COMMON FIXED POINT THEOREMS OF
CONTRACTIVE-TYPE MAPPINGS

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Using the concept of $D$-metric we prove some common fixed point theorems for generalized contractive mappings on a complete $D$-metric space. Our results extend, improve, and unify results of Fisher and Ćirić.

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1. Introduction. The Banach contraction mapping principle is well known. There are many generalizations of that principle to single- and multivalued mappings (see [1, 4, 5, 10, 11, 12]). The study of maps satisfying some contractive conditions has been the center of rigorous research activity since such mappings have many applications (see [2, 3, 9, 13, 14, 15]).

In 1998, Ćirić [6] proved a common fixed point theorem for nonlinear mappings on a complete metric space: let $(X,d)$ be a complete metric space and $S,T : X \rightarrow X$ self-maps such that $d(STx,TSy) \leq \max\{\varphi_1[(1/2)(d(x,Sy)+d(y,Tx))],\varphi_2[d(x,Tx)],\varphi_3[d(y,Sy)],\varphi_4[d(x,y)]\}$ for all $x,y$ in $X$, where $\varphi_i \in \Phi$ $(i = 1,2,3,4)$. If $S$ or $T$ is continuous, then $S$ and $T$ have a unique common fixed point. This result improved and extended a theorem of Fisher [8].

In this paper, using the concept of $D$-metric, we prove common fixed point theorems which extend, improve, and unify the corresponding theorems of Fisher [8] and Ćirić [6].

Throughout the paper, by $\Phi$ we denote the collection of functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ which are continuous from the right, nondecreasing, and which satisfy the condition $\varphi(t) < t$ for all $t > 0$. We denote by $\mathbb{N}$ the set of all positive integers.

2. Preliminaries. Before proving the main theorem, we will introduce some definitions and lemmas.

**Definition 2.1** [7]. Let $X$ be any nonempty set. A $D$-metric for $X$ is a function $D : X \times X \times X \rightarrow \mathbb{R}$ such that

1. $D(x,y,z) \geq 0$ for all $x,y,z \in X$ and equality holds if and only if $x = y = z$,
2. $D(x,y,z) = D(x,z,y) = D(y,x,z) = D(y,z,x) = D(z,x,y) = D(z,y,x)$ for all $x,y,z \in X$,
3. $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$ for all $x,y,z \in X$.

If $D$ is a $D$-metric for $X$, then the ordered pair $(X,D)$ is called a $D$-metric space or the set $X$, together with a $D$-metric, is called a $D$-metric space. We note that to
a given ordinary metric space \((X,d)\) there corresponds a \(D\)-metric space \((X,D)\), but the converse may not be true (see Example 3.3). In this sense the \(D\)-metric spaces are the generalizations of the ordinary metric space.

**Definition 2.2** [7]. A sequence \(\{x_n\}\) of points of a \(D\)-metric space \(X\) converges to a point \(x \in X\) if for an arbitrary \(\varepsilon > 0\), there exists an \(n_0 \in \mathbb{N}\) such that for all \(n > m \geq n_0\), \(D(x_m,x_n,x) < \varepsilon\).

**Definition 2.3** [7]. A sequence \(\{x_n\}\) of points of a \(D\)-metric space \(X\) is said to be a \(D\)-Cauchy sequence if for an arbitrary \(\varepsilon > 0\), there exists an \(n_0 \in \mathbb{N}\) such that for all \(p > n > m \geq n_0\), \(D(x_m,x_n,x_p) < \varepsilon\).

**Definition 2.4** [7]. A \(D\)-metric space \(X\) is a complete \(D\)-metric space if every \(D\)-Cauchy sequence \(\{x_n\}\) in \(X\) converges to a point \(x \in X\).

**Definition 2.5**. A real-valued function \(f\) defined on a metric space \(X\) is said to be lower semicontinuous at a point \(t \in X\) if \(\lim_{x \to t} \inf f(x) = \infty\) or \(\lim_{x \to t} \inf f(x) \geq f(t)\).

**Definition 2.6**. A real-valued function \(f\) defined on a metric space \(X\) is said to be upper semicontinuous at a point \(t \in X\) if \(\lim_{x \to t} \sup f(x) = \infty\) or \(\lim_{x \to t} \sup f(x) \leq f(t)\).

**Definition 2.7**. Let \(x_0 \in X\) and \(\varepsilon > 0\) be given. Then the open ball \(B(x_0,\varepsilon)\) in \(X\) centered at \(x_0\) of radius \(\varepsilon\) is defined by

\[
B(x_0,\varepsilon) = \left\{ y \in X \mid D(x_0,y,y) < \varepsilon \text{ if } y = x_0, \sup_{z \in X} D(x_0,y,z) < \varepsilon \text{ if } y \neq x_0 \right\}.
\]

Then the collection of all open balls \(\{B(x,\varepsilon) : x \in X\}\) defines the topology on \(X\) denoted by \(\tau\).

**Lemma 2.8** [7]. The \(D\)-metric for \(X\) is a continuous function on \(X \times X \times X\) in the topology \(\tau\) on \(X\).

**Lemma 2.9** [6]. If \(\varphi_1, \varphi_2 \in \Phi\), then there is some \(\varphi \in \Phi\) such that \(\max\{\varphi_1(t), \varphi_2(t)\} \leq \varphi(t)\) for all \(t > 0\).

**Lemma 2.10**. Let \((X,D)\) be a \(D\)-metric space. Let \(g : X \times X \to X\) be a mapping and let \(S,T : X \to X\) be mappings such that

\[
\max \{D(STx,TSy,g(STx,TSy)),D(TS\gamma,STx,g(TS\gamma,STx))\} \\
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (D(x,Sy,g(x,Sy)) + D(y,Tx,g(y,Tx))) \right], \varphi_2[D(x,Tx,g(x,Tx))], \varphi_3[D(y,Sy,g(y,Sy))], \varphi_4[D(x,y,g(x,y))] \right\}
\]

for all \(x,y \in X\), where \(\varphi_i \in \Phi\) \((i = 1,2,3,4)\),

\[
x = y \Rightarrow D(x,y,g(x,y)) = 0,
\]
and

\[
\max \{D(x, z, g(x, z)), D(x, y, g(x, z)), D(y, z, g(x, z))\} \\
\leq D(x, y, g(x, y)) + D(y, z, g(y, z))
\]

for all \(x, y, z \in X\). The sequence \(\{x_n\}\) is defined by \(x_0 \in X\), \(x_{2n+1} = Tx_{2n}\), and \(x_{2n+2} = Sx_{2n+1}\) for every \(n \in \mathbb{N} \cup \{0\}\). Then

(i) for an arbitrary \(\varepsilon > 0\), there exists a positive integer \(L\) such that \(L \leq n < m\) implies

\[
\max \{D(x_n, x_m, g(x_n, x_m)), D(x_m, x_n, g(x_m, x_n))\} < \varepsilon,
\]

(ii) a sequence \(\{x_n\}_{n=0}^{\infty}\) is a \(D\)-Cauchy sequence.

**Proof.** Let \(M = \max\{D(x_0, x_1, g(x_0, x_1)), D(x_1, x_2, g(x_1, x_2)), D(x_2, x_1, g(x_2, x_1))\}\).

Since all \(\varphi_i\) are nondecreasing functions by (2.2), (2.3), and (2.4),

\[
\max \{D(x_2, x_3, g(x_2, x_3)), D(x_3, x_2, g(x_3, x_2))\} \\
= \max \{D(STx_0, TSx_1, g(STx_0, TSx_1)), D(TSx_1, STx_0, g(TSx_1, STx_0))\} \\
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (D(x_0, Sx_1, g(x_0, Sx_1)) + D(x_1, Tx_0, g(x_1, Tx_0))) \right], \right. \\
\left. \varphi_2 [D(x_0, Tx_0, g(x_0, Tx_0))], \varphi_3 [D(x_1, Sx_1, g(x_1, Sx_1))], \right. \\
\left. \varphi_4 [D(x_0, x_1, g(x_0, x_1))] \right\} \\
\leq \max \left\{ \varphi_1(M), \varphi_2(M), \varphi_3(M), \varphi_4(M) \right\} \\
\leq \varphi(M),
\]

where \(\varphi \in \Phi\). Such \(\varphi\) exists from an extended version of Lemma 2.9. Therefore, we have \(\max \{D(x_2, x_3, g(x_2, x_3)), D(x_3, x_2, g(x_3, x_2))\} \leq \varphi(M)\). Again, from (2.2), (2.3), and (2.4), we get

\[
\max \{D(x_3, x_4, g(x_3, x_4)), D(x_4, x_3, g(x_4, x_3))\} \\
= \max \{D(STx_1, STx_2, g(STx_1, STx_2)), D(STx_2, STx_1, g(STx_2, STx_1))\} \\
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (D(x_2, Sx_1, g(x_2, Sx_1)) + D(x_1, Tx_2, g(x_1, Tx_2))) \right], \right. \\
\left. \varphi_2 [D(x_2, Tx_2, g(x_2, Tx_2))], \varphi_3 [D(x_1, Sx_1, g(x_1, Sx_1))], \right. \\
\left. \varphi_4 [D(x_2, x_1, g(x_2, x_1))] \right\} \\
\leq \max \left\{ \varphi_1(M), \varphi_2 [\varphi(M)], \varphi_3(M), \varphi_4(M) \right\} \\
\leq \varphi(M).
\]

Using the obtained relations \(\max \{D(x_2, x_3, g(x_2, x_3)), D(x_3, x_2, g(x_3, x_2))\} \leq \varphi(M)\) and \(\max \{D(x_3, x_4, g(x_3, x_4)), D(x_4, x_3, g(x_4, x_3))\} \leq \varphi(M)\), from (2.2), (2.3), and (2.4),
we get
\[
\max \left\{ D(x_4, x_5, g(x_4, x_5)), D(x_5, x_4, g(x_5, x_4)) \right\}
= \max \left\{ D(STx_2, TSx_3, g(STx_2, TSx_3)), D(TSx_3, STx_2, g(TSx_3, STx_2)) \right\}
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} \left( D(x_2, Sx_3, g(x_2, Sx_3)) + D(x_3, Tx_2, g(x_3, Tx_2)) \right) \right], \right.
\varphi_2 \left[ D(x_2, Tx_2, g(x_2, Tx_2)) \right], \varphi_3 \left[ D(x_3, Sx_3, g(x_3, Sx_3)) \right], \varphi_4 \left[ D(x_2, x_3, g(x_2, x_3)) \right] \right\},
\[
\leq \max \left\{ \varphi_1 [\varphi(M)], \varphi_2 [\varphi^2(M)], \varphi_3 [\varphi(M)], \varphi_4 [\varphi(M)] \right\}
\leq \varphi^2(M).
\]

Similarly, again from (2.2), (2.3), and (2.4), we get
\[
\max \left\{ D(x_5, x_6, g(x_5, x_6)), D(x_6, x_5, g(x_6, x_5)) \right\}
= \max \left\{ D(TSx_3, STx_4, g(TSx_3, STx_4)), D(STx_4, TSx_3, g(STx_4, TSx_3)) \right\}
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} \left( D(x_4, Sx_3, g(x_4, Sx_3)) + D(x_3, Tx_4, g(x_3, Tx_4)) \right) \right], \right.
\varphi_2 \left[ D(x_4, Tx_4, g(x_4, Tx_4)) \right], \varphi_3 \left[ D(x_3, Sx_3, g(x_3, Sx_3)) \right], \varphi_4 \left[ D(x_4, x_3, g(x_4, x_3)) \right] \right\},
\[
\leq \max \left\{ \varphi_1 [\varphi(M)], \varphi_2 [\varphi^2(M)], \varphi_3 [\varphi(M)], \varphi_4 [\varphi(M)] \right\}
\leq \varphi^2(M).
\]

In general, by induction, we get
\[
\max \left\{ D(x_n, x_{n+1}, g(x_n, x_{n+1})), D(x_{n+1}, x_n, g(x_{n+1}, x_n)) \right\} \leq \varphi^{[n/2]}(M) \tag{2.9}
\]
for \( n \geq 2 \), where \([n/2]\) stands for the greatest integer not exceeding \( n/2 \). Since \( \varphi \in \Phi \), by Singh and Meade [13, Lemma 1], it follows that \( \varphi^n(M) \to 0 \) as \( n \to +\infty \) for every \( M > 0 \). Thus, we obtain
\[
\max \left\{ D(x_n, x_{n+1}, g(x_n, x_{n+1})), D(x_{n+1}, x_n, g(x_{n+1}, x_n)) \right\} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.10}
\]

Suppose that (i) does not hold. Then there exists an \( \varepsilon > 0 \) such that for each \( i \in \mathbb{N} \), there exist positive integers \( n_i, m_i \), with \( i \leq n_i < m_i \), satisfying
\[
\varepsilon \leq \max \left\{ D(x_{n_i}, x_{m_i}, g(x_{n_i}, x_{m_i})), D(x_{m_i}, x_{n_i}, g(x_{m_i}, x_{n_i})) \right\},
\max \left\{ D(x_{n_i}, x_{m_i-1}, g(x_{n_i}, x_{m_i-1})), D(x_{m_i-1}, x_{n_i}, g(x_{m_i-1}, x_{n_i})) \right\} < \varepsilon \quad \text{for} \quad i = 1, 2, \ldots. \tag{2.11}
\]

Set
\[
\varepsilon_i = \max \left\{ D(x_{n_i}, x_{m_i}, g(x_{n_i}, x_{m_i})), D(x_{m_i}, x_{n_i}, g(x_{m_i}, x_{n_i})) \right\},
\rho_i = \max \left\{ D(x_{i+1}, x_{i+1}, g(x_{i+1}, x_{i+1})), D(x_{i+1}, x_{i+1}, g(x_{i+1}, x_{i+1})) \right\} \quad \text{for} \quad i = 1, 2, \ldots. \tag{2.12}
\]
Then we have
\[ \varepsilon \leq \varepsilon_i \]
\[ = \max \left\{ D(x_{n_i}, x_{m_i}, g(x_{n_i}, x_{m_i})), D(x_{m_i}, x_{n_i}, g(x_{m_i}, x_{n_i})) \right\} \]
\[ \leq \max \left\{ D(x_{n_i}, x_{m_i-1}, g(x_{n_i}, x_{m_i-1})), D(x_{m_i-1}, x_{n_i}, g(x_{m_i-1}, x_{n_i})) \right\} \]
\[ + \max \left\{ D(x_{m_i-1}, x_{m_i}, g(x_{m_i-1}, x_{m_i})), D(x_{m_i}, x_{m_i-1}, g(x_{m_i}, x_{m_i-1})) \right\} \]
\[ < \varepsilon + \rho_{m_i-1}, \quad i = 1, 2, \ldots. \]

Taking the limit as \( i \to +\infty \), we get \( \lim \varepsilon_i = \varepsilon \). On the other hand, by (2.2), (2.3), and (2.4),
\[ \xi_i = \max \left\{ D(x_{n_i}, x_{m_i}, g(x_{n_i}, x_{m_i})), D(x_{m_i}, x_{n_i}, g(x_{m_i}, x_{n_i})) \right\} \]
\[ \leq \max \left\{ D(x_{n_i}, x_{n_i+1}, g(x_{n_i}, x_{n_i+1})), D(x_{n_i+1}, x_{n_i}, g(x_{n_i+1}, x_{n_i})) \right\} \]
\[ + \max \left\{ D(x_{n_i+1}, x_{n_i+2}, g(x_{n_i+1}, x_{n_i+2})), D(x_{n_i+2}, x_{n_i+1}, g(x_{n_i+2}, x_{n_i+1})) \right\} \]
\[ + \max \left\{ D(x_{n_i+2}, x_{m_i+2}, g(x_{n_i+2}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2})) \right\} \]
\[ + \max \left\{ D(x_{m_i+2}, x_{m_i+1}, g(x_{m_i+2}, x_{m_i+1})), D(x_{m_i+1}, x_{m_i+2}, g(x_{m_i+1}, x_{m_i+2})) \right\} \]
\[ = \rho_{n_i} + \rho_{n_i+1} + \max \left\{ D(x_{n_i+2}, x_{m_i+2}, g(x_{n_i+2}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2})) \right\} \]
\[ + \rho_{m_i+1} + \rho_{m_i}, \quad \text{for } i = 1, 2, \ldots. \] (2.14)

We will now analyze the term \( \max \left\{ D(x_{n_i+2}, x_{m_i+2}, g(x_{n_i+2}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2})) \right\} \) based on the parity of the subscripts.

**Case 1.** \( n_i + 2 \) is even and \( m_i + 2 \) is odd. From (2.2), (2.3), and (2.4), we have
\[ \max \left\{ D(x_{n_i+2}, x_{m_i+2}, g(x_{n_i+2}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2})) \right\} \]
\[ = \max \left\{ D(STx_{n_i}, TSx_{n_i}, g(STx_{n_i}, TSx_{n_i})), D(TSx_{m_i}, STx_{m_i}, g(TSx_{m_i}, STx_{m_i})) \right\} \]
\[ \leq \max \left\{ \varphi_1 \left[ \frac{1}{2} \left( D(x_{n_i}, x_{m_i}, g(x_{n_i}, Sx_{m_i})), D(x_{m_i}, x_{n_i}, g(x_{m_i}, TTx_{n_i})) \right) \right], \varphi_2 \left[ D(x_{n_i}, x_{n_i}, g(x_{n_i}, TTx_{n_i})), \varphi_3 \left[ D(x_{n_i}, x_{m_i}, g(x_{m_i}, TTx_{n_i})), \right. \right. \right. \right. \]
\[ \varphi_4 \left[ D(x_{n_i}, x_{m_i}, g(x_{n_i}, TTx_{n_i})), \right] \right\} \]
\[ \leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (\varepsilon_i + \rho_{m_i} + \varepsilon_i + \rho_{n_i}) \right], \varphi_2 (\rho_{n_i}), \varphi_3 (\rho_{m_i}), \varphi_4 (\varepsilon_i) \right\} \]
\[ \leq \varphi (\varepsilon_i + \rho_{m_i} + \rho_{n_i}). \] (2.15)

Therefore, we have
\[ \max \left\{ D(x_{n_i+2}, x_{m_i+2}, g(x_{n_i+2}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2})) \right\} \leq \varphi (k_i), \] (2.16)
where \( k_i = \varepsilon_i + \rho m_i + \rho n_i \). Substituting (2.16) into (2.14), taking the limit as \( i \to +\infty \), and using the right continuity of \( \varphi \), we get

\[
\varepsilon = \lim_{i \to \infty} \varepsilon_i \leq \lim_{k_i \to \varepsilon^+} \varphi(k_i) = \varphi(\varepsilon) < \varepsilon,
\]

which is a contradiction.

**Case 2.** Both \( n_i + 2 \) and \( m_i + 2 \) are odd. Then, we have

\[
\max \{ D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1})) \} \\
\leq \max \{ D(x_{n_i+1}, x_{n_i+1}, g(x_{n_i+1}, x_{n_i+1})), D(x_{n_i+1}, x_{n_i+1}, g(x_{n_i+1}, x_{n_i+1})) \} \\
+ \max \{ D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+1})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1})) \} \\
= \rho n_i + 1 + \max \{ D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1})) \}.
\]

Since \( n_i + 1 \) is even and \( m_i + 2 \) is odd, from **Case 1**, we have

\[
\max \{ D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1})) \} \\
= \max \{ D(STx_{n_i-1}, TSx_{m_i}, g(STx_{n_i-1}, TSx_{m_i})), D(2TSx_{m_i}, STx_{n_i-1}, g(2TSx_{m_i}, STx_{n_i-1})) \} \\
\leq \max \{ \varphi_1 \left[ \frac{1}{2} (D(x_{n_i-1}, x_{n_i-1}, g(x_{n_i-1}, x_{n_i-1}))) + D(x_{m_i}, x_{m_i}) \right], \varphi_2 (D(x_{n_i-1}, x_{n_i-1}, g(x_{n_i-1}, x_{n_i-1}))) \} \\
\leq \max \{ \varphi_1 \left[ \frac{1}{2} (\rho_{n_i-1} + \varepsilon_i + \rho m_i + \varepsilon_i) \right], \varphi_2(\rho_{n_i-1}), \varphi_3(\rho m_i), \varphi_4(\rho_{n_i-1} + \varepsilon_i) \} \leq \varphi(\varepsilon_i + \rho m_i + \rho_{n_i-1}).
\]

Therefore, we get

\[
\max \{ D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1})) \} \leq \varphi(l_i),
\]

where \( l_i = \varepsilon_i + \rho m_i + \rho_{n_i-1} \). Hence, substituting (2.20) into (2.18), then putting (2.18) into (2.14), and taking the limit as \( i \to +\infty \), we have

\[
\varepsilon = \lim_{i \to \infty} \varepsilon_i \leq \lim_{l_i \to \varepsilon^+} \varphi(l_i) = \varphi(\varepsilon) < \varepsilon,
\]

which is a contradiction. In a similar manner, we get (2.17) and (2.21) for the cases in which \( n_i + 2 \) and \( m_i + 2 \) are both even, and \( n_i + 2 \) is odd and \( m_i + 2 \) is even. That is, all cases lead to a contradiction. Therefore (I) holds.
We claim that \( \{x_n\} \) is \( D \)-Cauchy. Let \( n, m, p \) \((n < m < p)\) be any positive integers. Then, by Definition 2.1 and (2.4),

\[
D(x_n, x_m, x_p) \leq D(x_n, x_m, g(x_n, x_m)) + D(x_n, x_p, g(x_n, x_m)) + D(x_m, x_p, g(x_m, x_m)) \\
\leq D(x_n, x_m, g(x_n, x_m)) + 2D(x_n, x_m, g(x_n, x_m)) + 2D(x_m, x_p, g(x_m, x_p)) \\
= 3D(x_n, x_m, g(x_n, x_m)) + 2D(x_m, x_p, g(x_m, x_p)).
\]

(2.22)

Since \( \lim_{n \to \infty} D(x_n, x_m, g(x_n, x_m)) = 0 \), we have \( \lim_{n \to \infty} D(x_n, x_m, x_p) = 0 \). Thus \( \{x_n\} \) is a \( D \)-Cauchy sequence.

3. Main results. Now we will prove the following fixed point theorems for a complete \( D \)-metric space.

**Theorem 3.1.** Let \((X, D)\) be a complete \( D \)-metric space. Let \( g : X \times X \to X \) be a function and let \( S \) and \( T \) be self-maps on \( X \) satisfying (2.2), (2.3), and (2.4) of Lemma 2.10. For any sequences \( \{u_n\}, \{v_n\} \) in \( X \) such that \( \lim_{n \to \infty} u_n = \alpha \) and \( \lim_{n \to \infty} v_n = \beta \), \( \lim_{n \to \infty} D(u_n, v_n, g(u_n, v_n)) = D(\alpha, \beta, g(\alpha, \beta)) \) for some \( \alpha, \beta \) in \( X \).

If \( S \) or \( T \) is continuous, then \( S \) and \( T \) have a unique common fixed point.

**Proof.** Let the sequence \( \{x_n\} \) be defined by \( x_0 \in X, x_{2n+1} = Tx_{2n} \) and \( x_{2n+2} = Sx_{2n+1} \) for every \( n \in \mathbb{N} \cup \{0\} \). Then, by Lemma 2.10(ii), it follows that \( \{x_n\} \) is a \( D \)-Cauchy sequence. Since \( X \) is a complete \( D \)-metric space, \( \{x_n\} \) is convergent to a limit \( u \) in \( X \). Suppose that \( S \) is continuous. Then

\[
u = \lim_{n \to \infty} x_{2n+2} = \lim_{n \to \infty} Sx_{2n+1} = S\left( \lim_{n \to \infty} x_{2n+1} \right) = Su.
\]

(3.1)

This implies that \( u \) is a fixed point of \( S \). From (2.2), (2.3), and (2.4), we get \( D(u, Su, g(u, Su)) = 0 \) and

\[
D(u, Tu, g(u, Tu)) = D(u, Tsu, g(u, Tsu)) \\
\leq D(u, x_{2n+2}, g(u, x_{2n+2})) + D(Tx_{2n}, Tsu, g(Tx_{2n}, Tsu)) \\
\leq D(u, x_{2n+2}, g(u, x_{2n+2})) \\
+ \max \left\{ \varphi_1 \left( \frac{1}{2} D(x_{2n}, Su, g(x_{2n}, Su)) + D(u, Tx_{2n}, g(u, Tx_{2n})) \right), \right. \\
\left. \varphi_2 [D(x_{2n}, Tx_{2n}, g(x_{2n}, Tx_{2n}))], \varphi_3 [D(u, Su, g(u, Su))], \right. \\
\left. \varphi_4 [D(x_{2n}, u, g(x_{2n}, u))] \right\}.
\]

(3.2)

Taking the limit when \( n \) tends to infinity, by hypothesis, we get \( D(u, Tu, g(u, Tu)) = 0 \). Thus, we have \( u = Su = Tu \). Therefore, \( u \) is the common fixed point of \( S \) and \( T \). The proof for \( T \) continuous is similar.
We will now show that \( u \) is unique. Suppose that \( v \) is also a common fixed point of \( S \) and \( T \). Then, from (2.2), (2.3), and (2.4),

\[
\max \{ D(u, v, g(u, v)), D(v, u, g(v, u)) \} \\
= \max \{ D(STu, TSv, g(STu, TSv)), D(TSv, STu, g(TSv, STu)) \} \\
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (D(u, Sv, g(u, Sv)) + D(v, Tu, g(v, Tu))) \right], \varphi_2 [D(u, Tu, g(u, Tu))], \varphi_3 [D(v, Sv, g(v, Sv))], \varphi_4 [D(u, v, g(u, v))] \right\} \\
= \max \left\{ \varphi_1 \left[ \frac{1}{2} (D(u, v, g(u, v)) + D(v, u, g(v, u))) \right], \varphi_2 [D(u, u, g(u, u))], \varphi_3 [D(v, v, g(v, v))], \varphi_4 [D(u, v, g(u, v))] \right\} \\
\leq \varphi(\max \{ D(u, v, g(u, v)), D(v, u, g(v, u)) \}).
\] (3.3)

We write \( \max \{ D(u, v, g(u, v)), D(v, u, g(v, u)) \} \leq \varphi(\max \{ D(u, v, g(u, v)), D(v, u, g(v, u)) \}) \), which implies that \( \max \{ D(u, v, g(u, v)), D(v, u, g(v, u)) \} = 0 \), that is, \( u = v \). Therefore, the common fixed point of \( S \) and \( T \) is unique. \( \square \)

**Remark 3.2.** Let \( X \) be a complete metric space with a metric \( d \). If we take \( D(x, y, z) = \max\{d(x, y), d(x, z), d(y, z)\} \) and \( g(x, y) = x \) for all \( x, y, z \in X \), then Theorem 3.1 is Ćirić’s [6, Theorem 2] which has extended a theorem of Fisher [8].

The following example shows that a \( D \)-metric is a proper extension of a metric \( d \).

**Example 3.3.** Let \( d \) be a metric on \( \mathbb{R} \). Define the function \( \varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by \( \varphi(x, y) = (x - y)^2 \) for all \( x, y \in \mathbb{R} \). Then, clearly, \( \varphi \) is not metric since \( \varphi(2, 1/2) > \varphi(2, 1) + \varphi(1, 1/2) \). Let \( G, H : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be functions such that \( G(x, y, z) = \max\{d(x, y), d(x, z), d(y, z)\} \) and \( H(x, y, z) = \max\{\varphi(x, y), \varphi(x, z), \varphi(y, z)\} \) for all \( x, y, z \in \mathbb{R} \). Then, clearly, \( G \) and \( H \) are \( D \)-metric for \( \mathbb{R} \). But \( H \) is a \( D \)-metric that is a proper extension of the metric \( d \). Therefore, a \( D \)-metric space is a proper extension of a metric space.

**Corollary 3.4.** Let \( (X, D) \) be a complete \( D \)-metric space. Let \( g : X \times X \to X \) be a function and let \( S \) and \( T \) be self-maps on \( X \) satisfying

\[
\max \{ D(STx, TSy, g(STx, TSy)), D(TSy, STx, g(TSy, STx)) \} \\
\leq c \cdot \max \left\{ \frac{1}{2} [D(x, Sy, g(x, Sy)) + D(y, Tx, g(y, Tx))], \frac{1}{2} [D(x, Tx, g(x, Tx)) + D(y, Sy, g(y, Sy))], D(x, y, g(x, y)) \right\} \\
\] (3.4)

for all \( x, y \in X \), where \( x = y \) implies \( D(x, y, g(x, y)) = 0 \) and \( \max\{D(x, z, g(x, z)), D(x, y, g(x, z)), D(y, z, g(x, z))\} \leq D(x, y, g(x, y)) + D(y, z, g(y, z)) \) for all \( x, y, z \in X \).
For any sequences \( \{u_n\}, \{v_n\} \) in \( X \) such that \( \lim_{n \to \infty} u_n = \alpha \) and \( \lim_{n \to \infty} v_n = \beta \), \( \lim_{n \to \infty} D(u_n, v_n, g(u_n, v_n)) = D(\alpha, \beta, g(\alpha, \beta)) \) for some \( \alpha, \beta \) in \( X \).

If \( S \) or \( T \) is continuous, then \( S \) and \( T \) have a unique common fixed point.

**Proof.** The proof follows by taking \( \varphi_1(t) = c \cdot t \) with \( 0 < c < 1 \) in Theorem 3.1.

We will prove the following corollary using another condition instead of continuity in Theorem 3.1.

**Corollary 3.5.** Let \( (X, D) \) be a complete \( D \)-metric space. Let \( g : X \times X \to X \) be a function, let \( S \) and \( T \) be self-maps on \( X \) satisfying (2.2), (2.3), and (2.4) of Lemma 2.10, and, for each \( u \in X \) with \( u \neq Su \) or \( u \neq Tu \), let

\[
\inf \{ D(x, u, g(x, u)) + D(x, Sx, g(x, Sx)) + D(y, Ty, g(y, Ty)) : x, y \in X \} > 0.
\]

(3.5)

For any sequences \( \{a_n\} \) and \( \{b_n\} \) in \( X \) such that \( \lim_{n \to \infty} a_n = u \) and \( \lim_{n \to \infty} b_n = v \), the following conditions hold:

1. \( \lim_{n \to \infty} D(a_n, b_n, g(a_n, b_n)) = D(u, v, g(u, v)) \),
2. \( \lim_{m \to \infty} D(a_n, b_m, g(a_n, b_m)) = D(a_n, v, g(a_n, v)) \) for each \( n \in \mathbb{N} \),
3. \( \lim_{m \to \infty} D(b_m, a_n, g(b_m, a_n)) = D(v, a_n, g(v, a_n)) \) for each \( n \in \mathbb{N} \).

Then \( S \) and \( T \) have a unique common fixed point.

**Proof.** From Lemma 2.10(i) and (ii), the sequence \( \{x_n\} \) defined by \( x_0 \in X, x_{2n+1} = Tx_{2n}, \) and \( x_{2n+2} = Sx_{2n+1} \) for every \( x \in \mathbb{N} \cup \{0\} \) is a \( D \)-Cauchy sequence. Since \( X \) is a complete \( D \)-metric space, there exists \( u \in X \) such that \( \{x_n\} \) converges to \( u \). Then we have

\[
D(x_{2n+1}, x_{2m+2}, g(x_{2n+1}, x_{2m+2})) = D(TSx_{2n-1}, STx_{2m}, g(TSx_{2n-1}, STx_{2m}))
\]

\[
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (D(x_{2m}, Sx_{2n-1}, g(x_{2m}, Sx_{2n-1})) + D(x_{2n-1}, Tx_{2m}, g(x_{2n-1}, Tx_{2m}))) \right], \varphi_2 [D(x_{2m}, Tx_{2m}, g(x_{2m}, Tx_{2m}))], \varphi_3 [D(x_{2n-1}, Sx_{2n-1}, g(x_{2n-1}, Sx_{2n-1}))], \varphi_4 [D(x_{2m}, x_{2n-1}, g(x_{2m}, x_{2n-1}))] \right\}
\]

\[
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (D(x_{2m}, x_{2n}, g(x_{2m}, x_{2n})) + D(x_{2n-1}, x_{2m+1}, g(x_{2n-1}, x_{2m+1}))) \right], \varphi_2 [D(x_{2m}, x_{2m+1}, g(x_{2m}, x_{2m+1}))], \varphi_3 [D(x_{2n-1}, x_{2n}, g(x_{2n-1}, x_{2n}))], \varphi_4 [D(x_{2m}, x_{2n-1}, g(x_{2m}, x_{2n-1}))] \right\}.
\]

(3.6)

Thus, we obtain \( \lim_{n \to \infty} D(x_{2n+1}, u, g(x_{2n+1}, u)) = 0 \). Assume that \( u \neq Su \) or \( u \neq Tu \).
Then, by hypothesis, we have

\[ 0 < \inf \{ D(x, u, g(x, u)) + D(x, Sx, g(x, Sx)) + D(y, Ty, g(y, Ty)) : x, y \in X \} \]
\[ \leq \inf \{ D(x_{2n+1}, u, g(x_{2n+1}, u)) + D(x_{2n+1}, Sx_{2n+1}, g(x_{2n+1}, Sx_{2n+1})) + D(x_{2n+2}, Tx_{2n+2}, g(x_{2n+2}, Tx_{2n+2})) : n \in \mathbb{N} \} \]
\[ = \inf \{ D(x_{2n+1}, u, g(x_{2n+1}, u)) + D(x_{2n+1}, x_{2n+2}, g(x_{2n+1}, x_{2n+2})) + D(x_{2n+2}, x_{2n+3}, g(x_{2n+2}, x_{2n+3})) : n \in \mathbb{N} \} \]
\[ = 0. \]  

This is a contradiction. Therefore, we have \( u = Su = Tu \).

On the other hand, we can prove the existence of a unique common fixed point of \( S \) and \( T \) by a method similar to that of Theorem 3.1.

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