SOME REFINEMENTS AND GENERALIZATIONS OF CARLEMAN'S INEQUALITY

DAH-YAN HWANG

Received 24 November 2003

We give some refinements and generalizations of Carleman's inequality with weaker condition for weight coefficient.

2000 Mathematics Subject Classification: 26D15.

1. Introduction. The following Carleman's inequality (see [6, Theorem 334]) is well known, unless (a_n) is null:

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n. \quad (1.1)$$

The constant is the best possible.

There is a vast literature which deals with alternative proofs, various generalizations and extensions, and numerous variants and applications in analysis of inequality (1.1); see [1, 2, 3, 5, 7, 9, 8, 10, 13, 14, 15, 16, 17, 18, 19] and the references given therein. According to Hardy (see [6, Theorem 349]), Carleman's inequality was generalized as follows. If $a_n \geq 0$, $\lambda_n \geq 0$, $\Lambda_n = \sum_{m=1}^n \lambda_m$ ($n \in \mathbb{N}$), and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \quad (1.2)$$

In [19], Yuan obtained the refined Carleman's inequality as follows. If $a_n \geq 0$, $n = 1, 2, \dots$, and $0 < \sum_{n=1}^{\infty} a_n < \infty$, then

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \frac{(1-\beta/n)}{(1+1/n)^\alpha} a_n, \quad (1.3)$$

where α, β satisfy $0 \leq \alpha \leq 1/\ln 2 - 1$, $0 \leq \beta \leq 1 - 2/e$, and $e\beta + 2^{1+\alpha} = e$.

Recently, Kim [10] established the following new extension of the refined Hardy's inequality in the spirit of the property of the power mean of n distinct positive numbers.

THEOREM 1.1. *If $0 < \lambda_{n+1} \leq \lambda_n$, $a_n \geq 0$, $\Lambda_n = \sum_{m=1}^n \lambda_m$, $\Lambda_n \geq 1$, $0 < p \leq 1$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} < \frac{e^p}{p} \sum_{n=1}^{\infty} \left(\frac{\Lambda_n^\alpha (\Lambda_n - \lambda_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n)^\alpha} \right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^n \lambda_k c_k^p a_k^p \right)^{(1-p)/p}, \quad (1.4)$$

where $c_k^{\lambda_k} = (\Lambda_{k+1})^{\Lambda_k} / (\Lambda_k)^{\Lambda_{k+1}}$, and α, β satisfy $0 \leq \alpha \leq 1/\ln 2 - 1$, $0 \leq \beta \leq 1 - 2/e$, and $e\beta + 2^{1+\alpha} = e$.

When $p = 1$ in inequality (1.4), there exists the following class of new refined Hardy's inequality:

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left(\frac{\Lambda_n^\alpha (\Lambda_n - \lambda_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n)^\alpha} \right) \lambda_n a_n. \quad (1.5)$$

In this paper, we will establish some new refinements and generalizations of Carleman's inequality. Also, our results correspond to Theorem 1.1 with a weaker condition for weight coefficient.

2. Results. The following results are new generalizations of Carleman's inequality.

THEOREM 2.1. *Let $\lambda_n > 0$, $v_n > 0$, $a_n \geq 0$, $\Lambda_n = \sum_{m=1}^n \lambda_m v_m$ ($n \in \mathbb{N}$), $0 < p \leq 1$, and $0 < \sum_{n=1}^{\infty} \lambda_n v_n a_n < \infty$. If*

$$\frac{\Lambda_{n+1} \Lambda_n}{\Lambda_n + \lambda_n v_n} \leq \Lambda_{n-1}^{(1-1/p)(\Lambda_{n-1}/\Lambda_n)} \Lambda_n^{1/p} v_n^{-(\Lambda_{n-1}/\Lambda_n)} v_{n+1}, \quad (2.1)$$

then

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \cdots a_n^{\lambda_n v_n} \right)^{1/\Lambda_n} \leq \frac{e^p}{p} \sum_{n=1}^{\infty} \left(\frac{\Lambda_n^\alpha (\Lambda_n - \lambda_n v_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n v_n)^\alpha} \right)^p \lambda_n v_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^n \lambda_k v_k (c_k a_k)^p \right)^{(1-p)/p}, \quad (2.2)$$

where

$$c_k = \left[\frac{(\Lambda_k^{(1-1/p)} \Lambda_{k+1})^{\Lambda_k} v_k^{\Lambda_{k-1}}}{(\Lambda_{k-1}^{(1-1/p)} \Lambda_k)^{\Lambda_{k-1}} v_{k+1}^{\Lambda_k}} \right]^{1/\Lambda_k v_k}, \quad (2.3)$$

and α, β satisfy $0 \leq \alpha \leq 1/\ln 2 - 1$, $0 \leq \beta \leq 1 - 2/e$, and $e\beta + 2^{1+\alpha} = e$.

PROOF. By the power mean inequality [11, page 15], we have

$$\alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n} \leq \left(\sum_{m=1}^n q_m \alpha_m^p \right)^{1/p} \quad (2.4)$$

for $\alpha_m \geq 0$, $p > 0$, and $q_m > 0$ ($m \in \mathbb{N}$) with $\sum_{m=1}^n q_m = 1$. Setting $c_m > 0$, $\alpha_m = c_m a_m$, and $q_m = \lambda_m v_m / \Lambda_n$, we obtain

$$(c_1 a_1)^{\lambda_1 v_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 v_2 / \Lambda_n} \cdots (c_n a_n)^{\lambda_n v_n / \Lambda_n} \leq \left(\sum_{m=1}^n \frac{\lambda_m v_m}{\Lambda_n} (c_m a_m)^p \right)^{1/p}. \quad (2.5)$$

Using the above inequality, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \cdots a_n^{\lambda_n v_n})^{1/\Lambda_n} \\ &= \sum_{n=1}^{\infty} \lambda_{n+1} \frac{(c_1 a_1)^{\lambda_1 v_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 v_2 / \Lambda_n} \cdots (c_n a_n)^{\lambda_n v_n / \Lambda_n}}{(c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n})^{1/\Lambda_n}} \\ &\leq \sum_{n=1}^{\infty} \left(\frac{\lambda_{n+1}}{(c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n})^{1/\Lambda_n}} \right) \left(\frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \right)^{1/p}. \end{aligned} \quad (2.6)$$

By using the inequality (see [4, 12])

$$\left(\sum_{m=1}^n z_m \right)^t \leq t \sum_{m=1}^n z_m \left(\sum_{k=1}^m z_k \right)^{t-1}, \quad (2.7)$$

where $t \geq 1$ is constant and $z_m \geq 0$ ($m \in \mathbb{N}$), it is easy to observe that

$$\left(\frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \right)^{1/p} \leq \frac{1}{p \Lambda_n^{1/p}} \sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(1-p)/p} \quad (2.8)$$

for $\Lambda_n \geq 1$ and $0 < p \leq 1$. Then, by (2.6) and (2.8), we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \cdots a_n^{\lambda_n v_n})^{1/\Lambda_n} \\ &\leq \frac{1}{p} \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n^{1/p} (c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n})^{1/\Lambda_n}} \sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(1-p)/p} \\ &= \frac{1}{p} \sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \sum_{n=m}^{\infty} \left(\frac{\lambda_{n+1}}{\Lambda_n^{1/p} (c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n})^{1/\Lambda_n}} \right) \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(1-p)/p}. \end{aligned} \quad (2.9)$$

Choosing $c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n} = (\Lambda_n^{(1-1/p)} \Lambda_{n+1} v_{n+1}^{-1})^{\Lambda_n}$ ($n \in \mathbb{N}$) and setting $\Lambda_0 = 1$, from (2.1), it follows that

$$\begin{aligned} c_n &= \left(\frac{(\Lambda_n^{(1-1/p)} \Lambda_{n+1} v_{n+1}^{-1})^{\Lambda_n}}{(\Lambda_{n-1}^{(1-1/p)} \Lambda_n v_n^{-1})^{\Lambda_{n-1}}} \right)^{1/\Lambda_n v_n} \\ &= \left(\frac{\Lambda_{n+1}}{\Lambda_{n-1}^{(1-1/p)(\Lambda_{n-1}/\Lambda_n)} \Lambda_n^{1/p} v_n^{-(\Lambda_{n-1}/\Lambda_n)} \cdot v_{n+1}} \right)^{\Lambda_n/\Lambda_n v_n} \Lambda_n \\ &\leq \left(1 + \frac{\lambda_n v_n}{\Lambda_n} \right)^{\Lambda_n/\Lambda_n v_n} \Lambda_n. \end{aligned} \quad (2.10)$$

This implies that

$$\begin{aligned} &\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \cdots a_n^{\lambda_n v_n})^{1/\Lambda_n} \\ &\leq \frac{1}{p} \sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \sum_{n=m}^{\infty} \frac{\lambda_{n+1} v_{n+1}}{\Lambda_n \Lambda_{n+1}} \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(1-p)/p} \\ &= \frac{1}{p} \sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \sum_{n=m}^{\infty} \left(\frac{1}{\Lambda_n} - \frac{1}{\Lambda_{n+1}} \right) \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(1-p)/p} \\ &= \frac{1}{p} \sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \frac{1}{\Lambda_n} \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(1-p)/p} \\ &\leq \frac{1}{p} \sum_{m=1}^n \left(1 + \frac{\lambda_m v_m}{\Lambda_m} \right)^{p \Lambda_m / \lambda_m v_m} \lambda_m v_m a_m^p \Lambda_m^{p-1} \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(1-p)/p}. \end{aligned} \quad (2.11)$$

Hence, using the inequality [19, Lemma 3.1]

$$\left(1 + \frac{1}{x} \right)^x \leq e \frac{(1-\beta/x)}{(1+1/x)^\alpha} \quad (2.12)$$

for $x > 1$, and α, β satisfying $0 \leq \alpha \leq 1/\ln 2 - 1$, $0 \leq \beta \leq 1 - 2/e$, and $e\beta + 2^{1+\alpha} = e$, we have (2.2). Thus Theorem 2.1 is proved. $\square$

Taking $v_n = 1$ ($n \in \mathbb{N}$) in Theorem 2.1, we have the following corollary.

COROLLARY 2.2. Let $\lambda_n > 0$, $a_n \geq 0$, $\Lambda_n = \sum_{m=1}^n \lambda_m$ ($n \in \mathbb{N}$), $0 < p \leq 1$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$. If

$$\frac{\Lambda_{n+1} \Lambda_n}{\Lambda_n + \lambda_n} \leq \Lambda_{n-1}^{(1-1/p)(\Lambda_{n-1}/\Lambda_n)} \Lambda_n^{1/p}, \quad (2.13)$$

then

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} \leq \frac{e^p}{p} \sum_{n=1}^{\infty} \left(\frac{\Lambda_n^\alpha (\Lambda_n - \lambda_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n)^\alpha} \right)^p \lambda_n a_n^p. \quad (2.14)$$

REMARK 2.3. Corollary 2.2 is a result corresponding to Theorem 1.1 with a weaker condition for λ_n ($n \in \mathbb{N}$). Further, setting $p = 1$ in Corollary 2.2, we obtain an inequality corresponding to inequality (1.5) with a weaker condition for Λ_n ($n \in \mathbb{N}$).

Setting $p = 1$ in Theorem 2.1, we obtain the following corollary.

COROLLARY 2.4. Let $\lambda_n > 0$, $\nu_n > 0$, $a_n \geq 0$, $\Lambda_n = \sum_{m=1}^n \lambda_m \nu_m$ ($n \in \mathbb{N}$), and $0 < \sum_{n=1}^{\infty} \lambda_n \nu_n a_n < \infty$. If

$$\frac{\Lambda_{n+1}}{\Lambda_n + \lambda_n \nu_n} \leq \nu_n^{-(\Lambda_{n-1}/\Lambda_n)} \nu_{n+1}, \quad (2.15)$$

then

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{1/\Lambda_n} \leq e \sum_{n=1}^{\infty} \left(\frac{\Lambda_n^\alpha (\Lambda_n - \lambda_n \nu_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n \nu_n)^\alpha} \right) \lambda_n \nu_n a_n. \quad (2.16)$$

REMARK 2.5. Inequality (2.16) is a new refinement and generalization of inequality (1.3).

COROLLARY 2.6. Under the assumptions of Theorem 2.1, if $\alpha = 0$, $\beta = 1 - 2/e$, then

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{1/\Lambda_n} \\ & \leq \frac{e^p}{p} \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n \nu_n (1 - 2/e)}{\Lambda_n} \right)^p \lambda_n \nu_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^n \lambda_k \nu_k (c_k a_k)^p \right)^{(1-p)/p}, \end{aligned} \quad (2.17)$$

where (2.3) holds.

REMARK 2.7. Corollary 2.6 is a new refinement and generalization of [10, Corollary 2.3]. Further, setting $\nu_n = 1$ ($n \in \mathbb{N}$) in Corollary 2.6, we obtain a result corresponding to [10, Corollary 2.3] with a weaker condition for λ_n ($n \in \mathbb{N}$).

COROLLARY 2.8. Under the assumptions of Theorem 2.1, if $\alpha = 1/\ln 2 - 1$, $\beta = 0$, then

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{1/\Lambda_n} \\ & \leq \frac{e^p}{p} \sum_{n=1}^{\infty} \left(\left(1 + \frac{\lambda_n \nu_n}{\Lambda_n} \right)^{1-1/\ln 2} \right)^p \lambda_n \nu_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^n \lambda_k \nu_k (c_k a_k)^p \right)^{(1-p)/p}, \end{aligned} \quad (2.18)$$

where (2.3) holds.

REMARK 2.9. Corollary 2.8 is a new refinement and generalization of [10, Corollary 2.4]. Further, setting $v_n = 1$ ($n \in \mathbb{N}$) in Corollary 2.8, we obtain a result corresponding to [10, Corollary 2.4] with a weaker condition for λ_n ($n \in \mathbb{N}$).

A new general refined Hardy's inequality is introduced to the following theorem.

THEOREM 2.10. Let $\lambda_n > 0$, $v_n > 0$, $a_n \geq 0$, $\Lambda_n = \sum_{m=1}^n \lambda_m v_m$ ($n \in \mathbb{N}$), and $0 < \sum_{n=1}^{\infty} \lambda_n v_n a_n < \infty$ for $0 < p \leq t < \infty$. If

$$\frac{\Lambda_{n+1} \Lambda_n}{\Lambda_n + \lambda_n v_n} \leq \Lambda_{n-1}^{((p-t)/p)(\Lambda_{n-1}/\Lambda_n)} \Lambda_n^{t/p} v_n^{-(\Lambda_{n-1}/\Lambda_n)} v_{n+1}, \quad (2.19)$$

then

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \cdots a_n^{\lambda_n v_n})^{t/\Lambda_n} \\ & \leq \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \left(\frac{\Lambda_n^{\alpha} (\Lambda_n - \lambda_n v_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n v_n)^{\alpha}} \right)^{p/t} \lambda_n v_n a_n^p \Lambda_n^{(p-t)/t} \left(\sum_{k=1}^n \lambda_k v_k (c_k a_k)^p \right)^{(t-p)/p}, \end{aligned} \quad (2.20)$$

where

$$c_k = \left[\frac{(\Lambda_k^{(1-t/p)} \Lambda_{k+1})^{\Lambda_k} v_k^{\Lambda_{k-1}}}{(\Lambda_{k-1}^{(1-t/p)} \Lambda_k)^{\Lambda_{k-1}} v_{k+1}^{\Lambda_k}} \right]^{1/\lambda_k v_k t}, \quad (2.21)$$

and α, β satisfy $0 \leq \alpha \leq 1/\ln 2 - 1$, $0 \leq \beta \leq 1 - 2/e$, and $e\beta + 2^{1+\alpha} = e$.

PROOF. The proof is almost the same as in Theorem 2.1. By the power mean inequality [11, page 15], we have (2.4) for $\alpha_m \geq 0$, $p > 0$, and $q_m > 0$ ($m \in \mathbb{N}$) with $\sum_{m=1}^n q_m = 1$. By (2.4), we obtain

$$(\alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n})^t \leq \left(\sum_{m=1}^n q_m \alpha_m^p \right)^{t/p} \quad (2.22)$$

for $t > 0$. Taking $c_m > 0$, $\alpha_m = c_m a_m$, and $q_m = \lambda_m v_m / \Lambda_n$, we have

$$\left((c_1 a_1)^{\lambda_1 v_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 v_2 / \Lambda_n} \cdots (c_n a_n)^{\lambda_n v_n / \Lambda_n} \right)^t \leq \left(\sum_{m=1}^n \frac{\lambda_m v_m}{\Lambda_n} (c_m a_m)^p \right)^{t/p}. \quad (2.23)$$

Using the above inequality, we have

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \dots a_n^{\lambda_n v_n})^{t/\Lambda_n} \\
 &= \sum_{n=1}^{\infty} \lambda_{n+1} \frac{((c_1 a_1)^{\lambda_1 v_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 v_2 / \Lambda_n} \dots (c_n a_n)^{\lambda_n v_n / \Lambda_n})^t}{(c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \dots c_n^{\lambda_n v_n})^{t/\Lambda_n}} \\
 &\leq \sum_{n=1}^{\infty} \left(\frac{\lambda_{n+1}}{(c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \dots c_n^{\lambda_n v_n})^{t/\Lambda_n}} \right) \left(\sum_{m=1}^n \frac{\lambda_m v_m}{\Lambda_n} (c_m a_m)^p \right)^{t/p}.
 \end{aligned} \tag{2.24}$$

By using the inequality (see [4, 12])

$$\left(\sum_{m=1}^n z_m \right)^t \leq t \sum_{n=1}^{\infty} z_m \left(\sum_{k=1}^m z_k \right)^{t-1}, \tag{2.25}$$

where $t \geq 1$ is constant and $z_m > 0$ ($n \in \mathbb{N}$), it is easy to observe that

$$\frac{1}{\Lambda_n^{t/p}} \left(\sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \right)^{t/p} \leq \frac{t}{p \Lambda_n^{t/p}} \sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(t-p)/p} \tag{2.26}$$

for $0 < p \leq t$. Then, by (2.24) and (2.26), we obtain

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \dots a_n^{\lambda_n v_n})^{t/\Lambda_n} \\
 &\leq \frac{t}{p} \sum_{n=1}^{\infty} \left(\frac{\lambda_{n+1}}{\Lambda_n^{t/p} (c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \dots c_n^{\lambda_n v_n})^{t/\Lambda_n}} \right) \sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(t-p)/p}.
 \end{aligned} \tag{2.27}$$

Choosing $c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \dots c_n^{\lambda_n v_n} = (\Lambda_n^{(1-t/p)} \Lambda_{n+1} v_{n+1}^{-1})^{\Lambda_n/t}$ ($n \in \mathbb{N}$) and setting $\Lambda_0 = 1$, from (2.19), it follows that

$$\begin{aligned}
 c_n &= \left(\frac{(\Lambda_n^{(1-t/p)} \Lambda_{n+1} v_{n+1}^{-1})^{\Lambda_n/t}}{(\Lambda_{n-1}^{(1-t/p)} \Lambda_n v_n^{-1})^{\Lambda_{n-1}/t}} \right)^{1/\lambda_n v_n} \\
 &= \left(\frac{\Lambda_{n+1} v_n^{(\Lambda_{n-1}/\Lambda_n)}}{\Lambda_{n-1}^{(1-t/p)(\Lambda_{n-1}/\Lambda_n)} \Lambda_n^{t/p} v_{n+1}} \right)^{\Lambda_n/\lambda_n v_n t} \Lambda_n^{1/t} \\
 &\leq \left(1 + \frac{\lambda_n v_n}{\Lambda_n} \right)^{\Lambda_n/\lambda_n v_n t} \Lambda_n^{1/t}.
 \end{aligned} \tag{2.28}$$

This implies that

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \cdots a_n^{\lambda_n v_n} \right)^{t/\Lambda_n} \\
 & \leq \frac{t}{p} \sum_{m=1}^n \lambda_m v_m (c_m a_m)^p \sum_{n=m}^{\infty} \frac{\lambda_{n+1} v_{n+1}}{\Lambda_n \Lambda_{n+1}} \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(t-p)/p} \\
 & \leq \frac{t}{p} \sum_{m=1}^n \left(1 + \frac{\lambda_m v_m}{\Lambda_m} \right)^{p \Lambda_m / \lambda_m v_m t} \lambda_m v_m a_m^p \Lambda_n^{p/t} \\
 & \quad \times \sum_{n=m}^{\infty} \left(\frac{1}{\Lambda_n} - \frac{1}{\Lambda_{n+1}} \right) \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(t-p)/p} \\
 & \leq \frac{t}{p} \sum_{m=1}^{\infty} \left(1 + \frac{\lambda_m v_m}{\Lambda_m} \right)^{p \Lambda_m / \lambda_m v_m t} \lambda_m v_m a_m^p \Lambda_m^{(p-t)/t} \left(\sum_{k=1}^m \lambda_k v_k (c_k a_k)^p \right)^{(t-p)/p}.
 \end{aligned} \tag{2.29}$$

Hence, using inequality (2.12) [19, Lemma 3.1] for $x > 1$, and α, β satisfy $0 \leq \alpha \leq 1/\ln 2 - 1$, $0 \leq \beta \leq 1 - 2/e$, and $e\beta + 2^{1+\alpha} = e$, we have (2.20). Thus Theorem 2.10 is proved. $\square$

REMARK 2.11. Theorem 2.10 reduces to Theorem 2.1 when $t = 1$. Hence, inequality (2.20) is a new generalization of Hardy's inequality. Taking $v_n = 1$ ($n \in \mathbb{N}$) in Theorem 2.10, we obtain a result corresponding to [10, Theorem 2.6] with a weak condition for λ_n ($n \in \mathbb{N}$). Also assuming that $\lambda_n = 1$ in Theorem 2.10, we have an extension of [10, Corollary 2.7] as in the following corollary.

COROLLARY 2.12. Let $v_n > 0$, $a_n \geq 0$, $\Lambda_n = \sum_{m=1}^n v_m$ ($n \in \mathbb{N}$), and $0 < \sum_{n=1}^{\infty} v_n a_n < \infty$ for $0 < p \leq t < \infty$. If

$$\frac{\Lambda_{n+1} \Lambda_n}{\Lambda_n + v_n} \leq \Lambda_{n-1}^{((p-t)/p)(\Lambda_n/\Lambda_n)} \Lambda_n^{t/p} v_n^{-(\Lambda_{n-1}/\Lambda_n)} v_{n+1}, \tag{2.30}$$

then

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \left(a_1^{v_1} a_2^{v_2} \cdots a_n^{v_n} \right)^{t/\Lambda_n} \\
 & \leq \frac{t e^{p/t}}{p} \sum_{n=1}^{\infty} \left(\frac{\Lambda_n^{\alpha} (\Lambda_n - v_n \beta)}{\Lambda_n (\Lambda_n + v_n)^{\alpha}} \right)^{p/t} v_n a_n^p \Lambda_n^{(p-t)/t} \left(\sum_{k=1}^n v_k (c_k a_k)^p \right)^{(t-p)/p},
 \end{aligned} \tag{2.31}$$

where

$$c_k = \left(\frac{(\Lambda_k^{(1-t/p)} \Lambda_{k+1})^{\Lambda_k} v_k^{\Lambda_{k-1}}}{(\Lambda_k^{(1-t/p)} \Lambda_k)^{\Lambda_{k-1}} v_{k+1}^{\Lambda_k}} \right)^{1/v_k t}. \tag{2.32}$$

COROLLARY 2.13. Under the assumptions of [Theorem 2.10](#), if $\alpha = 0, \beta = 1 - 2/e$, then

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{t/\Lambda_n} \leq \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n \nu_n (1 - 2/e)}{\Lambda_n} \right)^{p/t} \lambda_n \nu_n a_n^p \Lambda_n^{(p-t)/t} \left(\sum_{k=1}^n \lambda_k \nu_k (c_k a_k)^p \right)^{(t-p)/p}, \quad (2.33)$$

where

$$c_k^{\lambda_k \nu_k t} = \frac{\left(\Lambda_k^{(1-t/p)} \Lambda_{k+1} \right)^{\Lambda_k} \nu_k^{\Lambda_{k-1}}}{\left(\Lambda_{k-1}^{(1-t/p)} \Lambda_k \right)^{\Lambda_{k-1}} \nu_{k+1}^{\Lambda_k}}. \quad (2.34)$$

COROLLARY 2.14. Under the assumptions of [Theorem 2.1](#), if $\alpha = 1/\ln 2 - 1, \beta = 0$, then

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{t/\Lambda_n} \leq \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \left(\left(1 + \frac{\lambda_n \nu_n}{\Lambda_n} \right)^{1-1/\ln 2} \right)^{p/t} \lambda_n \nu_n a_n^p \Lambda_n^{(p-t)/t} \left(\sum_{k=1}^n \lambda_k \nu_k (c_k a_k)^p \right)^{(t-p)/p}, \quad (2.35)$$

where [\(2.34\)](#) holds.

REMARK 2.15. Corollaries [2.13](#) and [2.14](#) are new refinements and generalizations of [[10](#), Corollaries 2.8 and 2.9], respectively. Further, taking $\nu_n = 1$ ($n \in \mathbb{N}$) in Corollaries [2.13](#) and [2.14](#), we obtain results corresponding to [[10](#), Corollaries 2.8 and 2.9] with a weaker condition for λ_n ($n \in \mathbb{N}$).

REFERENCES

- [1] H. Alzer, *On Carleman's inequality*, Portugal. Math. **50** (1993), no. 3, 331–334.
- [2] ———, *Refinement of a Carleman-type inequality*, Studia Sci. Math. Hungar. **32** (1996), no. 3–4, 361–366.
- [3] ———, *A refinement of Carleman's inequality*, J. Approx. Theory **95** (1998), no. 3, 497–499.
- [4] G. S. Davies and G. M. Petersen, *On an inequality of Hardy's. II*, Quart. J. Math. Oxford Ser. (2) **15** (1964), 35–40.
- [5] G. Hardy, *Note on a theorem of Hilbert*, Math. Zeitschr. **6** (1920), 314–317.
- [6] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, London, 1952.
- [7] D. Y. Hwang and G. S. Yang, *Note on discrete Hardy's inequality*, Tamkang J. Math. **21** (1990), no. 4, 333–336.
- [8] Y.-H. Kim, *Refinements and extensions of an inequality*, J. Math. Anal. Appl. **245** (2000), no. 2, 628–632.
- [9] ———, *On Carleman's inequality and its improvement*, Korean J. Comput. Appl. Math. **8** (2001), 1021–1026.

- [10] ———, *Refinements of Carleman's inequality and its generalizations*, Indian J. Pure Appl. Math. **33** (2002), no. 12, 1775–1784.
- [11] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Mathematics and Its Applications, East European Series, vol. 61, Kluwer Academic Publishers, Dordrecht, 1993.
- [12] J. Németh, *Generalizations of the Hardy-Littlewood inequality*, Acta Sci. Math. (Szeged) **32** (1971), 295–299.
- [13] B. G. Pachpatte, *A note on Copson's inequality involving series of positive terms*, Tamkang J. Math. **21** (1990), no. 1, 13–19.
- [14] P. Yan and G. Sun, *A strengthened Carleman's inequality*, J. Math. Anal. Appl. **240** (1999), no. 1, 290–293.
- [15] B. Yang, *On Hardy's inequality*, J. Math. Anal. Appl. **234** (1999), no. 2, 717–722.
- [16] B. Yang and L. Debnath, *Some inequalities involving the constant e , and an application to Carleman's inequality*, J. Math. Anal. Appl. **223** (1998), no. 1, 347–353.
- [17] X. Yang, *Approximations for constant e and their applications*, J. Math. Anal. Appl. **262** (2001), no. 2, 651–659.
- [18] ———, *On Carleman's inequality*, J. Math. Anal. Appl. **253** (2001), no. 2, 691–694.
- [19] B.-Q. Yuan, *Refinements of Carleman's inequality*, J. Inequal. Pure Appl. Math. **2** (2001), no. 2, Article 21, <http://jipam.vu.edu.au/article.php?sid=137>.

Dah-Yan Hwang: Center for General Education, Kuang Wu Institute of Technology, Peito, Taipei 11271, Taiwan

E-mail address: dyhuang@kwit.edu.tw