SUPPORT FUNCTIONALS AND THEIR RELATION TO THE RADON-NIKODYM PROPERTY

I. SADEQI

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In this paper, we examine the Radon-Nikodym property and its relation to the Bishop-Phelps theorem for complex Banach spaces. We also show that the Radon-Nikodym property implies the Bishop-Phelps property in the complex case.


1. Introduction. Let $X$ be a complex Banach space and let $C$ be a closed convex subset of $X$. The set of support points of $C$, written as $\text{supp} C$, is the collection of all points $z \in C$ for which there exists nontrivial $f \in X^*$ such that $\sup_{x \in C} |f(x)| = |f(z)|$. Such an $f$ is called support functional. The point $z \in \text{supp} C$ is called a strongly exposed point of $C$ if for all sequences $\{z_n\} \subset C$, $\lim_{n \to \infty} (\text{Re} f(z_n)) = \sup_{x \in C} \text{Re}(f)$ implies that $z_n \to z$, where Re denotes the real part.

In this paper, we will show that the unit ball of an infinite-dimensional function algebra has no strongly exposed points. Lomonosov [4] constructed a closed, bounded, and convex subset $C$ of a complex Banach space such that the set of support points of $C$ is empty. This means that the Bishop-Phelps theorem fails to hold in the complex case. We show below that for Hardy spaces, the Bishop-Phelps theorem does hold.

Bourgain [1] proved that if $X$ is a real Banach space, then the Radon-Nikodym property (RNP) and the Bishop-Phelps property (BPP) are equivalent. The precise definitions are given below.

A subset $C$ of a Banach space is called dentable if for every $\epsilon > 0$, there exists an $x \in C$ such that $x \not\in \overline{\text{co}}(C - \mathbb{N}(x, \epsilon))$, where $\overline{\text{co}}$ denotes “closed convex hull” and $\mathbb{N}(x, \epsilon)$ is the...
open $\epsilon$-neighborhood of $x$. In the final part of this paper, we discuss uncountability of
the set of normalized support functionals, a question posed by L. Zajicek in 1999.

2. The RNP for complex Banach spaces. Bourgain proved in [1] that if the real Banach space $X$ has the RNP and if $C$ is a closed, convex, and bounded subset of $X$, then the set of support functionals that strongly expose some point of $C$ is dense in $X^*$. Suppose that the complex Banach space $X$ has the RNP and that $C$ is a bounded, closed, and convex subset of $X$. Put

$$H := \{ e^{i\theta}x : 0 \leq \theta < 2\pi, \ x \in C \}. \tag{2.1}$$

Let $B$ denote the closed convex hull of $H$. Then, as cited above, the set of real parts of linear functionals which strongly expose some point of $B$ forms a dense subset of $X_r^*$. Here $X_r$ is the underlying real Banach space. By the standard isometry $f \rightarrow \text{Re} \ f$ between $X^*$ and $X^*_r$, these are the real parts of a dense subset of $X^*$. The strongly exposed points of $B$ are contained in $H$ since $H$ is closed, so by the theorem of Phelps,

$$\sup |f|(C) = \sup \text{Re} \ f(H) \tag{2.2}$$

and the support functionals are dense in the complex case. Therefore, using the two theorems of Phelps and Bourgain, the RNP implies the Bishop-Phelps theorem in the complex case. If we show that the RNP implies the BPP in the complex case, the Bishop-Phelps theorem clearly holds for complex Banach spaces with the RNP without recourse to the theorems of Bourgain [1] and Phelps [6].

**Definition 2.1.** Let $B$ be a nonempty, bounded, closed, and convex subset of the complex Banach space $X$. Let $Y$ be a Banach space and $T \in L(X,Y)$. Say that $T$ is an $F$-strongly exposing operator for the set $B$ if there exists some point $x \in B$ depending on $T$ such that every sequence $(x_n) \subset B$ satisfying

$$\sup \|T\|(B) = \lim_{n \to \infty} \|T(x_n)\| \tag{2.3}$$

has a subsequence converging to $\alpha x$ for some complex number $\alpha$ with $|\alpha| = 1$.

If $X$ is a real Banach space, then the definition of an $F$-strongly exposing operator becomes Bourgain's definition of an $R$-strongly exposing operator, as follows.

Let $B$ be a nonempty, bounded, closed, and convex subset of the complex Banach space $X$. Let $Y$ be a Banach space and $T \in L(X,Y)$. $T$ is called an $R$-strongly exposing operator for the set $B$ if there exists some point $x \in B$ depending on $T$ such that every sequence $(x_n) \subset B$ satisfying (2.3) has a subsequence converging to $x$ or $-x$ [1].

**Theorem 2.2** (Bourgain). Let $B$ be a nonempty, bounded, closed, and convex subset of the real Banach space $X$. Assume that every nonempty subset of $B$ is dentable. Then for any Banach space $Y$, the set of all $R$-strongly exposing operators $T \in L(X,Y)$ for the set $B$ is a dense subset of $L(X,Y)$.

In the following discussion, we show why we need the definition of an $F$-strongly exposing operator to prove Bourgain's theorem in the complex case. Let $X$ be a complex
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Banach space and let $C$ be a circled, bounded, closed, and convex subset of $X$ (i.e., $\alpha C \subseteq C$ if $|\alpha| = 1$). Suppose that $T$ achieves its max norm $\|T\|(C)$ on $C$. Then there exists $x_0$ in $C$ such that

$$\sup \|T\|(C) = \|T(x_0)\|. \quad (2.4)$$

Choose $\alpha_n = i$; it is clear that

$$\lim_{n \to \infty} \|T(\alpha_n x_0)\| = \sup \|T\|(C), \quad (2.5)$$

and $(\alpha_n x_0)$ has no subsequence converging to $x_0$ or $-x_0$. So the circled subsets of $X$ have no $R$-strongly exposing operator. Also let $X_r$ and $Y_r$ be the underlying real part of the complex Banach spaces $X$ and $Y$, respectively. Since $L(X_r,Y_r)$ is not isomorphic to $L(X,Y)$ in general, we cannot directly use Bourgain's definition in the complex case.

**Theorem 2.3.** Let $X$ be a complex Banach space with the RNP. Then $X$ possesses the BPP.

**Proof.** Let $C$ be a nonempty, closed, convex, and bounded subset of $X$. Let $Y$ be a complex Banach space. We must show that for $T \in L(X,Y)$, there is an approximating sequence $(T_n)$ for which each $(T_n)$ achieves its max norm on $C$. For $n \in \mathbb{N}$, define

$$K_n := \left\{ T \in L(X,Y) : \exists \xi \geq 0, \exists t \in X, S(T,\xi) \subseteq \bigcup_{|\alpha| = 1} \mathbb{N}(\alpha t, n^{-1}) \right\}, \quad (2.6)$$

where $S(T,\xi) := \{x \in C : \|Tx\| \geq \|T\|(C) - \xi\}$. Since $X$ has the RNP, every nonempty and bounded subset of $X$ is dentable, so $K_n$ is dense in $L(x,y)$ (see [1]). If $T \in K_n$ for each $n \in \mathbb{N}$, then there exists a sequence $(x_n)$ in $C$ such that $\lim_{n \to \infty} \|Tx_n\| = \|T\|(C)$. There exists also $t \in X$ such that $(x_n)$ is contained in $F := \{\alpha t : |\alpha| = 1\}$. It follows that $x_n = \alpha_n t$, and there is a subsequence of $(x_n)$ converging to $\alpha_0 t$ for some $\alpha_0$ with $|\alpha_0| = 1$. Therefore, $T$ is an $F$-strongly exposing operator and achieves its max norm on $C$ at $\alpha_0 t$. It follows that $T$ is an $F$-strongly exposing operator if and only if $T \in K_n$ for every $n \in \mathbb{N}$. Put $K := \bigcap_{n=1}^\infty K_n$. Since any $K_n$ is dense in $L(X,Y)$ if we show that any $K_n$ is open, we conclude that $K$ is dense in $L(X,Y)$. Let $T \in K_n$ and suppose that $F \in L(X,Y)$ with $\|F - T\| \leq \xi/3$. It is easy to see that $S(F,\xi/3) \subseteq S(T,\xi/3)$, so $F \in K_n$ and $K_n$ is open in $L(X,Y)$. The fact that each $T \in K$ is an $F$-strongly exposing operator and achieves its max norm on $C$ completes the proof.

**Corollary 2.4.** The RNP implies the Bishop-Phelps theorem in the complex case.

**Proof.** Since the RNP implies the BPP in the complex case, the proof is clear. □

That the BPP implies the RNP can be easily checked through Bourgain's proof in [1], so we have the following result.

**Theorem 2.5.** The RNP and the BPP are equivalent for complex Banach spaces.

We are ready to examine the RNP for some complex Banach spaces. Let $\tau := \{e^{i\theta} : 0 \leq \theta \leq \pi\}$ and $L^1(\tau)$ denote the summable complex functions on $\tau$. The following discussion shows that some subspaces of $L^1(\tau)$ have the RNP so that the Bishop-Phelps
theorem is true for them. The complex Bishop-Phelps theorem is still open for $L^1$; however, we can verify this theorem for some subspaces of $L^1$. Let $C(\tau)$ be the set of all continuous functions on $\tau$ and let $C^0$ denote the functions in $C(\tau)$ which are analytic with mean value zero. It is well known that

$$(C(\tau)/C^0)^* \approx H^1(\tau). \quad (2.7)$$

Since $H^1(\tau)$ is a separable space with a predual, it has the RNP and, hence, the Bishop-Phelps theorem is true for $H^1(\tau) \subset L^1(\tau)$. Also, all $H^p \ (1 \leq p < \infty)$ are separable and have a predual. The following theorem is an immediate consequence of this discussion.

**Theorem 2.6.** The Bishop-Phelps theorem is satisfied for the Hardy spaces $H^p \ (1 \leq p < \infty)$ in the complex case.

As mentioned above, a separable dual Banach space has the RNP. Since $H^\infty$ is not separable, it is still unknown whether the Bishop-Phelps theorem is true for $H^\infty$. Hensgen (see [3]) proved that the unit ball of $H^\infty$ has no strongly exposed points. Therefore, $H^\infty$ is guaranteed to lack both the RNP and the BPP. In the following, we will show that the unit ball of an infinite-dimensional uniform algebra has no strongly exposed point. As a result, such spaces do not have the RNP. So it is natural to ask whether the Bishop-Phelps theorem holds for infinite-dimensional uniform algebras.

Let $X$ be a nonempty set and let $K$ be a normed linear algebra. We denote by $\ell^\infty(X,K)$ the normed linear algebra of all bounded mappings of $X$ into $K$ with pointwise addition and scalar multiplication and with the uniform norm. For a bounded mapping $f$, the uniform norm is defined by

$$\|f\|_\infty = \sup \{||f(x)|| : x \in X\}. \quad (2.8)$$

A uniform algebra of functions on $X$ is a subalgebra of the Banach algebra $\ell^\infty(X,F)$, where $F$ is either the real or complex numbers. Given a nonempty topological space $X$, $C(X,K)$ denotes the linear space of all continuous mappings of $X$ into $K$ with pointwise addition and scalar multiplication. When $X$ is compact, $C(X,K)$ is a closed linear subspace of $\ell^\infty(X,K)$, and in particular, $C(X,F)$ is a uniform algebra of functions. The notation $C(X,F)$ is abbreviated to $C(X)$.

**Lemma 2.7.** Let $X$ be a compact Hausdorff space such that $C(X)$ is infinite-dimensional. Then the unit ball of $C(X)$ has no strongly exposed point.

**Proof.** If $X$ admits no Baire diffuse measure (a nonnegative measure $\mu$ on $X$ is said to be diffuse if $\mu(V) > 0$ for every nonempty open subset $V$ of $X$), then the unit ball of $C(X)$ contains no exposed point (see [5]). So the proof is clear in this case. Let $X$ admit a diffuse measure and suppose that $\lambda \in C(X)^*$ exposes $f \in U$, where $U$ is the unit ball of $C(X)$. Then $|f(t)| = 1 \ (t \in X)$. Since $X$ admits a diffuse measure and $\|f\| = 1$, a result of Eberlein’s guarantees that accumulation points can be approximated by sequences (see [7]). We can assume that there exists a sequence $(t_n) \subset X$ such that $f(t_n) \to 1$. Choose pairwise disjoint open sets $U_n$ in $X$ such that $t_n \in U_n$. Also choose $h_n \in C(X)$ with $0 \leq h_n \leq 1$, $h_n(t_n) = 1$, and $h_n = 0$ on $K \setminus U_n$. Put $g_n = 1 - h_n$; clearly $\|g_n\| \leq 1$.
and $g_n(t) \to 1$ for each $t \in X$. Hence $g_n f(t) \to f(t)$, $t \in X$. Also by the Hahn-Banach theorem, it is easy to see that $\lambda$ must be of the form

$$\lambda(g) = \int_X g \, d\mu.$$  

(2.9)

It is clear that $\lambda(g_n f) \to \lambda(f)$, but $\|g_n f - f\| = \|h_n f\| \geq 1$. So $f$ cannot be a strongly exposed point. Also since any uniform algebra is a commutative $B^*$-algebra, by the Gelfand-Naimark theorem, $A$ is an isometric isomorphism of $C(\Delta A)$, where $\Delta A$ is the maximal ideal space of $A$. Therefore, as cited above, $A$ has no strongly exposed point, and we have the following result.

**Theorem 2.8.** The unit ball of an infinite-dimensional $C^*$-algebra has no strongly exposed point.

It is well known that if $X$ is a complex Banach space with the RNP, then the set of support functionals that expose some points of the unit ball of $X$ is norm dense in $X^*$ [1].

**Corollary 2.9.** An infinite-dimensional separable $C^*$-algebra possesses neither the RNP nor the BPP; hence it has no predual.

### 3. The set of normalized support functionals

**Problem 3.1.** Suppose that $X$ is a real Banach space with dim $X > 1$ and $C \subseteq X$ is a bounded, closed, and convex subset of $X$. Is the set of normalized support functionals $\Sigma(C)$ an uncountable subset of $S_{X^*}$? (Due to L. Zajíček).

Phelps has made the following observations. Suppose that $\Sigma(C)$ is countable; then it has no interior point, so $S_{X^*} \setminus \Sigma(C)$ is a dense $G_\delta$ set in $S_{X^*}$. Since $\Sigma(C)$ is dense in $S_{X^*}$, from the Bishop-Phelps theorem, we conclude that $X^*$ is separable. If $X$ is reflexive, then any linear functional of $S_{X^*}$ supports $C$, which is a contradiction. Since $\Sigma(C)$ is countable, $X$ must be a nonreflexive space with a separable dual space. Also $C$ must have an empty interior because otherwise we may assume that $0 \in \text{int } C$, and since dim $X > 1$, it is possible to have a two-dimensional subspace $M$ of $X$. So if $f \in S_{M^*}$, then it supports $M \cap C$. By the Hahn-Banach theorem, there is an extension $f''$ of $f$ in $\Sigma(C)$. That is, there is one-to-one map from the uncountable set $\Sigma(M \cap C)$ into $\Sigma(C)$, which is impossible.

**Theorem 3.2.** Let $X$ be a weakly sequentially complete (w.s.c.) Banach space and let $C$ be a closed, convex, and bounded subset of $X$. Then $\Sigma(C)$ is uncountable.

**Proof.** If $\Sigma(C)$ is countable, since $\Sigma(C)$ is dense in $S_{X^*}$, $S_{X^*}$ is separable. By Dunford-Pettis theorem (see [2]), $X^*$ possesses the RNP. Let $(x_n)$ be a bounded sequence in $X$. Then $Y$, the closed linear span of $(x_n)$, is a separable subspace of $X$. Since $X^*$ has the RNP, $Y^*$ is separable. By a classical result of Banach, $(x_n)$ has a weak Cauchy subsequence in $Y$ (again denoted by $(x_n)$), which is also a weak Cauchy sequence in $X$. Since $X$ is w.s.c., then $(x_n)$ is a weakly convergent sequence, therefore, any bounded sequence $(x_n)$ is a weakly convergent sequence in $X$, and so $X$ is a reflexive space. Thus $\Sigma(C) = S_{X^*}$, an uncountable set.
Remark 3.3. If any nonempty subset of a closed, convex, and bounded set $C$ is dentable, then by a result of Bourgain (see [6]), for such a set $C$, the set of support functionals is $G_\sigma$ dense in $X^*$. And by a classical theorem that the Banach space $X$ has no countable $G_\sigma$ dense subset, we conclude that $\Sigma(C)$ is uncountable.

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References


I. Sadeqi: Departement of Mathematics, Sahand University of Technology, P.O. Box 51335-1996, Tabriz, Iran

E-mail address: esadeqi@sut.ac.ir
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