SOME EXACT INEQUALITIES OF HARDY-LITTLEWOOD-POLYA TYPE FOR PERIODIC FUNCTIONS

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We investigate the following problem: for a given $A \geq 0$, find the infimum of the set of $B \geq 0$ such that the inequality $\|x^{(k)}\|_2^2 \leq A\|x^{(r)}\|_2^2 + B\|x\|_2^2$, for $k, r \in \mathbb{N} \cup \{0\}$, $0 \leq k < r$, holds for all sufficiently smooth functions.

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1. Introduction. Let $G = \mathbb{R}$ or $G = \mathbb{T} = [0, 2\pi)$. By $L_2(G)$, we will denote the spaces of all measurable functions $x : G \rightarrow \mathbb{R}$ such that

$$\|x\|_2 = \|x\|_{L_2(G)} := \left\{ \int_G |x(t)|^2 dt \right\}^{1/2} < \infty. \quad (1.1)$$

Denote by $L_2^r(G)$ ($r \in \mathbb{N}$) the space of all functions x such that $x^{(r-1)}$ are locally absolutely continuous and $x^{(r)} \in L_2(G)$, and set $L_{2,2}^r(G) = L_2(G) \cap L_2^r(G)$ (in the case $G = \mathbb{T}$, we mean that spaces $L_2(G)$ and $L_2^r(G)$ consist of 2π -periodic functions). Note that $L_2^r(G) \subset L_2(G)$ if $G = \mathbb{T}$.

It is well known that the exact inequality of Hardy [3]

$$\|x^{(k)}\|_2^2 \leq \|x\|_2^{2(1-k/r)} \|x^{(r)}\|_2^{2(k/r)}, \quad k \in \mathbb{N}, \quad 0 < k < r, \quad (1.2)$$

holds for every function $x \in L_{2,2}^r(\mathbb{R})$.

For any $A > 0$ and any $x \in L_{2,2}^r(\mathbb{R})$, from inequality (1.2), we get

$$\|x^{(k)}\|_2^2 \leq \left\{ \left(\frac{k}{Ar} \right)^{k/(r-k)} \|x\|_2^2 \right\}^{(r-k)/r} \left\{ \frac{Ar}{k} \|x^{(r)}\|_2^2 \right\}^{k/r}. \quad (1.3)$$

Using Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 \leq p < \infty, \quad a, b > 0, \quad (1.4)$$

with $p = r/(r-k)$ and $p' = r/k$, we get, for any $A > 0$ and any $x \in L_{2,2}^r(\mathbb{R})$, the following inequality:

$$\|x^{(k)}\|_2^2 \leq A\|x^{(r)}\|_2^2 + \frac{r-k}{r} \left(\frac{k}{Ar} \right)^{k/(r-k)} \|x\|_2^2. \quad (1.5)$$

This inequality is the best possible in the next sense: for a given $A > 0$, the infimum of constants B such that the inequality

$$\|x^{(k)}\|_2^2 \leq A\|x^{(r)}\|_2^2 + B\|x\|_2^2 \quad (1.6)$$

holds for all functions $x \in L_{2,2}^r(\mathbb{R})$ is equal to

$$\frac{r-k}{r} \left(\frac{k}{Ar} \right)^{k/(r-k)}. \quad (1.7)$$

As is well known, inequality (1.2) (and consequently (1.5)) holds true for any function $x \in L_{2,2}^r(\mathbb{T})$. However, the constant (1.7) is not the best possible in general (for a given constant A). Therefore, the main problem which we will study in this paper is the following.

For a given $A \geq 0$, find the infimum of constants B such that inequality (1.6) holds for all functions $x \in L_{2,2}^r(\mathbb{T})$.

We will denote this infimum by $\Psi(\mathbb{T}; r, k; A)$. We will investigate also the analogous problem in the presence of some restrictions on the spectrum of functions $x \in L_{2,2}^r(\mathbb{T})$.

Note that Babenko and Rassias [1] investigated the problem on exact inequalities for functions $x \in L_{2,2}^r(\mathbb{T})$. They have found, for a given $A \geq 0$, the infimum of constants B such that the inequality

$$\|x^{(k)}\|_2^2 \leq A\|x\|_2^2 + B\|x^{(r)}\|_2^2 \quad (1.8)$$

holds for all functions $x \in L_{2,2}^r(\mathbb{T})$.

For more information related to this subject, see, for example, [2, 4, 5, 6].

2. Main results

THEOREM 2.1. *Let $k, r \in \mathbb{N}$, $k < r$. Then for any $A \geq 0$ and any $x \in L_{2,2}^r(\mathbb{T})$,*

$$\|x^{(k)}\|_2^2 \leq A\|x^{(r)}\|_2^2 + (v_0^{2k} - Av_0^{2r})\|x\|_2^2 = A\|x^{(r)}\|_2^2 + \varphi(A, v_0)\|x\|_2^2 \quad (2.1)$$

holds if v_0 is such that $\eta(v_0 + 1) \leq A \leq \eta(v_0)$, where

$$\eta(v) = \frac{v^{2k} - (v-1)^{2k}}{v^{2r} - (v-1)^{2r}}. \quad (2.2)$$

Given A , the constant $\varphi(A, v_0)$ in (2.1) is the best possible; that is,

$$\Psi(\mathbb{T}; r, k; A) = (v_0^{2k} - Av_0^{2r}), \quad (2.3)$$

where v_0 is such that $\eta(v_0 + 1) \leq A \leq \eta(v_0)$.

PROOF. Let

$$\begin{aligned} e_v(t) &:= \frac{1}{2\pi} e^{ivt}, \quad v \in \mathbb{Z}, t \in \mathbb{R}, \\ c_v(x) &= \int_0^{2\pi} x(t) e_v(t) dt \end{aligned} \quad (2.4)$$

be Fourier coefficients of a function x , and let

$$\sum_{v \in \mathbb{Z}} c_v(x) e_v(t) \quad (2.5)$$

be the Fourier series of a function x .

For any $x \in L_{2,2}^r(\mathbb{T})$, $0 < k < r$, and any $A \geq 0$, using Parseval's equality, we get

$$\begin{aligned} \|x^{(k)}\|_2^2 &= \sum_{\substack{v \in \mathbb{Z} \\ v \neq 0}} |c_v(x)|^2 v^{2k} \\ &= A \sum_{\substack{v \in \mathbb{Z} \\ v \neq 0}} |c_v(x)|^2 v^{2r} + \sum_{\substack{v \in \mathbb{Z} \\ v \neq 0}} |c_v(x)|^2 v^{2r} \left[\frac{v^{2k}}{v^{2r}} - A \right] \\ &= A \|x^{(r)}\|_2^2 + \sum_{\substack{v \in \mathbb{Z} \\ v \neq 0}} |c_v(x)|^2 [v^{2k} - A v^{2r}] \\ &\leq A \|x^{(r)}\|_2^2 + \max_{v \in \mathbb{N}} [v^{2k} - A v^{2r}] \sum_{\substack{v \in \mathbb{Z} \\ v \neq 0}} |c_v(x)|^2 \\ &= A \|x^{(r)}\|_2^2 + \max_{v \in \mathbb{N}} [v^{2k} - A v^{2r}] \|x\|_2^2. \end{aligned} \quad (2.6)$$

Set

$$\varphi(A, v) := v^{2k} - A v^{2r}; \quad (2.7)$$

then the last inequality can be written in the form

$$\|x^{(k)}\|_2^2 \leq A \|x^{(r)}\|_2^2 + \max_{v \in \mathbb{N}} \varphi(A, v) \|x\|_2^2. \quad (2.8)$$

Our goal now is to find for a given $A \geq 0$ the value of

$$\max_{v \in \mathbb{N}} \varphi(A, v). \quad (2.9)$$

We consider the difference

$$\begin{aligned} \delta_v &= \varphi(A, v) - \varphi(A, v-1) \\ &= v^{2k} - A v^{2r} - (v-1)^{2k} + A (v-1)^{2r} \\ &= A [(v-1)^{2r} - v^{2r}] - [(v-1)^{2k} - v^{2k}] \\ &= [(v)^{2r} - (v-1)^{2r}] \left[\frac{v^{2k} - (v-1)^{2k}}{v^{2r} - (v-1)^{2r}} - A \right]. \end{aligned} \quad (2.10)$$

Set, for $v \in \mathbb{N}$,

$$\eta(v) := \frac{v^{2k} - (v-1)^{2k}}{v^{2r} - (v-1)^{2r}}; \quad (2.11)$$

then the last equality can be written in the form

$$\delta_v = [(v)^{2r} - (v-1)^{2r}] [\eta(v) - A]. \quad (2.12)$$

It is not difficult to see that

$$\operatorname{sgn} \delta_v = \operatorname{sgn} [\eta(v) - A]. \quad (2.13)$$

We now study the function $\eta(v)$.

Note that $\eta(1) = 1$, $\eta(v) \rightarrow 0$ as $v \rightarrow \infty$ (since $k < r$), and, for $v \geq 1$,

$$\eta(v) > \eta(v+1). \quad (2.14)$$

Indeed, using Cauchy's theorem,

$$\eta(v) = \frac{k}{r} \frac{\theta_v^{2k}}{\theta_v^{2r}}, \quad v-1 < \theta_v < v. \quad (2.15)$$

Thus, inequality (2.14) is equivalent to the inequality

$$\frac{k}{r} \frac{\theta_v^{2k}}{\theta_v^{2r}} > \frac{k}{r} \frac{\theta_{v+1}^{2k}}{\theta_{v+1}^{2r}} \quad (2.16)$$

or

$$\left(\frac{\theta_v}{\theta_{v+1}} \right)^{2r-2k} < 1. \quad (2.17)$$

The last inequality is true since $\theta_v < \theta_{v+1}$ and $2r-2k > 0$.

If, for a given $A \geq 0$, the value v_0 is such that $\eta(v_0+1) \leq A \leq \eta(v_0)$, then for $v \leq v_0$, taking into account equality (2.13), we obtain that $\delta_v \geq 0$, and consequently,

$$\varphi(A, 1) \leq \varphi(A, 2) \leq \cdots \leq \varphi(A, v_0). \quad (2.18)$$

In the case $v > v_0$, we get $\delta_v \leq 0$ and then

$$\varphi(A, v_0) \geq \varphi(A, v_0+1) \geq \cdots. \quad (2.19)$$

Therefore,

$$\max_{v \in \mathbb{N}} \varphi(A, v) = \max_{v \in \mathbb{N}} [v^{2k} - Av^{2r}] = \varphi(A, v_0) \quad (2.20)$$

if $\eta(v_0+1) \leq A \leq \eta(v_0)$. Thus inequality (2.1) is proved.

We now show the evidence of equality (2.3). Let $x(t) = \cos v_0 t$. Then the inequality becomes an equality since

$$\|x^{(k)}\|_2^2 = \pi v_0^{2k}, \quad \|x\|_2^2 = \pi, \quad \|x^{(r)}\|_2^2 = \pi v_0^{2r}. \quad (2.21)$$

□

The function $\Psi(\mathbb{T}; r, k; A)$ defined by (2.3) is continuous, linear on any interval $[\eta(v+1), \eta(v)]$, and for any $v \geq 1$,

$$\Psi(\mathbb{T}; r, k; \eta(v+1)) = \frac{v^{2k}(v+1)^{2r} - v^{2r}(v+1)^{2k}}{(v+1)^{2r} - v^{2r}}. \quad (2.22)$$

Claim that

$$v_0^{2k} - Av_0^{2r} < \frac{r-k}{r} \left(\frac{k}{Ar} \right)^{k/(r-k)}. \quad (2.23)$$

To do this, we will consider the function

$$f(A) = \frac{r-k}{r} \left(\frac{k}{Ar} \right)^{k/(r-k)} - v^{2k} + Av^{2r}. \quad (2.24)$$

Differentiating the function f , we get

$$f'(A) = v^{2r} - \left(\frac{k}{r} \right)^{r/(r-k)} \left(\frac{1}{A} \right)^{r/(r-k)} \quad (2.25)$$

and the condition $f'(A) = 0$ implies

$$A_0 = \frac{k}{r} v^{2k-2r}. \quad (2.26)$$

Now we have $f(A_0) = 0$ and our statement is proved.

Let Π_{2n+1} be the set of trigonometric polynomials of order less than or equal to n . Then in view of the Bernstein-type inequality, we have, for any $\tau \in \Pi_{2n+1}$ and any $k \in \mathbb{N}$,

$$\|\tau^{(k)}\|_2^2 \leq n^{2k} \|\tau\|_2^2. \quad (2.27)$$

Therefore, for $x = \tau$, inequality (1.6) holds with $A = 0$ and $B = n^{2k}$. Let now $A > 0$. By repeating (with obvious modifications) the proof of [Theorem 2.1](#), we obtain that for any $k, r \in \mathbb{N}$, $k < r$, and any $\tau \in \Pi_{2n+1}$, the following holds:

$$\|\tau^{(k)}\|_2^2 \leq A \|\tau^{(r)}\|_2^2 + B \|\tau\|_2^2 = A \|\tau^{(r)}\|_2^2 + \max_{\substack{v \in \mathbb{N} \\ v \leq n}} \varphi(A, v) \|\tau\|_2^2. \quad (2.28)$$

We now compute the value

$$\max_{\substack{v \in \mathbb{N} \\ v \leq n}} \varphi(A, v). \quad (2.29)$$

Let $\eta(v_0 + 1) \leq A \leq \eta(v_0)$, where $v_0 \leq n$. Then

$$\max_{\substack{v \in \mathbb{N} \\ v \leq n}} \varphi(A, v) = \varphi(A, v_0) = \max_{v \in \mathbb{N}} \varphi(A, v). \quad (2.30)$$

If $\eta(v_0 + 1) \leq A \leq \eta(v_0)$, where $v_0 \geq n + 1$, we get, taking into account the relations

$$\varphi(A, 1) \leq \varphi(A, 2) \leq \dots \leq \varphi(A, n) \leq \dots \leq \varphi(A, v_0), \quad (2.31)$$

that

$$\max_{\substack{v \in \mathbb{N} \\ v \leq n}} \varphi(A, v) = \varphi(A, n) = n^{2k} - An^{2r} \quad (2.32)$$

if $A \leq \eta(n)$. Therefore, we have proved the following theorem.

THEOREM 2.2. For any $k, n, r \in \mathbb{N}$, $k < r$, any $\tau \in \Pi_{2n+1}$, and any $A \geq 0$,

$$\|\tau^{(k)}\|_2^2 \leq A \|\tau^{(r)}\|_2^2 + B \|\tau\|_2^2, \quad (2.33)$$

where

$$B = \varphi(A, v_0) \quad (2.34)$$

if $\eta(v_0 + 1) \leq A \leq \eta(v_0)$, $v_0 \leq n$, and

$$B = \varphi(A, n) \quad (2.35)$$

if $A \leq \eta(n)$. Inequality (2.33) is the best possible for any $A \geq 0$.

Consider the set of functions $x \in L_{2,2}^r(\mathbb{T})$ such that $c_v(x) = 0$ for $|v| \leq n-1$ (we will denote this set of functions by $L_{2,2}^r(\mathbb{T}; n)$). The following inequality is well known for functions $x \in L_{2,2}^r(\mathbb{T}; n)$:

$$\|x\|_2^2 \leq \frac{1}{n^{2r}} \|x^{(r)}\|_2^2. \quad (2.36)$$

Thus, for any $k < r$,

$$\|x^{(k)}\|_2^2 \leq \frac{1}{n^{2r-2k}} \|x^{(r)}\|_2^2. \quad (2.37)$$

Then inequality (1.6) for functions $x \in L_{2,2}^r(\mathbb{T}; n)$ holds with $B = 0$ and $A \geq 1/n^{2r-2k}$.

By repeating (with obvious modifications) the proof of Theorem 2.1, we obtain that for any $k, r \in \mathbb{N}$, $k < r$, any $x \in L_{2,2}^r(\mathbb{T}; n)$, and any $0 \leq A \leq 1/n^{2r-2k}$,

$$\|x^{(k)}\|_2^2 \leq A \|x^{(r)}\|_2^2 + \max_{\substack{v \in \mathbb{N} \\ v \geq n}} \varphi(A, v) \|x\|_2^2. \quad (2.38)$$

We need to find the value of

$$\max_{\substack{v \in \mathbb{N} \\ v \geq n}} \varphi(A, v). \quad (2.39)$$

Note that

$$\eta(n) = \frac{n^{2k} - (n-1)^{2k}}{n^{2r} - (n-1)^{2r}} \leq \frac{n^{2k}}{n^{2r}}. \quad (2.40)$$

To show this, assume that

$$\eta(n) > \frac{n^{2k}}{n^{2r}}, \quad (2.41)$$

then we get

$$\left(\frac{n}{n-1}\right)^{2r} < \left(\frac{n}{n-1}\right)^{2k} \quad (2.42)$$

which is impossible since $n/(n-1) > 1$ and $k < r$.

First let $\eta(v_0 + 1) \leq A \leq \eta(v_0)$ where $v_0 \leq n$. Then

$$\varphi(A, n) \geq \varphi(A, n+1) \geq \dots \quad (2.43)$$

and therefore

$$\max_{\substack{v \in \mathbb{N} \\ v \geq n}} \varphi(A, v) = \varphi(A, n) \quad (2.44)$$

if $\eta(v_0 + 1) \leq A \leq n^{2k-2r}$.

Let now $\eta(v_0 + 1) \leq A \leq \eta(v_0)$ where $v_0 \geq n + 1$. In this case, we get

$$\max_{\substack{v \in \mathbb{N} \\ v \geq n}} \varphi(A, v) = \max_{v \in \mathbb{N}} \varphi(A, v) = \varphi(A, v_0). \quad (2.45)$$

Thus we have proved the following theorem.

THEOREM 2.3. *For any $k, n, r \in \mathbb{N}$, $k < r$, any $x \in L_{2,2}^r(\mathbb{T}; n)$, and any $0 \leq A \leq n^{2k-2r}$, inequality (1.6) holds where $B = \varphi(A, n)$ if $\eta(v_0 + 1) \leq A \leq n^{2k-2r}$, $v_0 \leq n$, and $B = \varphi(A, v_0)$ if $\eta(v_0 + 1) \leq A \leq \eta(v_0)$, $v_0 \geq n + 1$. Inequality (1.6) is the best possible for any $0 \leq A \leq n^{2k-2r}$.*

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