EDGE-DISJOINT HAMILTONIAN CYCLES IN TWO-DIMENSIONAL TORUS

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The torus is one of the popular topologies for the interconnecting processors to build high-performance multicomputers. This paper presents methods to generate edge-disjoint Hamiltonian cycles in 2D tori.

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1. Introduction. A multicomputer system consists of multiple nodes that communicate by exchanging messages through an interconnection network. At a minimum, each node normally has one or more processing elements, a local memory, and a communication module. A popular topology for the interconnection network is the *torus*. Also called a *wrap-around mesh* or a *toroidal mesh*, this topology includes the k -ary n -cube which is an n -dimensional torus with the restriction that each dimension is of the same size, k , and the hypercube, which is a k -ary n -cube with $k = 2$; a mesh is a subgraph of a torus.

Several parallel machines, both commercial and experimental, have been designed with a toroidal interconnection network. Included among these machines are the following: the iWarp (torus) [5], Cray T3D and T3E (3D torus) [13], the Mosaic (k -ary n -cube) [14], and the Tera parallel computer (torus) [2].

Some topological properties of torus and k -ary n -cubes based on Lee distance are given in [6, 7]. The existence of disjoint Hamiltonian cycles in the cross-product of various graphs has been discussed in [1, 4, 8, 9, 10, 11, 15]; however, a straightforward way of generating such cycles was not known until the results in [3], where some simple ways of generating edge-disjoint Hamiltonian cycles in k -ary n -cubes are presented. In this paper, some simple solutions to this problem are described for 2D torus. For example, Figure 1.1 gives two edge-disjoint cycles in $C_3 \times C_4$.

The rest of the paper is organized as follows. Section 2 gives some preliminaries about the definition of torus. Section 3 discusses the results on edge-disjoint Hamiltonian cycles on the 2D torus. Section 4 is the conclusion of this paper.

2. Preliminaries. This section contains definitions and mathematical background that will be useful in subsequent sections.

2.1. Lee distance, cross-product, and torus. Let $A = a_{n-1}a_{n-2} \cdots a_0$ be an n -dimensional mixed-radix vector over Z_K , where $K = k_{n-1} \times k_{n-2} \times \cdots \times k_0$, that is, all $a_i \in Z_{k_i}$,

FIGURE 1.1. Two disjoint Hamiltonian cycles in $C_3 \times C_4$.

for $i = 0, 1, \dots, n-1$. The Lee weight of A in mixed-radix notation is defined as

$$W_L(A) = \sum_{i=0}^{n-1} |a_i|, \quad (2.1)$$

where $|a_i| = \min(a_i, k_i - a_i)$, for $i = 0, 1, \dots, n-1$.

The Lee distance between the two vectors A and B is denoted by $D_L(A, B)$ and is defined to be $W_L(A - B)$. That is, the Lee distance between the two vectors is the Lee weight of their digitwise difference. In other words, $D_L(A, B) = \sum_{i=0}^{n-1} \min(a_i - b_i, b_i - a_i)$, where $a_i - b_i$ and $b_i - a_i$ are mod k_i operations. For example, when $K = 4 \times 6 \times 3$, $W_L(321) = \min(3, 4-3) + \min(2, 6-2) + \min(1, 3-1) = 1 + 2 + 1 = 4$, and $D_L(123, 321) = W_L(123 - 321) = W_L(202) = 3$.

A k -ary n -cube graph (C_k^n) and an n -dimensional torus ($T_{k_1, k_2, \dots, k_n}$) are $2n$ -regular graphs containing k^n and $k_1 k_2 \cdots k_n$ nodes, respectively; it is assumed that $k \geq 3$ and $k_i \geq 3$ for $i = 1, 2, \dots, n$. Each node in a C_k^n is labeled with a distinct n -digit radix- k vector while each node in a $T_{k_1, k_2, \dots, k_n}$ is labeled with a distinct n -digit mixed-radix vector. If u and v are two nodes in the graph, then there is an edge between them *if and only if* $D_L(u, v) = 1$. From the definition of Lee distance, it can be seen that every node in a C_k^n or a $T_{k_1, k_2, \dots, k_n}$ shares an edge with two nodes in every dimension, resulting in a regular graph of degree $2n$.

Since the Hamming distance, $D_H(A, B)$, between the two vectors A and B is the number of positions in which A and B differ, $D_L(A, B) = D_H(A, B)$ when $k_i = 2$ or 3 , for all i , and $D_L(A, B) \geq D_H(A, B)$ when some $k_i > 3$.

The k -ary n -cube and the torus can also be seen as the cross-product of cycles. The *cross-product* of G_1 and G_2 , denoted by $G = G_1 \otimes G_2$, is defined as follows [6, 12]:

$$\begin{aligned} V &= \{(u, v) \mid u \in V_1, v \in V_2\}, \\ E &= \{((u_1, v_1), (u_2, v_2)) \mid ((u_1, u_2) \in E_1 \text{ and } v_1 = v_2) \text{ or } (u_1 = u_2 \text{ and } (v_1, v_2) \in E_2)\}, \end{aligned} \quad (2.2)$$

where $G = (V, E)$, $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$. A cycle of length k is denoted by C_k , and each node in C_k is labeled with a radix k number, $0, \dots, k-1$. There is an edge between vertices u and v *if and only if* $D_L(u, v) = 1$. Thus, a k -ary n -cube (C_k^n) and an n -dimensional torus ($T_{k_1, k_2, \dots, k_n}$) can be defined as a product of cycles as follows:

$$\begin{aligned} C_k^n &= \underbrace{C_k \otimes C_k \otimes \cdots \otimes C_k}_{n \text{ times}} = \otimes_{i=1}^n C_k, \\ T_{k_1, k_2, \dots, k_n} &= C_{k_1} \otimes C_{k_2} \otimes \cdots \otimes C_{k_n}. \end{aligned} \quad (2.3)$$

FIGURE 3.1. Basic mappings.

3. Edge-disjoint Hamiltonian cycles in a 2D torus. When edge-disjoint Hamiltonian cycles are used in a communication algorithm, their effectiveness is improved if more than one cycle exists. As mentioned earlier, the existence of disjoint Hamiltonian cycles in the cross-product of various graphs has been discussed in the literature [1, 4, 8, 9, 10, 11, 15]; however, all these methods do not give a straightforward way of generating such disjoint cycles. This section contains the functions that generate these disjoint cycles for the 2D torus.

Two Gray codes, G_1 and G_2 , over Z_k^n are said to be *independent* if two words, a and b , are adjacent in G_1 (or G_2), then they are not adjacent in G_2 (or G_1). If $k \geq 3$, we can have at most n sets of independent Gray codes; for all $k_i = 2$, this number is $\lfloor n/2 \rfloor$. Note that the independent Gray codes form the edge-disjoint cycles.

In order to develop generating functions for 2D torus, we first define several basic mappings: *spiral* and *maze*, and some basic vector operations: *reverse*, *mod*, and *translate*.

Using these basic building blocks, we first show the generating functions of the edge-disjoint Hamiltonian cycles in 2D torus (T_{k_1, k_0}) whose sides are either both even or both odd, and then extend the result to the general 2D torus. This 2D Hamiltonian decomposition can also be used as a basis for 3D Hamiltonian decomposition.

Let X be an integer, $x_1 = \lfloor X/k \rfloor$, and $x_0 = X \bmod k$.

BASIC OPERATIONS.

- (i) $\text{reverse}((x_1, x_0)) = (x_1, x_0)^R = (x_0, x_1)$.
- (ii) $(x_1, x_0) \bmod (k_1, k_0) = (x_1 \bmod k_1, x_0 \bmod k_0)$.
- (iii) $\text{translate}(+) : (x_1, x_0) + (d_1, d_0) = (x_1 + d_1, x_0 + d_0)$.
- (iv) $\text{translate}(-) : (x_1, x_0) - (d_1, d_0) = (x_1 - d_1, x_0 - d_0)$.

SPIRAL MAPPING. Figure 3.1(a) is the graphical view of this mapping. The cycle is produced by generating $G_s(X; k)$ for successive values of X starting at $X = 0$:

$$\begin{aligned} G_s(X; k) &= G_s((x_1, x_0); k) = (x_1, (x_0 - x_1) \bmod k), \\ G_s^{-1}((y_1, y_0); k) &= (y_1, (y_1 + y_0) \bmod k). \end{aligned} \quad (3.1)$$

MAZE MAPPING. Figure 3.1(b) is the graphical view of this mapping. The cycle is produced by generating $G_m(X; k)$ for successive values of X starting at $X = 0$:

$$\begin{aligned} G_m(X; k) &= G_m((x_1, x_0); k) = \begin{cases} (x_1, x_0), & \text{if } x_1 \text{ is even,} \\ (x_1, k-1-x_0), & \text{if } x_1 \text{ is odd,} \end{cases} \\ G_m^{-1}(Y; k) &= G_m^{-1}((y_1, y_0); k) = \begin{cases} (y_1, y_0), & \text{if } y_1 \text{ is even,} \\ (y_1, k-1-y_0), & \text{if } y_1 \text{ is odd.} \end{cases} \end{aligned} \quad (3.2)$$

MAZE WITH FEEDBACK. Figure 3.1(c) is the graphical view of this mapping. The cycle is produced by generating $G_{mf}(X; k)$ for successive values of X starting at $X = 0$:

$$G_{mf}(X; k_1, k_0) = \begin{cases} G_m(X; k_0 - 1), & \text{if } x < k_1 k_0 - k_1, \\ -X, & \text{if } k_1 k_0 - k_1 \leq x. \end{cases} \quad (3.3)$$

3.1. Special case 1: $k_1 = k_0 = k$. In a $T_{k,k}$, the Hamiltonian cycles can be described as follows:

$$\begin{aligned} h_0(X, T_{k,k}) &= (x_1, x_0 - x_1) \bmod(k, k) = G_s(X, k), \\ h_1(X, T_{k,k}) &= h_0^R(X, T_{k,k}) = (x_0 - x_1, x_1) \bmod(k, k), \end{aligned} \quad (3.4)$$

where $x_1 = \lfloor X/k \rfloor$ and $x_0 = X \bmod k$.

3.2. Special case 2: $k_1 = mk_0$ and $\text{GCD}(k_1, k_0 - 1) = 1$

THEOREM 3.1. In a mixed-radix number system $Z_{k_1 \times k_0}$, if $\text{GCD}(k_1, k_0 - 1) = 1$ and $k_1 = mk_0$, for $m \geq 1$, then the following two functions generate the independent Gray codes:

$$\begin{aligned} f_0(X; k_1, k_0) &= (x_1 \bmod k_1, (x_0 + (k_0 - 1)x_1) \bmod k_0), \\ f_1(X; k_1, k_0) &= ((x_0 + (k_0 - 1)x_1) \bmod k_1, x_1 \bmod k_0), \end{aligned} \quad (3.5)$$

where $X = (x_1, x_0)$, $x_1 \in Z_{k_1}$, and $x_0 \in Z_{k_0}$.

PROOF. The proof has three parts.

(1) If $X' \neq X''$, then it is required to prove that $f_0(X'; k_1, k_0) \neq f_0(X''; k_1, k_0)$ and $f_1(X'; k_1, k_0) \neq f_1(X''; k_1, k_0)$. Let $X' = (x'_1, x'_0)$, $X'' = (x''_1, x''_0)$, $x'_1, x''_1 \in Z_{k_1}$, and $x'_0, x''_0 \in Z_{k_0}$.

(a) Suppose $f_0(X'; k_1, k_0) = f_0(X''; k_1, k_0)$. Since $x'_1 = x''_1 \bmod k_1$ and $x'_1, x''_1 \in Z_{k_1}$, $x'_1 = x''_1$. For the second component, $x'_0 + (k_0 - 1)x'_1 = x''_0 + (k_0 - 1)x''_1 \bmod k_0$, and hence $x'_0 = x''_0$. Thus $f_0(X'; k_1, k_0) \neq f_0(X''; k_1, k_0)$ if $X' \neq X''$.

(b) Suppose $f_1(X'; k_1, k_0) = f_1(X''; k_1, k_0)$, $x'_1 = x''_1 \bmod k_0$, that is, $k_0 | (x'_1 - x''_1)$, and $x'_0 + (k_0 - 1)x'_1 = (x''_0 + (k_0 - 1)x''_1) \bmod k_1$, that is, $x'_0 - x''_0 + (k_0 - 1)(x'_1 - x''_1) = 0 \bmod k_1$. Since $|x'_0 - x''_0| < k_0$ and $k_0 | (x'_1 - x''_1)$, $x'_0 - x''_0 = 0 \bmod k_0$. Further, $x'_1 - x''_1 = 0 \bmod k_1$ because $\text{GCD}(k_0 - 1, k_1) = 1$. Thus $f_1(X'; k_1, k_0) \neq f_1(X''; k_1, k_0)$ if $X' \neq X''$.

This implies that f_0 and f_1 are one-to-one mappings over Z_k^2 .

(2) f_0 and f_1 generate cycles H_0 and H_1 , respectively. In other words, the mappings of two numbers X and $X+1$ by f_0 or f_1 must generate an edge in H_0 or H_1 . There would be the following subcases.

- (a) Case $X = (x_1, x_0)$ and $X+1 = (x_1, x_0+1)$. Since $f_0(X) = (x_1, x_0 + (k_0-1)x_1)$ and $f_1(X+1) = (x_1, x_0+1 + (k_0-1)x_1)$, we have $e_0 = (f_0(X), f_0(X+1)) = ((x_1, x_0 + (k_0-1)x_1), (x_1, x_0+1 + (k_0-1)x_1))$ and $D_L(f_0(X), f_0(X+1)) = 1$. Thus e_0 is an edge of H_0 .
- (b) Case $X' = (x'_1, k_0-1)$ and $X'+1 = (x'_1+1, 0)$. Similar to case (a), we have $e_1 = ((x'_1, (k_0-1)(x'_1+1)), (x'_1+1, (k_0-1)(x'_1+1)))$ and $D_L(f_0(X'), f_0(X'+1)) = 1$. Thus e_1 is an edge of H_0 .
- (c) Case $X'' = (x''_1, x''_0)$ and $X''+1 = (x''_1, x''_0+1)$. Similarly, we have $e_2 = (f_1(X''), f_1(X''+1)) = ((x''_0 + (k_0-1)x''_1, x''_1), (x''_0+1 + (k_0-1)x''_1, x''_1))$ and $D_L(f_1(X''), f_1(X''+1)) = 1$. Thus e_2 is an edge of H_1 .
- (d) Case $X''' = (x'''_1, k_0-1)$ and $X''' + 1 = (x'''_1+1, 0)$. We have $e_3 = (f_1(X'''), f_1(X''' + 1)) = (((k_0-1)(x'''_1+1), x'''_1), ((k_0-1)(x'''_1+1), (x'''_1+1)))$ and $D_L(f_1(X'''), f_1(X''' + 1)) = 1$. Thus e_3 is an edge of H_1 .

Since f_0 is one-to-one and $D_L(f_0(X), f_0(X+1)) = 1$, f_0 generates a Hamiltonian cycle. Similarly, f_1 also generates another Hamiltonian cycle.

(3) The edges which are generated by f_0 and f_1 must be unique so that the cycles, H_0 and H_1 , become edge-disjoint. In other words, the edges e_0 , e_1 , e_2 , and e_3 (described in the above case (2)) must be different. The proof is by contradiction. Suppose that e_0 (in H_0) is the same as e_2 (in H_1). For that, one of (3.6a) or (3.6b) must hold:

$$\begin{aligned} x_1 &= x''_0 + (k_0-1)x''_1, \\ x_0 + (k_0-1)x_1 &= x''_1, \\ x_1 &= x''_0 + 1 + (k_0-1)x''_1, \\ x_0 + 1 + (k_0-1)x_1 &= x''_1, \end{aligned} \tag{3.6a}$$

$$\begin{aligned} x_1 &= x''_0 + 1 + (k_0-1)x''_1, \\ x_0 + (k_0-1)x_1 &= x''_1, \\ x_1 &= x''_0 + (k_0-1)x''_1, \\ x_0 + 1 + (k_0-1)x_1 &= x''_1. \end{aligned} \tag{3.6b}$$

However, either (3.6a) or (3.6b) cannot be true. Similarly, one of e_0 and e_1 (in H_0) cannot be the same as one of e_2 and e_3 (in H_1). Therefore H_0 and H_1 are edge-disjoint. $\square$

The inverses of f_0 and f_1 are as follows:

$$\begin{aligned} f_0^{-1}(Y; k_1, k_0) &= (y_1 \bmod k_1, (y_0 - (k_0-1)y_1) \bmod k_0), \\ f_1^{-1}(Y; k_1, k_0) &= ((y_0 - (k_0-1)y_1) \bmod k_1, y_1 \bmod k_0), \end{aligned} \tag{3.7}$$

where $Y = (y_1, y_0)$, $y_1 \in Z_{k_1}$, and $y_0 \in Z_{k_0}$.

FIGURE 3.2. Edge-disjoint Hamiltonian cycles in $T_{9,3}$ produced by f_0 and f_1 of Theorem 3.1.

COROLLARY 3.2. *There are two independent Gray codes in $T_{k^r,k}$ for $k \geq 3$ and $r \geq 1$ and are generated by the functions h_0 and h_1 , where*

$$\begin{aligned} h_0(x_1, x_0) &= (a_1, a_0) = (x_1, (x_0 - x_1) \bmod k), \\ h_1(x_1, x_0) &= (b_1, b_0) = ((x_0 + (k-1)x_1) \bmod k^r, x_1 \bmod k). \end{aligned} \quad (3.8)$$

The inverse functions are given by

$$\begin{aligned} h_0^{-1}(a_1, a_0) &= (x_1, x_0) = (a_1, (a_1 + a_0) \bmod k), \\ h_1^{-1}(b_1, b_0) &= (x_1, x_0) = ((b_1 - x_0)(k-1)^{-1} \bmod k^r, (b_1 - b_0(k-1)) \bmod k) \\ &= ((b_1 - x_0)(k-1)^{-1} \bmod k^r, (b_1 + b_0) \bmod k) \\ &= ((b_1 - ((b_1 + b_0) \bmod k))(k-1)^{-1} \bmod k^r, (b_1 + b_0) \bmod k), \end{aligned} \quad (3.9)$$

where $(k-1)^{-1}$ is the multiplicative inverse of $(k-1)$ under $\bmod k^r$ (note that for $k \geq 3$, $k-1$ and k^r are relatively prime and so the inverse exists).

EXAMPLE 3.3. Figure 3.2 shows the two edge-disjoint Hamiltonian cycles in $T_{9 \times 3}$ produced by f_0 and f_1 of Theorem 3.1.

3.3. $k_1 = k + 2r$ and $k_0 = k + 2s$. Without loss of generality, we assume $k_1 = k + 2r$ and $k_0 = k + 2s$ for some $k \geq 3$, $r \geq 0$, and $s \geq 0$.

DEFINITION 3.4. If $k_1 = k + 2r$ and $k_0 = k + 2s$ for some $k, r \geq 0$, and $s \geq 0$, define a function $h_0(X; T_{k_1, k_0})$ as follows:

$$\begin{aligned} h'_0(X; T_{k_1, k_0}) &= \begin{cases} G_s(X + 2s; k_0) - (0, 2s), & \text{if } 0 \leq X < p_\alpha, \\ G_m^R(X - p_\alpha; k_1 - k + 2) + (k-1, k), & \text{if } p_\alpha \leq X < p_\beta, \\ G_m(X - p_\beta; k) + (k, 0), & \text{if } p_\beta \leq X, \end{cases} \\ h_0(X; T_{k_1, k_0}) &= h'_0(X; T_{k_1, k_0}) \bmod (k_1, k_0), \end{aligned} \quad (3.10)$$

where $p_\alpha = (k-2)k_0 + 2k - 1$ and $p_\beta = k_1k_0 - k(k_1 - k)$.

Note that $h_0(p_\alpha; T_{k_1, k_0}) = (k-1, k \bmod k_0)$ and $h_0(p_\beta; T_{k_1, k_0}) = (k \bmod k_1, 0)$.

THEOREM 3.5. *The function $h_0(X; T_{k_1, k_0})$ generates a Hamiltonian cycle, $H_0(T_{k_1, k_0})$, in a 2D torus (T_{k_1, k_0}) .*

PROOF. The proof has two parts.

(1) If $X \neq X'$, then $h_0(X; T_{k_1, k_0}) \neq h_0(X'; T_{k_1, k_0})$.

Assume $Y = (y_1, y_0) = h_0(X; T_{k_1, k_0})$. By [Definition 3.4](#), the range of Y can be found from the range of X . If these ranges are disjoint, then the claim will be true:

$$\begin{aligned} R_1 &= \{h_0(X) \mid 0 \leq X < p_\alpha\} \\ &= \{(y_1, y_0) \mid y_1 = 0, 0 \leq y_0 < k\} \cup \{(y_1, y_0) \mid 0 < y_1 < k-1, 0 \leq y_0 < k_0\} \\ &\quad \cup \{(y_1, y_0) \mid y_1 = k-1, 1 \leq y_0 < k\}, \\ R_2 &= \{h_0(X) \mid p_\alpha \leq X < p_\beta\} \\ &= \{(y_1, y_0) \mid y_1 = 0, k \leq y_0 < k_0\} \cup \{(y_1, y_0) \mid k-1 \leq y_1 < k_1, k \leq y_0 < k_0\}, \\ R_3 &= \{h_0(X) \mid p_\beta \leq X\} = \{(y_1, y_0) \mid k \leq y_1 < k_1, 0 \leq y_0 < k\}. \end{aligned} \quad (3.11)$$

Since R_1, R_2 , and R_3 are mutually exclusive, the claim is true.

(2) $D_L(h_0(X; T_{k_1, k_0}), h_0(X'; T_{k_1, k_0})) = 1$ if $X = X' + 1$. In each subrange, the proof is trivial. The only case that needs to be considered is the situation where there are transitions from one subrange to another subrange. $h_0(p_\alpha - 1; T_{k_1, k_0}) = (k-1, k-1)$, $h_0(p_\alpha; T_{k_1, k_0}) = (k-1, k)$, $h_0(p_\beta - 1; T_{k_1, k_0}) = (k-1, 0)$, and $h_0(p_\beta; T_{k_1, k_0}) = (k, 0)$. $\square$

COROLLARY 3.6 (inverse of $h_0(X; T_{k_1, k_0})$). $h_0^{-1}((y_1, y_0); T_{k_1, k_0})$ is described as follows:

$$h_0^{-1}(Y; T_{k_1, k_0}) = \begin{cases} G_s^{-1}(Y + (0, 2s); k_0) - 2s, & \text{if } Y \in R_1, \\ G_m^{-1}((Y - (k-1, k))^R; k_1 - k + 2) + p_\alpha, & \text{if } Y \in R_2, \\ G_m^{-1}(Y - (k, 0); k) + p_\beta, & \text{if } Y \in R_3. \end{cases} \quad (3.12)$$

THEOREM 3.7. *If $H_0(T_{k_1, k_0})$ and $H_1(T_{k_1, k_0})$ are the edge-disjoint Hamiltonian cycles, and $H_0(T_{k_1, k_0})$ is generated by the function $h_0(X; T_{k_1, k_0})$, then the generator function $h_1(X; T_{k_1, k_0})$ for $H_1(T_{k_1, k_0})$ is defined as*

$$h_1(X; T_{k_1, k_0}) = h_0^R(X; T_{k_0, k_1}). \quad (3.13)$$

PROOF. We prove the theorem by induction on k_1 and k_0 .

BASE STEP. Consider $T_{k, k}$. We get $h_0(X; T_{k, k}) = G_s(X; k)$ and $h_1^R(X; T_{k, k}) = G_s^R(X; k)$. Thus $H_0(T_{k, k})$ and $H_1(T_{k, k})$ are edge-disjoint.

INDUCTIVE STEP. Assume that a T_{k_1, k_0} , where $k_1 = k + 2r$ and $k_0 = k + 2s$, has two edge-disjoint Hamiltonian cycles generated by $h_0(X; T_{k_1, k_0})$ and $h_1(X; T_{k_1, k_0})$.

CASE 1. $k'_1 = k_1 + 2$: [Figure 3.3](#) illustrates the process of adding two rows to a $T_{3,7}$.

CASE 2. $k'_0 = k_0 + 2$: similar to the previous case. $\square$

EXAMPLE 3.8. [Figure 3.3](#) shows two edge-disjoint Hamiltonian cycles in $T_{5,7}$ produced by [Theorem 3.7](#).

FIGURE 3.3. H_0 and H_1 in $T_{5,7}$ ($k = 3$). FIGURE 3.4. H_0 and H_1 in $T_{6,7}$ ($k = 3$).

3.4. $k_1 = 4 + 2r$ and $k_0 = 3 + 2s$. Without loss of generality, we assume $k_1 = 4 + 2r$ and $k_0 = 3 + 2s$. We treat T_{k_1, k_0} as a torus obtained after inserting a row to T_{k_1-1, k_0} . For example, $T_{6,7}$ is the torus obtained from $T_{3,3}$ after inserting rows and columns as in Figure 3.4.

COROLLARY 3.9. $H_0(T_{k_1, k_0})$ from $H_0(T_{k_1-1, k_0})$: by inserting a row in between the third and the fourth row of T_{k_1-1, k_0} , the function $h_0(X; T_{k_1, k_0})$ which generates a Hamiltonian cycle, $H_0(T_{k_1, k_0})$, can be described as follows:

$$h'_0(X; T_{k_1, k_0}) = \begin{cases} G_s(X + 2s; k_0) - (0, 2s), & \text{if } 0 \leq X < k_0 + 3, \\ G_m^R(X - k_0 - 3; 2) + (2, 1), & \text{if } k_0 + 3 \leq X < k_0 + 7, \\ G_m^R(X - k_0 - 7; k_1 - 1) + (2, 3), & \text{if } k_0 + 7 \leq X < p_\beta, \\ G_m(X - p_\beta; 3) + (4, 0), & \text{if } p_\beta \leq X, \end{cases} \quad (3.14)$$

$$h_0(X; T_{k_1, k_0}) = h'_0(X; T_{k_1, k_0}) \bmod (k_1, k_0),$$

where $p_\beta = k_1 k_0 - 3(k_1 - 4)$.

Note that $h_0(p_\gamma; T_{k_1, k_0}) = (1, 0)$. Further note that $p_\alpha = p_\beta$ if $k_0 = 3$, and $0 = p_\beta$ if $k_1 = 4$, and we get

$$\begin{aligned} h_0(0, T_{k_1, k_0}) &= (0, 0), \\ h_0(p_\alpha, T_{k_1, k_0}) &= (4 \bmod k_1, 3 \bmod k_0), \\ h_0(p_\beta, T_{k_1, k_0}) &= (4 \bmod k_1, 0). \end{aligned} \quad (3.15)$$

EXAMPLE 3.10. Figure 3.4 shows two edge-disjoint Hamiltonian cycles in $T_{6,7}$ produced from $T_{5,7}$ by the above corollaries.

COROLLARY 3.11. $H_1(T_{k_1, k_0})$ from $H_1(T_{k_1-1, k_0})$: the second function $h_1(X; T_{k_1, k_0})$, where k_1 is even and k_0 is odd, is as follows:

$$h'_1(X; T_{k_1, k_0}) = \begin{cases} G_s^R(X; 3), & \text{if } 0 \leq X < 4, \\ G_s^R(p_\alpha - 1 - X; k_1) + (3, 1), & \text{if } 4 \leq X < p_\alpha, \\ G_m(X - p_\alpha + k_0 - 1; k_0 - 1) + (2, 2), & \text{if } p_\alpha \leq X < p_\beta, \\ G_m^R(X - p_\beta; 3) + (0, 3), & \text{if } p_\beta \leq X, \end{cases} \quad (3.16)$$

$$h_1(X; T_{k_1, k_0}) = h'_1(X; T_{k_1, k_0}) \bmod (k_1, k_0),$$

where $p_\alpha = k_1 + 5$, $p_\beta = k_1 k_0 - 3(k_0 - 3)$.

Note that $p_\alpha = p_\beta$ if $k_0 = 4$, and $0 = p_\beta$ if $k_1 = 3$, and we get

$$\begin{aligned} h_0(0, T_{k_1, k_0}) &= (0, 0), \\ h_0(p_\alpha, T_{k_1, k_0}) &= (3 \bmod k_1, 4 \bmod k_0), \\ h_0(p_\beta, T_{k_1, k_0}) &= (3 \bmod k_1, 0). \end{aligned} \quad (3.17)$$

COROLLARY 3.12. $h_0(X, T_{k_1, k_0})$ and $h_1(X, T_{k_1, k_0})$, by Corollaries 3.9 and 3.11, generate two edge-disjoint Hamiltonian cycles in T_{k_1, k_0} , where k_1 is even and k_0 is odd.

Similar to Corollary 3.6, we can obtain the inverses of $h_0(X, T_{k_1, k_0})$ and $h_1(X, T_{k_1, k_0})$.

4. Conclusion. In this paper, we present methods to generate the edge-disjoint Hamiltonian cycles in 2D torus. These methods can be used to generate edge-disjoint Hamiltonian cycles in higher-dimensional torus networks. For example, consider a 4D torus $T = (C_{k_1} \otimes C_{k_2} \otimes C_{k_3} \otimes C_{k_4})$. This can be decomposed as $T = (H_1 \oplus H_2) \otimes (H_3 \oplus H_4)$, where H_0 and H_1 are disjoint cycles obtained from $(C_{k_1} \otimes C_{k_2})$; so also H_3 and H_4 from $(C_{k_3} \otimes C_{k_4})$. Then, T can be written as

$$T = (H_1 \otimes H_3) \oplus (H_2 \otimes H_4) = H'_1 \oplus H'_2 \oplus H'_3 \oplus H'_4. \quad (4.1)$$

All these four cycles (H'_i s) are disjoint and of length $(k_1 \times k_2 \times k_3 \times k_4)$. Some simple functions to generate these cycles need further research.

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