
\]

The proof of Theorem 3.1 is parallel to that of Theorem 1.2; we just have to use the above estimates for the operators.

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