ON A HIGHER-ORDER EVOLUTION EQUATION
WITH A STEPA诺V-BOUNDED SOLUTION

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We study strong solutions \( u : \mathbb{R} \to X \), a Banach space \( X \), of the \( n \)-th-order evolution equation

\[
 u^{(n)} - A u^{(n-1)} = f,
\]

an infinitesimal generator of a strongly continuous group \( A : D(A) \subseteq X \to X \), and a given forcing term \( f : \mathbb{R} \to X \). It is shown that if \( X \) is reflexive, \( u \) and \( u^{(n-1)} \) are Stepaνov-bounded, and \( f \) is Stepaνov almost periodic, then \( u \) and all derivatives \( u', \ldots, u^{(n-1)} \) are strongly almost periodic. In the case of a general Banach space \( X \), a corresponding result is obtained, proving weak almost periodicity of \( u, u', \ldots, u^{(n-1)} \).

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1. Introduction. In this paper, we are concerned with an \( n \)-th-order evolution equation of the form

\[
 u^{(n)} - A u^{(n-1)} = f.
\]

Here \( A : D(A) \subseteq X \to X \) is an infinitesimal generator of a strongly continuous group, \( f : \mathbb{R} \to X \) a given forcing term, \( X \) a Banach space with scalar field \( \mathbb{C} \), \( n \) a positive integer, and \( \mathbb{R} \) denotes the set of reals. We will give suitable assumptions to ensure that almost periodicity of the forcing term \( f \) carries over to the solution \( u \) and its derivatives up to order \( (n-1) \).

The reason for studying this rather special evolution equation may be classified as a first pilot study of the issue of higher-order evolution equations, which probably has not been studied before.

We first recall the relevant concepts. A continuous function \( f : \mathbb{R} \to X \) is said to be strongly (or Bochner) almost periodic if, for every given \( \varepsilon > 0 \), there is an \( r > 0 \) such that any interval in \( \mathbb{R} \) of length \( r \) contains a point \( \tau \) for which

\[
 \sup_{t \in \mathbb{R}} \| f(t + \tau) - f(t) \| \leq \varepsilon.
\]

Here \( \| \cdot \| \) denotes the norm in \( X \).

A function \( f : \mathbb{R} \to X \) is called weakly almost periodic if \( x^* f(\cdot) : \mathbb{R} \to \mathbb{C} \) is continuous and almost periodic for every \( x^* \) in the dual space \( X^* \) of \( X \).

We will call a function \( f \in L^1_{loc}(\mathbb{R}, X) \) Stepaνov-bounded or briefly \( S \)-bounded if

\[
 \| f \|_S := \sup_{t \in \mathbb{R}} \int_t^{t+1} \| f(s) \| ds < \infty.
\]
We will call a function $f \in L^{1}_{\text{loc}}(\mathbb{R}, X)$ Stepanov almost periodic or briefly $S$-almost periodic if, for every given $\varepsilon > 0$, there is an $r > 0$ such that any interval in $\mathbb{R}$ of length $r$ contains a point $\tau$ for which
\[
\sup_{t \in \mathbb{R}} \int_{t}^{t+1} ||f(s + \tau) - f(s)||ds \leq \varepsilon.
\]
(1.4)

We denote by $L(X, X)$ the set of all bounded linear operators on $X$ into itself. An operator-valued function $T : \mathbb{R} \rightarrow L(X, X)$ will be called a strongly continuous group if
\[
T(t_1 + t_2) = T(t_1)T(t_2) \quad \forall t_1, t_2 \in \mathbb{R},
\]
(1.5)
\[
T(0) = I = \text{the identity operator on } X,
\]
(1.6)
\[
T(\cdot)x : \mathbb{R} \rightarrow X \text{ is continuous for every } x \in X.
\]
(1.7)

We recall (e.g., from Dunford and Schwartz [4]) that the infinitesimal generator $A : D(A) \subseteq X \rightarrow X$ of a strongly continuous group $T : \mathbb{R} \rightarrow L(X, X)$ is a densely defined, closed linear operator.

An operator-valued function $T : \mathbb{R} \rightarrow L(X, X)$ is said to be strongly (weakly) almost periodic if $T(\cdot)x : \mathbb{R} \rightarrow X$ is strongly (weakly) almost periodic for every $x \in X$.

Suppose $A : D(A) \subseteq X \rightarrow X$ is a densely defined, closed linear operator, and $f : \mathbb{R} \rightarrow X$ is a continuous function. Then a strong solution of the evolution equation
\[
u^{(n)}(t) - Au^{(n-1)}(t) = f(t) \quad \text{a.e. for } t \in \mathbb{R}
\]
(1.8)
is an $n$ times strongly differentiable function $u : \mathbb{R} \rightarrow X$ with $u^{(n-1)}(t) \in D(A)$ for all $t \in \mathbb{R}$, and satisfies problem (1.8).

Our first result is as follows (see Zaidman [7, 8] for first-order evolution equations).

**Theorem 1.1.** Let $X$ be reflexive, $f : \mathbb{R} \rightarrow X$ continuous, $S$-almost periodic, $A$ an infinitesimal generator of a strongly almost periodic group $T : \mathbb{R} \rightarrow L(X, X)$. In this case, if, for the strong solution $u : \mathbb{R} \rightarrow X$ of problem (1.8), both $u$ and $u^{(n-1)}$ are $S$-bounded on $\mathbb{R}$, then $u, u', \ldots, u^{(n-1)}$ are all strongly almost periodic.

Our second result refers to a weak variant of our first theorem in the case of a general—not necessarily reflexive—Banach space $X$.

**Theorem 1.2.** Suppose $f : \mathbb{R} \rightarrow X$ is an $S$-almost periodic (or a weakly almost periodic) continuous function, $A$ an infinitesimal generator of a strongly continuous group $T : \mathbb{R} \rightarrow L(X, X)$ such that the conjugate operator group $T^* : \mathbb{R} \rightarrow L(X^*, X^*)$ is strongly almost periodic. If, for the strong solution $u : \mathbb{R} \rightarrow X$ of problem (1.8), both $u$ and $u^{(n-1)}$ are $S$-bounded on $\mathbb{R}$, then $u, u', \ldots, u^{(n-1)}$ are all weakly almost periodic.

**Remark 1.3.** For some examples of first-order and higher-order evolution equations with strongly almost periodic solutions, the reader may wish to consult Cooke [3] and Zaidman [9].
2. Lemmas

**Lemma 2.1.** If $A$ is the infinitesimal generator of a strongly continuous group $G : \mathbb{R} \to L(X,X)$, then the $(n-1)$th derivative of any solution of (1.8) has the representation

$$u^{(n-1)}(t) = G(t)u^{(n-1)}(0) + \int_0^t G(t-s)f(s)ds \quad \text{for } t \in \mathbb{R}. \tag{2.1}$$

**Proof.** For an arbitrary but fixed $t \in \mathbb{R}$, we have

$$\frac{d}{ds} [G(t-s)u^{(n-1)}(s)] = G(t-s)[u^{(n)}(s) - Au^{(n-1)}(s)]$$

$$= G(t-s)f(s) \quad \text{a.e. for } s \in \mathbb{R}, \text{ by (1.8)}. \tag{2.2}$$

Now, integrating (2.2) from 0 to $t$, we obtain

$$\int_0^t \frac{d}{ds} [G(t-s)u^{(n-1)}(s)] ds = \int_0^t G(t-s)f(s)ds,$$

which gives the desired representation, by (1.6). \hfill \Box

**Lemma 2.2.** If $g : \mathbb{R} \to X$ is a strongly almost periodic function, and $G : \mathbb{R} \to L(X,X)$ is a strongly (weakly) almost periodic operator-valued function, then $G(\cdot)g(\cdot) : \mathbb{R} \to X$ is a strongly (weakly) almost periodic function.

For the proof of Lemma 2.2, see [6, Theorem 1] for weak almost periodicity.

**Lemma 2.3.** If $g : \mathbb{R} \to X$ is an $S$-almost periodic continuous function, and $G : \mathbb{R} \to L(X,X)$ is a weakly almost periodic operator-valued function, then $x^*G(\cdot)g(\cdot) : \mathbb{R} \to C$ is an $S$-almost periodic continuous function for every $x^* \in X^*$.

**Proof.** By our assumption, for an arbitrary but fixed $x^* \in X^*$, the function $x^*G(\cdot)x : \mathbb{R} \to C$ is almost periodic, and so is bounded on $\mathbb{R}$, for every $x \in X$. Hence, by the uniform-boundedness principle,

$$\sup_{t \in \mathbb{R}} \|x^*G(t)\| = K < \infty. \tag{2.4}$$

We note that the function $x^*G(\cdot)g(\cdot)$ is continuous on $\mathbb{R}$ (see [6, proof of Theorem 1]).

Consider the functions $g_\eta$ given by

$$g_\eta(t) = \frac{1}{\eta} \int_0^\eta g(t+s)ds \quad \text{for } \eta > 0, \ t \in \mathbb{R}. \tag{2.5}$$

Since $g$ is $S$-almost periodic from $\mathbb{R}$ to $X$, $g_\eta$ is strongly almost periodic from $\mathbb{R}$ to $X$ for every fixed $\eta > 0$. Further, as shown for $C$-valued functions in [2, pages 80-81], we can prove that $g_\eta \to g$ as $\eta \to 0^+$ in the $S$-sense, that is,

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} \|g(s) - g_\eta(s)\|ds \to 0 \quad \text{as } \eta \to 0^+. \tag{2.6}$$

Now we have

$$x^*G(s)g(s) = x^*G(s)[g(s) - g_\eta(s)] + x^*G(s)g_\eta(s) \quad \text{for } s \in \mathbb{R}, \tag{2.7}$$
and, by (2.4) and (2.6),
\[
\sup_{t \in \mathbb{R}} \int_{t}^{t+1} \left| x^* G(s) [g(s) - g_{\eta}(s)] \right| ds \\
\leq K \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \left\| g(s) - g_{\eta}(s) \right\| ds \to 0 \quad \text{as } \eta \to 0^+.
\] (2.8)

By Lemma 2.2, the functions \( x^* G(\cdot) g_{\eta}(\cdot) \) are almost periodic from \( \mathbb{R} \) to \( C \). Therefore, it follows from (2.7)-(2.8) that \( x^* G(\cdot) g(\cdot) \) is \( S \)-almost periodic from \( \mathbb{R} \) to \( C \).

**Lemma 2.4.** If \( g : \mathbb{R} \to X \) is an \( S \)-almost periodic continuous function, and \( G : \mathbb{R} \to L(X,X) \) is a strongly almost periodic operator-valued function, then \( G(\cdot) g(\cdot) : \mathbb{R} \to X \) is an \( S \)-almost periodic continuous function.

The proof of this lemma parallels that of Lemma 2.3 and may therefore be safely omitted.

**Lemma 2.5.** In a reflexive space \( X \), assume \( h : \mathbb{R} \to X \) is an \( S \)-almost periodic continuous function, and
\[
H(t) = \int_{0}^{t} h(s) ds \quad \text{for } t \in \mathbb{R}.
\] (2.9)
If \( H \) is \( S \)-bounded, then it is strongly almost periodic from \( \mathbb{R} \) to \( X \).

For the proof of Lemma 2.5, see [5, Notes (ii)].

**Lemma 2.6.** For an operator-valued function \( G : \mathbb{R} \to L(X,X) \), suppose \( G^*(t) \) is the conjugate (adjoint) of the operator \( G(t) \) for \( t \in \mathbb{R} \). If \( G^* : \mathbb{R} \to L(X^*,X^*) \) is strongly almost periodic, and \( g : \mathbb{R} \to X \) is weakly almost periodic, then \( G(\cdot) g(\cdot) : \mathbb{R} \to X \) is weakly almost periodic.

For the proof of Lemma 2.6, see [6, Remarks (iii)].

3. Proof of Theorem 1.1. By (2.1), we have
\[
T(-t) u^{(n-1)}(t) = u^{(n-1)}(0) + \int_{0}^{t} T(-s) f(s) ds \quad \text{for } t \in \mathbb{R}.
\] (3.1)
Evidently, \( T(\cdot) : \mathbb{R} \to L(X,X) \) is a strongly almost periodic group. Therefore, \( T(\cdot) x : \mathbb{R} \to X \) is strongly almost periodic, and so is bounded on \( \mathbb{R} \), for every \( x \in X \). Hence, by the uniform-boundedness principle,
\[
\sup_{t \in \mathbb{R}} \| T(-t) \| < \infty.
\] (3.2)
Consequently, \( T(\cdot) u^{(n-1)}(\cdot) \) is \( S \)-bounded on \( \mathbb{R} \) (by our assumption, \( u^{(n-1)} \) is \( S \)-bounded on \( \mathbb{R} \)).

Moreover, by Lemma 2.4, \( T(\cdot) f(\cdot) : \mathbb{R} \to X \) is an \( S \)-almost periodic continuous function. So, by Lemma 2.5, \( T(\cdot) u^{(n-1)}(\cdot) \) is strongly almost periodic from \( \mathbb{R} \) to \( X \). Hence, by Lemma 2.2, \( u^{(n-1)}(\cdot) = T(\cdot) [T(\cdot) u^{(n-1)}(\cdot)] \) is strongly almost periodic from \( \mathbb{R} \) to \( X \).
Now consider a sequence \((\alpha_k)_{k=1,2,...}\) of infinitely differentiable nonnegative functions on \(\mathbb{R}\) such that
\[
\alpha_k(t) = 0 \quad \text{for} \ |t| \geq \frac{1}{k}, \quad \int_{-1/k}^{1/k} \alpha_k(t) \, dt = 1. \tag{3.3}
\]
The convolution of \(u\) and \(\alpha_k\) is defined by
\[
(u * \alpha_k)(t) = \int_{\mathbb{R}} u(t-s) \alpha_k(s) \, ds = \int_{\mathbb{R}} u(s) \alpha_k(t-s) \, ds \quad \text{for} \ t \in \mathbb{R}. \tag{3.4}
\]
We set
\[
C_{\alpha_k} = \max_{|t| \leq 1/k} \alpha_k(t). \tag{3.5}
\]
Then we have
\[
\| (u * \alpha_k)(t) \| = \left\| \int_{-1}^{1} u(t-s) \alpha_k(s) \, ds \right\| \leq C_{\alpha_k} \int_{-1}^{1} \| u(\rho) \| \, d\rho \leq 2C_{\alpha_k} \| u \|_{S} \quad \text{for} \ t \in \mathbb{R}, \text{by (1.3)}. \tag{3.6}
\]
That is, \(u * \alpha_k\) is bounded on \(\mathbb{R}\).

We note that, for \(m = 1,2,\ldots,n-1\) and \(k = 1,2,\ldots\),
\[
(u * \alpha_k)^{(m)}(t) = (u^{(m)} * \alpha_k)(t) \quad \text{for} \ t \in \mathbb{R}. \tag{3.7}
\]
Further, since \(u^{(n-1)}\) is strongly almost periodic from \(\mathbb{R}\) to \(X\), \((u * \alpha_k)^{(n-1)} = (u^{(n-1)} * \alpha_k)\) is strongly almost periodic from \(\mathbb{R}\) to \(X\). Consequently, by [3, corollary to Lemma 5], \(u * \alpha_k, u' * \alpha_k, \ldots, u^{(n-2)} * \alpha_k\) are all strongly almost periodic from \(\mathbb{R}\) to \(X\).

With \(u^{(n-1)}\) being bounded on \(\mathbb{R}\), \(u^{(n-2)}\) is uniformly continuous on \(\mathbb{R}\). Therefore, the sequence of convolutions \((u^{(n-2)} * \alpha_k)(t) - u^{(n-2)}(t)\) as \(k \to \infty\), uniformly for \(t \in \mathbb{R}\). Hence \(u^{(n-2)}\) is strongly almost periodic from \(\mathbb{R}\) to \(X\). We thus conclude successively that \(u^{(n-2)}, \ldots, u', u\) are all strongly almost periodic from \(\mathbb{R}\) to \(X\), completing the proof of the theorem.

4. Proof of Theorem 1.2. By our assumption, for an arbitrary but fixed \(x^* \in X^*\), \(x^* T(\cdot) = T^*(\cdot)x^* : \mathbb{R} \to X^*\) is strongly almost periodic, and so \(x^* T(\cdot)x : \mathbb{R} \to C\) is almost periodic for every \(x \in X\). Therefore, it follows that \(T : \mathbb{R} \to L(X;X)\) is a weakly almost periodic group.

By (3.1), we have
\[
x^* T(\cdot) u^{(n-1)}(t) = x^* u^{(n-1)}(0) + \int_{0}^{t} x^* T(-s) f(s) \, ds \quad \text{for} \ t \in \mathbb{R}. \tag{4.1}
\]
By Lemma 2.3, \(x^* T(\cdot) f(\cdot) : \mathbb{R} \to C\) is an \(S\)-almost periodic continuous function. By (2.4), \(x^* T(\cdot) u^{(n-1)}(\cdot)\) is \(S\)-bounded on \(\mathbb{R}\), and so, by Lemma 2.5, is almost periodic from \(\mathbb{R}\) to \(C\). That is, \(T(\cdot) u^{(n-1)}(\cdot)\) is weakly almost periodic from \(\mathbb{R}\) to \(X\). Consequently, by Lemma 2.6, \(u^{(n-1)}(\cdot) = T(\cdot)[T(\cdot) u^{(n-1)}(\cdot)]\) is weakly almost periodic from \(\mathbb{R}\) to \(X\).
For the sequence \((\alpha_k)_{k=1,2,...}\) defined by (3.3), \((x^* u^* \alpha_k) = x^*(u^* \alpha_k)\) is bounded on \(\mathbb{R}\) (by (3.6)). Further, for \(m = 1,2,...,n-1\) and \(k = 1,2,...\), we have

\[
(x^* u^* \alpha_k)^{(m)}(t) = (x^* u^{(m)} \alpha_k)(t) \quad \text{for } t \in \mathbb{R}.
\]

(4.2)

Now the rest of the proof is obvious.

If \(f : \mathbb{R} \to X\) is weakly almost periodic, then by Lemma 2.6, \(T(\cdots f(\cdot)) : \mathbb{R} \to X\) is weakly almost periodic.

**Remark 4.1.** If \(T(t) \equiv I\) for \(t \in \mathbb{R}\), and so \(A = 0\), then problem (1.8) reduces to

\[
u^{(n)}(t) = f(t) \quad \text{a.e. for } t \in \mathbb{R}.
\]

(4.3)

(i) In a reflexive space \(X\), suppose \(f\) is defined as in Theorem 1.1. If \(u : \mathbb{R} \to X\) is an \(S\)-bounded strong solution of problem (4.3), then \(u,u',...,u^{(n-1)}\) are all strongly almost periodic from \(\mathbb{R}\) to \(X\).

(ii) Assume \(f : \mathbb{R} \to X\) is a weakly almost periodic continuous function. If \(u : \mathbb{R} \to X\) is an \(S\)-bounded strong solution of problem (4.3), then \(u,u',...,u^{(n-1)}\) are all weakly almost periodic from \(\mathbb{R}\) to \(X\).

These states are clearly special cases of Theorems 1.1 and 1.2 if we take into account that the assumption \(u^{(n-1)} \text{ is } S\)-bounded can be omitted, since, by (4.3), \(u^{(n)}\) is \(S\)-almost periodic, and so \(u^{(n-1)}\) is strongly (weakly) uniformly continuous on \(\mathbb{R}\) (by Amerio and Prouse [1, Theorem 8, page 79]).

**References**


