CONVERGENCY OF THE FUZZY VECTORS IN THE PSEUDO-FUZZY VECTOR SPACE OVER $F^1_p(1)$

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In 2003, we considered the pseudo-fuzzy vector space SFR over $F^1_p(1)$. Here, we further discuss the convergency of the fuzzy vectors in SFR.

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1. Introduction. In this paper, we discuss the convergency of the fuzzy space over $F^1_p(1)$ (see [4]). In [4, Section 2], we stated the pseudo-fuzzy vector space SFR over $F^1_p(1)$ as follows: for two points $P = (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ and $Q = (y^{(1)}, y^{(2)}, \ldots, y^{(n)})$ on $\mathbb{R}^n$, we have the crisp vector $\overrightarrow{PQ} = (y^{(1)} - x^{(1)}, y^{(2)} - x^{(2)}, \ldots, y^{(n)} - x^{(n)})$ in a pseudo-fuzzy vector space $F^n_p(1) = \{ (a^{(1)}, a^{(2)}, \ldots, a^{(n)}) | \forall (a^{(1)}, a^{(2)}, \ldots, a^{(n)}) \in \mathbb{R}^n \}$.

There is a one-to-one onto mapping $p$ such that $\overrightarrow{PQ} = (x^{(1)} - x^{(1)}, y^{(2)} - x^{(2)}, \ldots, y^{(n)} - x^{(n)})$, therefore, we can define the fuzzy vector $\tilde{PQ} = (\tilde{y}^{(1)} - \tilde{x}^{(1)}, \tilde{y}^{(2)} - \tilde{x}^{(2)}, \ldots, \tilde{y}^{(n)} - \tilde{x}^{(n)})$.

1. In Section 3, we will discuss the convergency of the fuzzy vectors in SFR.

2. Preparation. In [4], we discussed the pseudo-fuzzy vector space SFR over $F^1_p(1)$.

1. In order to discuss the convergency of the fuzzy vectors in SFR, we need to know some definitions.

**Definition 2.1.** (1) The fuzzy set $\tilde{A}$ on $\mathbb{R} = (-\infty, \infty)$ is convex if and only if every ordinary set $A(\alpha) = \{ x | \mu_{\tilde{A}} (x) \geq \alpha \ \forall \alpha \in [0, 1] \}$ is convex, and hence $A(\alpha)$ is a closed interval of $\mathbb{R}$.

(2) The fuzzy set $\tilde{A}$ on $\mathbb{R}$ is normal if and only if $\bigvee_{x \in \mathbb{R}} \mu_{\tilde{A}} (x) = 1$.

Next, we extend this definition to $\mathbb{R}^n$ by saying that the membership function of the fuzzy set $\tilde{D}$ on $\mathbb{R}^n$ is $\mu_{\tilde{D}} (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in [0, 1]$ for all $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n$.

**Definition 2.2.** The $\alpha$-cut $(0 \leq \alpha \leq 1)$ of the fuzzy set $\tilde{D}$ on $\mathbb{R}^n$ is defined by $D(\alpha) = \{ (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) | \mu_{\tilde{D}} (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \geq \alpha \}$.

**Definition 2.3.** (1) The fuzzy set $\tilde{D}$ on $\mathbb{R}^n$ is convex if and only if every ordinary set $D(\alpha) = \{ (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) | \mu_{\tilde{D}} (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \geq \alpha \ \forall \alpha \in [0, 1] \}$ is a convex closed subset of $\mathbb{R}^n$. 
(2°) The fuzzy set \( \tilde{D} \) is normal if and only if
\[ \bigvee_{(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n} \mu_{\tilde{D}}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = 1. \]

Let the family of the fuzzy sets on \( \mathbb{R}^n \) satisfying Definition 2.3 (1°), (2°) be \( F_c \).

**Definition 2.4** (Pu and Liu [3]). The fuzzy set \( a_\alpha \) \((0 \leq \alpha \leq 1)\) on \( \mathbb{R} \) is called a level \( \alpha \) fuzzy point on \( \mathbb{R} \) if its membership function \( \mu_{a_\alpha}(x) \) is
\[
\mu_{a_\alpha}(x) = \begin{cases} 
\alpha, & x = a, \\
0, & x \neq a.
\end{cases} \tag{2.1}
\]

Let the family of all level \( \alpha \) fuzzy points on \( \mathbb{R} \) be \( F^1(\alpha) = \{a_\alpha \forall \alpha \in \mathbb{R}\}, 0 \leq \alpha \leq 1. \)

**Definition 2.5.** The fuzzy set \( (a(1), a(2), \ldots, a(n))_\alpha \) \((0 \leq \alpha \leq 1)\) is called a level \( \alpha \) fuzzy point on \( \mathbb{R}^n \) if its membership function is
\[
\mu((a(1), a(2), \ldots, a(n))_\alpha) (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = \begin{cases} 
\alpha, & (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = (a(1), a(2), \ldots, a(n)), \\
0, & \text{elsewhere.}
\end{cases}
\tag{2.2}
\]

Let the family of all level \( \alpha \) fuzzy points on \( \mathbb{R}^n \) be
\[
F_p^n(\alpha) = \{(a(1), a(2), \ldots, a(n))_\alpha \forall (a(1), a(2), \ldots, a(n)) \in \mathbb{R}^n\}, 0 \leq \alpha \leq 1,
\tag{2.3}
\]

For each \( a_\alpha \in F^1(\alpha) \), regard \( a_\alpha = (a, a, \ldots, a)_\alpha \) as a special level \( \alpha \) fuzzy point on \( \mathbb{R}^n \) degenerated from a level \( \alpha \) fuzzy point \( (a(1), a(2), \ldots, a(n)) \) with \( a(1) = a(2) = \ldots = a(n) = a \). Hence, we have the following expression:
\[
\mu((a, a, \ldots, a)_\alpha) (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = \begin{cases} 
\alpha, & (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = (a, a, \ldots, a), \\
0, & (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \neq (a, a, \ldots, a),
\end{cases}
\tag{2.4}
\]

**Definition 2.6.** For \( D \subset \mathbb{R}^n \), call \( D_\alpha, 0 \leq \alpha \leq 1 \), a level \( \alpha \) fuzzy domain on \( \mathbb{R}^n \) if its membership function is
\[
\mu_{D_\alpha}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = \begin{cases} 
\alpha, & (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in D, \\
0, & (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \notin D.
\end{cases}
\tag{2.5}
\]

Let the family of all the level \( \alpha \) fuzzy domains on \( \mathbb{R}^n \) be \( FD^*_\alpha = \{E_\alpha \forall E \subset \mathbb{R}^n\} \), and let the family of all subsets of \( \mathbb{R}^n \) be \( \wp(\mathbb{R}^n) = \{E \forall E \subset \mathbb{R}^n\} \).
Then there is a one-to-one mapping $\eta$ between $\tilde{\mathcal{P}}(\mathbb{R}^n)$ and $FD^*$:

$$
E \in \tilde{\mathcal{P}}(\mathbb{R}^n) \rightarrow \eta(E) = E_\alpha \in FD^*, \\
\eta^{(-1)}(E_\alpha) = E, \quad \alpha \in [0, 1].
$$  \hbox{(2.6)}

Since $\tilde{D} \in F_c$, the $\alpha$-cut $D(\alpha)$ ($0 \leq \alpha \leq 1$) of $\tilde{D}$ can be mapped to $D(\alpha)_\alpha$.

Thus, we have the following decomposition principle:

$$
\forall \tilde{D} \in F_c, \quad \tilde{D} = \bigcup_{\alpha \in [0, 1]} D(\alpha)_\alpha.
$$  \hbox{(2.7)}

From Kaufmann and Gupta [2], we have for $D, E \subset \mathbb{R}^n$, $k \in \mathbb{R}$,

$$
D(+)E = \{(x^{(1)} + y^{(1)}, x^{(2)} + y^{(2)}, \ldots, x^{(n)} + y^{(n)}) \\
\forall (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in D, (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in E\},
$$  \hbox{(2.8)}

$$
D(-)E = \{(x^{(1)} - y^{(1)}, x^{(2)} - y^{(2)}, \ldots, x^{(n)} - y^{(n)}) \\
\forall (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in D, (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in E\},
$$  \hbox{(2.9)}

$$
k(\cdot)D = \{(kx^{(1)}, kx^{(2)}, \ldots, kx^{(n)}) \forall (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in D\}.  \quad \hbox{(2.10)}
$$

From (2.6), (2.7), (2.8), (2.9), (2.10), and the definition of the $\alpha$-cut, we have that

(i) the $\alpha$-cut of $\tilde{D}(+)\tilde{E}$ is $D(\alpha) + E(\alpha)$,

$$
\tilde{D} \oplus \tilde{E} = \bigcup_{0 \leq \alpha \leq 1} (D(\alpha)(+)E(\alpha))_\alpha.
$$  \hbox{(2.11)}

(ii) the $\alpha$-cut of $\tilde{D}(-)\tilde{E}$ is $D(\alpha) - E(\alpha)$,

$$
\tilde{D} \ominus \tilde{E} = \bigcup_{0 \leq \alpha \leq 1} (D(\alpha)(-)E(\alpha))_\alpha.
$$  \hbox{(2.12)}

(iii) the $\alpha$-cut of $k_1 \otimes wtD$ is $k(\cdot)D(\alpha)$,

$$
k_1 \otimes \tilde{D} = \bigcup_{0 \leq \alpha \leq 1} (k(\cdot)D(\alpha))_\alpha.
$$  \hbox{(2.13)}

In the crisp case on $\mathbb{R}^n$, we can consider the $n$-dimensional vector space $E^n$ over $\mathbb{R}$.

Let $P = (p^{(1)}, p^{(2)}, \ldots, p^{(n)}), Q = (q^{(1)}, q^{(2)}, \ldots, q^{(n)}), A = (a^{(1)}, a^{(2)}, \ldots, a^{(n)}), B = (b^{(1)}, b^{(2)}, \ldots, b^{(n)}) \in \mathbb{R}^n; k \in \mathbb{R}$.

Define the crisp vectors $\overrightarrow{PQ}, \overrightarrow{AB} + \overrightarrow{PQ}$, and $k \cdot \overrightarrow{PQ}$ as follows:

$$
\overrightarrow{PQ} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)}) = Q(-)P,  \quad \hbox{(2.14)}
$$

$$
\overrightarrow{AB} + \overrightarrow{PQ} = (b^{(1)} + q^{(1)} - a^{(1)} - p^{(1)}, b^{(2)} + q^{(2)} - a^{(2)} - p^{(2)},  \ldots, b^{(n)} + q^{(n)} - a^{(n)} - p^{(n)}),  \quad \hbox{(2.15)}
$$

$$
k \cdot \overrightarrow{PQ} = (kq^{(1)} - kp^{(1)}, kq^{(2)} - kp^{(2)}, \ldots, kq^{(n)} - kp^{(n)}).  \quad \hbox{(2.16)}
$$
Let $O = (0,0,\ldots,0) \in \mathbb{R}^n$, $\overrightarrow{OP} = (p^{(1)}, p^{(2)}, \ldots, p^{(n)})$, $\overrightarrow{OO} = (0,0,\ldots,0)$, and let $E^n = \{\overrightarrow{PQ} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)}) \forall P,Q \in \mathbb{R}^n\}$. This is an $n$-dimensional vector space over $\mathbb{R}$. There is a one-to-one onto mapping between the point $(a^{(1)}, a^{(2)}, \ldots, a^{(n)})$ on $\mathbb{R}^n$ and the level 1 fuzzy point $(a^{(1)}, a^{(2)}, \ldots, a^{(n)})_1$ on $F^n_p(1)$:

$$\rho: (a^{(1)}, a^{(2)}, \ldots, a^{(n)}) \in \mathbb{R}^n \mapsto \rho(a^{(1)}, a^{(2)}, \ldots, a^{(n)}) = (a^{(1)}, a^{(2)}, \ldots, a^{(n)})_1 \in F^n_p(1). \quad (2.17)$$

Let $\tilde{P} = (p^{(1)}, p^{(2)}, \ldots, p^{(n)})_1$, $\tilde{Q} = (q^{(1)}, q^{(2)}, \ldots, q^{(n)})_1 \in F^n_p(1)$. From (2.14) and (2.17), we have the following definition:

$$\overrightarrow{\tilde{P}\tilde{Q}} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)})_1 = \tilde{Q} \odot \tilde{P}. \quad (2.18)$$

Let $FE^n = \{\overrightarrow{P}\overrightarrow{Q} \forall \tilde{P}, \tilde{Q} \in F^n_p(1)\}$. From (2.14) and (2.18), we have the one-to-one onto mappings

$$\overrightarrow{P\tilde{Q}} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)})_1 \leadsto \rho(\overrightarrow{P\tilde{Q}}) = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)})_1 \in \overrightarrow{\tilde{P}\tilde{Q}} \in FE^n,$$

$$\overrightarrow{A\tilde{B} + \overrightarrow{P\tilde{Q}}} = (b^{(1)} + q^{(1)} - a^{(1)} - p^{(1)}, b^{(2)} + q^{(2)} - a^{(2)} - p^{(2)}, \ldots, b^{(n)} + q^{(n)} - a^{(n)} - p^{(n)})_1$$

$$\leadsto (b^{(1)} + q^{(1)} - a^{(1)} - p^{(1)}, b^{(2)} + q^{(2)} - a^{(2)} - p^{(2)}, \ldots, b^{(n)} + q^{(n)} - a^{(n)} - p^{(n)})_1 = \overrightarrow{AB} \oplus \overrightarrow{\tilde{P}\tilde{Q}}, \quad (2.19)$$

$$k \cdot \overrightarrow{\tilde{P}\tilde{Q}} = (kq^{(1)} - kp^{(1)}, kq^{(2)} - kp^{(2)}, \ldots, kq^{(n)} - kp^{(n)})_1$$

$$\leadsto (kq^{(1)} - kp^{(1)}, kq^{(2)} - kp^{(2)}, \ldots, kq^{(n)} - kp^{(n)})_1 = k_1 \odot \overrightarrow{\tilde{P}\tilde{Q}}.$$

Therefore, $FE^n = \{\overrightarrow{\tilde{P}\tilde{Q}} \forall \tilde{P}, \tilde{Q} \in F^n_p(1)\}$ is a vector space over $F^n_p(1)$ in fuzzy sense.

In [4], we further extend $FE^n$ as follows. For $\tilde{X}, \tilde{Y} \in F_c$, define $\overrightarrow{\tilde{X}\tilde{Y}} = \tilde{Y} \odot \tilde{X}$ and call $\overrightarrow{\tilde{X}\tilde{Y}}$ a fuzzy vector. Let $SFR = \{\overrightarrow{\tilde{X}\tilde{Y}} \forall \tilde{X}, \tilde{Y} \in F_c\}$. In [4], we proved that the following properties hold.
**Property 2.7.** For \( \overrightarrow{XY}, \overrightarrow{WZ} \in \text{SFR}, \)

\[
\overrightarrow{XY} = \overrightarrow{WZ} \iff \overrightarrow{Y} \oplus \overrightarrow{X} = \overrightarrow{Z} \oplus \overrightarrow{W}. \tag{2.20}
\]

**Property 2.8.** For \( \overrightarrow{XY}, \overrightarrow{WZ} \in \text{SFR}, k \in \mathbb{R}, \)

(1°) \( \overrightarrow{XY} \oplus \overrightarrow{WZ} = \overrightarrow{AB}, \) where \( \overrightarrow{A} = \overrightarrow{X} \oplus \overrightarrow{W}, \overrightarrow{B} = \overrightarrow{Y} \oplus \overrightarrow{Z}; \)

(2°) \( k_1 \odot \overrightarrow{XY} = \overrightarrow{CD}, \) where \( \overrightarrow{C} = k_1 \odot \overrightarrow{X}, \overrightarrow{D} = k_1 \odot \overrightarrow{Y}. \)

**Property 2.9.** For \( \overrightarrow{XY}, \overrightarrow{WZ}, \overrightarrow{UV} \in \text{SFR}, k, t \in \mathbb{R}, \)

(1°) \( \overrightarrow{XY} \oplus \overrightarrow{WZ} = \overrightarrow{WZ} \oplus \overrightarrow{XY}; \)

(2°) \( (\overrightarrow{XY} \oplus \overrightarrow{WZ}) \oplus \overrightarrow{UV} = \overrightarrow{XY} \oplus (\overrightarrow{WZ} \oplus \overrightarrow{UV}); \)

(3°) \( \overrightarrow{XY} \oplus \overrightarrow{OO} = \overrightarrow{XY}, \) where \( \overrightarrow{O} = (0, 0, \ldots, 0); \)

(4°) \( k_1 \odot (t_1 \odot \overrightarrow{XY}) = (kt_1) \odot \overrightarrow{XY}; \)

(5°) \( k_1 \odot (\overrightarrow{XY} \oplus \overrightarrow{WZ}) = (k_1 \odot \overrightarrow{XY}) \oplus (k_1 \odot \overrightarrow{WZ}); \)

(6°) \( 1 \odot \overrightarrow{XY} = \overrightarrow{XY}. \)

In SFR, the following do not hold.

(7°) For \( \overrightarrow{XY} \in \text{SFR} \) and \( \overrightarrow{XY} \neq \overrightarrow{OO}, \) there exists \( \overrightarrow{WZ} (\neq \overrightarrow{OO}) \in \text{SFR} \) such that \( \overrightarrow{XY} \oplus \overrightarrow{WZ} = \overrightarrow{OO}; \)

(8°) \( (k + t) \odot \overrightarrow{XY} = (k_1 \odot \overrightarrow{XY}) \oplus (t_1 \odot \overrightarrow{XY}). \)

From Property 2.9, we know that SFR satisfies all the conditions that the vector space required, except \((7°)\) and \((8°)\). Therefore, in [4], we called SFR a pseudo-fuzzy vector space over \( F_p^1(1). \)

**Example 2.10** (a moving station carrying a missile on it). This car left from point \( P = (2, 5) \) passing through point \( Q = (4, 6), \) arrived at \( R = (8, 9), \) and aiming at the target \( Z = (100, 200). \) As we can see, the missile usually falls in the vicinity of \( Z, \) say \( \tilde{Z}, \) instead of hitting at \( Z \) exactly.

Let the membership function of \( \tilde{Z} \) be

\[
\mu_{\tilde{Z}}(x^{(1)}, x^{(2)}) = \begin{cases} 
\frac{1}{25}(25 - (x^{(1)} - 100)^2 - (x^{(2)} - 200)^2), & \text{if } (x^{(1)} - 100)^2 + (x^{(2)} - 200)^2 \leq 25, \\
0, & \text{elsewhere.}
\end{cases} \tag{2.21}
\]

Consider the level 1 fuzzy points \( \tilde{P} = (2, 5)_1, \tilde{Q} = (4, 6)_1, \) and \( \tilde{R} = (8, 9)_1. \) We have the fuzzy routes

\[
\tilde{P} \rightarrow \tilde{Q} \rightarrow \tilde{R} \rightarrow \tilde{Z} \tag{2.22}
\]
and hence the fuzzy vectors $\overrightarrow{PQ} = (2,1), \overrightarrow{QR} = (4,3), \overrightarrow{RZ} = \hat{Z} \otimes \hat{R}$, and $\overrightarrow{PZ} = \hat{Z} \otimes \hat{P}$. By extension theory, the membership function of $\overrightarrow{RZ} = \hat{Z} \otimes \hat{R}$ is

$$
\mu_{\overrightarrow{RZ}}(z^{(1)},z^{(2)}) = \sup_{z^{(j)} = v^{(j)} - u^{(j)}, j=1,2} \mu_{\hat{R}}(u^{(1)},u^{(2)}) \wedge \mu_{\overrightarrow{v}}(v^{(1)},v^{(2)}) \\
= \mu_{\hat{Z}}(z^{(1)} + 8, z^{(2)} + 9) \\
= \begin{cases} 
\frac{1}{25} (25 - (z^{(1)} - 92)^2 - (z^{(2)} - 191)^2), & \text{if } (z^{(1)} - 92)^2 + (z^{(2)} - 191)^2 \leq 25, \\
0, & \text{elsewhere}. 
\end{cases}
(2.23)
$$

Similarly,

$$
\mu_{\overrightarrow{PZ}}(z^{(1)},z^{(2)}) = \begin{cases} 
\frac{1}{25} (25 - (z^{(1)} - 98)^2 - (z^{(2)} - 195)^2), & \text{if } (z^{(1)} - 98)^2 + (z^{(2)} - 195)^2 \leq 25, \\
0, & \text{elsewhere}. 
\end{cases}
(2.24)
$$

Let $S = (98,202)$. It is clear that $(98,202)$ is within the circle of center $(100,200)$ and radius 5. The crisp vector which starts with the point $P = (2,5)$ and ends at $S = (98,202)$ is $\overrightarrow{PS} = (96,197)$. Its grade of membership in $\overrightarrow{PS}$ from (2.23) is $\mu_{\overrightarrow{PS}}(96,197) = (1/25)(25 - 2^2 - 2^2) = 0.68$, that is, the grade of membership of the fuzzy vector $\overrightarrow{PS}$ for the crisp vector $\overrightarrow{PS}$ is 0.68. Let the aim be $T = (100,200)$. The crisp vector beginning at $P = (2,5)$ and aiming at $T = (100,200)$ is $\overrightarrow{PT} = (98,195)$. Its grade of membership in $\overrightarrow{PT}$, again from (2.23), is $\mu_{\overrightarrow{PT}}(98,195) = (1/25)(25 - 0^2 - 0^2) = 1$, that is, the grade of membership of the fuzzy vector $\overrightarrow{PT}$ for the crisp vector $\overrightarrow{PT}$ is 1.

**Example 2.11.** In a shooting practice, let $C((10,30),1+1/m) = \{(x,y) \mid (x-10)^2 + (y-30)^2 \leq (1+1/m)^2\}$, always shooting at (1,2) and aiming at $Z = (10,30)$. At the first time, the bullet was falling in $C((10,30),2 = 1 + 1)$. At the second time, it was falling in $C((10,30),1+1/2)$. At the $m$th time, it was falling in $C((10,30),1+1/m)$. In other words, the bullet was more and more closer to $C((10,30),1)$, that is, more and more accurate.

Let the fuzzy aim be $\hat{Z}_m$, its membership function is

$$
\mu_{\hat{Z}_m} = \begin{cases} 
\frac{1}{(1+1/m)^2} \left[ \left(1 + \frac{1}{m}\right)^2 - (x-10)^2 - (y-30)^2 \right], & \text{if } (x-10)^2 + (y-30)^2 \leq \left(1 + \frac{1}{m}\right)^2, \\
0, & \text{elsewhere}. 
\end{cases}
(2.25)
$$

Thus, we have the $m$th fuzzy vector $\overrightarrow{Q\hat{Z}_m}$, $m = 1,2,\ldots$, where $\overrightarrow{Q} = (1,2)$. In the next section, we will discuss the convergency of the fuzzy vectors in SFR and find out the limit fuzzy vector $\lim_{n \to \infty} \overrightarrow{Q\hat{Z}_m}$. 

3. The convergency of the vectors in SFR. Before we try to investigate the convergency of the fuzzy vectors in SFR, we first define the following open set in $\mathbb{R}^n$ and discuss some properties (Properties 3.4, 3.7, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, and 3.17). Let

\[
O((a^{(1,1)}, a^{(1,2)}), \ldots, (a^{(n,1)}, a^{(n,2)})) = \{(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \mid a^{(j,1)} < x^{(j)} < a^{(j,2)}, j = 1, 2, \ldots, n\}.
\]

From (2.8), (2.9), and (2.10), we have

\[
O((a^{(1,1)}, a^{(1,2)}), \ldots, (a^{(n,1)}, a^{(n,2)})) (+) O((b^{(1,1)}, b^{(1,2)}), \ldots, (b^{(n,1)}, b^{(n,2)})) = \{(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \mid z^{(j)} = x^{(j)} + y^{(j)}, a^{(j,1)} < x^{(j)} < a^{(j,2)}, b^{(j,1)} < y^{(j)} < b^{(j,2)}, j = 1, 2, \ldots, n\}.
\]

\[
O((a^{(1,1)} + b^{(1,1)}, a^{(1,2)} + b^{(1,2)}), \ldots, (a^{(n,1)} + b^{(n,1)}, a^{(n,2)} + b^{(n,2)}))
\]

\[
O((a^{(1,1)}, a^{(1,2)}), \ldots, (a^{(n,1)}, a^{(n,2)})) (-) O((b^{(1,1)}, b^{(1,2)}), \ldots, (b^{(n,1)}, b^{(n,2)})) = \{(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \mid z^{(j)} = x^{(j)} - y^{(j)}, a^{(j,1)} < x^{(j)} < a^{(j,2)}, b^{(j,1)} < y^{(j)} < b^{(j,2)}, j = 1, 2, \ldots, n\}.
\]

\[
O((a^{(1,1)} - b^{(1,1)}, a^{(1,2)} - b^{(1,2)}), \ldots, (a^{(n,1)} - b^{(n,1)}, a^{(n,2)} - b^{(n,2)})).
\]

Let $k > 0$,

\[
k(\cdot)O((a^{(1,1)}, a^{(1,2)}), \ldots, (a^{(n,1)}, a^{(n,2)})) = \{(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \mid z^{(j)} = kx^{(j)}, a^{(j,1)} < x^{(j)} < a^{(j,2)}, j = 1, 2, \ldots, n\}.
\]

If $k < 0$,

\[
k(\cdot)O((a^{(1,1)}, a^{(1,2)}), \ldots, (a^{(n,1)}, a^{(n,2)})) = \{(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \mid z^{(j)} = kx^{(j)}, a^{(j,1)} < x^{(j)} < a^{(j,2)}, j = 1, 2, \ldots, n\}.
\]

Let $\mathcal{B} = \{O((a^{(1,1)}, a^{(1,2)}), \ldots, (a^{(n,1)}, a^{(n,2)})) \mid \forall a^{(j,1)} < a^{(j,2)}, a^{(j,1)}, a^{(j,2)} \in \mathbb{R}, j = 1, 2, \ldots, n; 0 \leq \alpha \leq 1\}.$

Let $\mathcal{B}^*$ be the family of fuzzy sets in $\mathcal{B}$ or any arbitrary unions of these fuzzy sets.

**Remark 3.1.** Any intersection of two fuzzy sets in $\mathcal{B}$ belongs to $\mathcal{B}$, and when two fuzzy sets in $\mathcal{B}$ have no intersection, we call their intersection $\emptyset$.

From (2.3), let $F = F^n_p \cup F_c \cup \mathcal{B}^*$. In order to consider the problem of convergency, we first consider the topology for $F$. 
**Definition 3.2.** $\tilde{Q} \in F$ is an open fuzzy set if and only if for each $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})_\alpha \subset \tilde{Q}$, there exists $\tilde{O} \in \mathcal{B}$ such that $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})_\alpha \subset \tilde{O} \subset \tilde{Q}$.

Let $T_F$ be the family of all open fuzzy sets satisfying Definition 3.2. Obviously, $\mathcal{B}^* \subset T_F$.

**Definition 3.3** (Chang [1]). $T$ is a family of fuzzy sets in the space $X$ satisfying the following:

- $(1^*) \emptyset, X \in T$,
- $(2^*) \tilde{A}, \tilde{B} \in T$, then $\tilde{A} \cap \tilde{B} \in T$,
- $(3^*) \tilde{A}_j \in T$, $j \in I$ (any index set), then $\bigcup_{j \in I} \tilde{A}_j \in T$.

$T$ is called a fuzzy topology for $X$ and $(X, T)$ is called a fuzzy topological space (abbreviated as FTS).

**Property 3.4.** $T_F$ is a fuzzy topology for $\mathbb{R}^n$, $(\mathbb{R}^n, T_F)$ are fuzzy topological sets in $\mathbb{R}^n$ that are restricted in $F$.

**Proof.** $(1^*)$ It is obvious that $\mathbb{R}^n \in T_F$. Definition 3.3 $(1^*)$ is fulfilled.

$(2^*)$ For $\tilde{D}, \tilde{E} \in T_F$ and $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})_\alpha \subset \tilde{D} \cap \tilde{E}$, we have $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})_\alpha \subset \tilde{D}$ and $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})_\alpha \subset \tilde{E}$. From Definition 3.2, there exist $\tilde{I}, \tilde{J} \in \mathcal{B}$ such that $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})_\alpha \subset \tilde{I} \subset \tilde{D}$ and $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})_\alpha \subset \tilde{J} \subset \tilde{E}$. Therefore, $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})_\alpha \subset \tilde{I} \cap \tilde{J}$. Hence, $\tilde{I} \cap \tilde{J} \subset \tilde{D} \cap \tilde{E}$. Thus, $\tilde{D} \cap \tilde{E} \in T_F$. Definition 3.3 $(2^*)$ is fulfilled.

$(3^*)$ For $\tilde{D}_j \in T_F$, $j \in I$, and each $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})_\alpha \subset \bigcup_{j \in I} \tilde{D}_j$, there exists $m \in I$ such that $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})_\alpha \subset \tilde{D}_m$. By Definition 3.2, there is a $\tilde{J} \in \mathcal{B}$ such that $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})_\alpha \subset \tilde{J} \subset \tilde{D}_m \subset \bigcup_{j \in I} \tilde{D}_j \subset T_F$. Thus, Definition 3.3 $(3^*)$ is fulfilled.

Hence, from Definition 3.3, $T_F$ is a fuzzy topology for $\mathbb{R}^n$ and $(\mathbb{R}^n, T_F)$ is a fuzzy topological space, that is, if we set $X = \mathbb{R}^n$, $T = T_F$ in Definition 3.3, then the definition holds. Therefore, Definitions 3.5, 3.6 and Property 3.7 can all be applied.

**Definition 3.5** (Chang [1, Definition 2.3]). A fuzzy set $\tilde{U}$ in an FTS $(X, T)$ is a neighborhood of a fuzzy set $\tilde{A}$ if and only if there exists a fuzzy set $\tilde{O} \in T$ such that $\tilde{A} \subset \tilde{O} \subset \tilde{U}$.

**Definition 3.6** (Chang [1, Definition 3]). If a sequence of fuzzy sets $\{\tilde{A}_n, n = 1, 2, \ldots\}$ is in an FTS $(X, T)$, then this sequence converges to a fuzzy set $\tilde{A}$ if and only if it is eventually contained in each neighborhood of $\tilde{A}$ (i.e., if $\tilde{B}$ is any neighborhood of $\tilde{A}$, there is a positive integer $m$ such that whenever $n \geq m$, $\tilde{A}_n \subset \tilde{B}$).

**Property 3.7.** $\{\tilde{A}_n\}$ are increasing fuzzy sets, $\tilde{A}_1 \subset \tilde{A}_2 \subset \cdots \subset \tilde{A}$, and

$$\lim_{n \to \infty} \mu_{\tilde{A}_n}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = \mu_{\tilde{A}}(x^{(1)}, x^{(2)}, \ldots, x^{(n)})$$

(3.6)

for all $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n$. Then the sequence $\{\tilde{A}_n, n = 1, 2, \ldots\}$ converges to $\tilde{A}$, denoted by $\lim_{n \to \infty} \tilde{A}_n = \tilde{A}$.

**Proof.** The proof follows from Definition 3.6 easily. □
**Definition 3.8.** \( \bigcup_{\alpha \in [0,1]} O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\ldots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_{\alpha} \in T_F \) is a neighborhood of \( \tilde{D} = \bigcup_{\alpha \in [0,1]} D(\alpha)_{\alpha} \in F_c \) if and only if for each \( \alpha \in [0,1] \), there exists \( O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\ldots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_{\alpha} \in \mathbb{B} \) such that \( D(\alpha)_{\alpha} \subset O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\ldots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_{\alpha} \). 

**Definition 3.9.** In \( F_c \), the sequence of fuzzy sets \( \tilde{D}_k = \bigcup_{\alpha \in [0,1]} D_k(\alpha)_{\alpha} \), \( k = 1,2,\ldots \), converges to \( \tilde{D} = \bigcup_{\alpha \in [0,1]} D(\alpha)_{\alpha} \) if and only if for each neighborhood \( \bigcup_{\alpha \in [0,1]} O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\ldots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_{\alpha} \) of \( \tilde{D} \), there exists a natural number \( m \) such that whenever \( k \geq m \), \( D_k(\alpha)_{\alpha} \subset O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\ldots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_{\alpha} \), denoted by \( \lim_{k \to \infty} \tilde{D}_k = \tilde{D} \).

Since \( D \subset \mathbb{R}^n \) and \( D_{\alpha} \in (FD^*) \) is a one-to-one onto mapping, from Definition 3.9, we can get the following property.

**Property 3.10.** In \( F_c \), the sequence of fuzzy sets \( \tilde{D}_k = \bigcup_{\alpha \in [0,1]} D_k(\alpha)_{\alpha} \), \( k = 1,2,\ldots \), converges to \( \tilde{D} = \bigcup_{\alpha \in [0,1]} D(\alpha)_{\alpha} \) if and only if for each \( \alpha \in [0,1] \) and every neighborhood \( O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\ldots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_{\alpha} \) of \( D(\alpha)_{\alpha} \), there exists a natural number \( m \) such that whenever \( k \geq m \), \( D_k(\alpha)_{\alpha} \subset O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\ldots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_{\alpha} \) if and only if for each \( \alpha \in [0,1] \) and every neighborhood \( O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\ldots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_{\alpha} \) of \( D(\alpha)_{\alpha} \), there exists \( m \) such that whenever \( k \geq m \), \( D_k(\alpha)_{\alpha} \subset O((a^{(1,1)}(\alpha),a^{(1,2)}(\alpha)),\ldots,(a^{(n,1)}(\alpha),a^{(n,2)}(\alpha)))_{\alpha} \).

The convergency of fuzzy vectors needs the following property.

**Property 3.11.** For each \( \alpha \in [0,1] \), the \( \alpha \)-cuts \( D_k(\alpha)_{\alpha} \), \( E_k(\alpha)_{\alpha} \), \( k = 1,2,\ldots,n \), of \( \tilde{D}_k \), \( \tilde{E}_k \) in \( F_c \) satisfy the following:

1. \( D_k(\alpha)(+)E_k(\alpha)_{\alpha} = D_k(\alpha)_{\alpha} \oplus E_k(\alpha)_{\alpha} \)
2. \( D_k(\alpha)(-)E_k(\alpha)_{\alpha} = D_k(\alpha)_{\alpha} \ominus E_k(\alpha)_{\alpha} \)
3. Each \( \alpha \)-cut of \( \bigcup_{k=1}^{m}[\tilde{D}_k \ominus \tilde{E}_k] \) is \( \bigcup_{k=1}^{m} D_k(\alpha)(+)\tilde{E}_k(\alpha) \)
4. Each \( \alpha \)-cut of \( \bigcup_{k=1}^{m}[\tilde{D}_k \ominus \tilde{E}_k] \) is \( \bigcup_{k=1}^{m} D_k(\alpha)(+)\tilde{E}_k(\alpha) \)

**Proof.** By extension principle (1°)
\[
\begin{align*}
&= \alpha, \quad \text{if } (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in D_k(\alpha), \\
&= \alpha, \quad \text{if } (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in D_k(\alpha) + E_k(\alpha), \\
&= \mu(D_k(\alpha) + E_k(\alpha))_\alpha (z^{(1)}, z^{(2)}, \ldots, z^{(n)}), \quad \forall (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n. \\
\end{align*}
\] (3.7)

(2°) The proof is similar to that of (1°).

(3°) Let \( \tilde{S}_k = \tilde{D}_k \oplus \tilde{E}_k \); from (2.11), we have

\[
\bigcup_{k=1}^{m} \tilde{S}_k = \bigcup_{k=1}^{m} \bigcup_{\alpha \in [0,1]} (D_k(\alpha) + E_k(\alpha))_\alpha = \bigcup_{\alpha \in [0,1]} \bigcup_{k=1}^{m} (D_k(\alpha) + E_k(\alpha))_\alpha. \tag{3.8}
\]

Therefore, the \( \alpha \)-cut of \( \bigcup_{k=1}^{m} \tilde{S}_k(\alpha) \) is \( \bigcup_{k=1}^{m} S_k(\alpha) \) for some \( k \in \{1, 2, \ldots, m\} \).

(3°-1) For each \( \alpha \in [0,1] \), the subset \( \bigcup_{k=1}^{m} S_k(\alpha) \) of \( \mathbb{R}^n \) corresponds to the fuzzy set \( \bigcup_{k=1}^{m} S_k(\alpha)_\alpha = \bigcup_{k=1}^{m} (D_k(\alpha) + E_k(\alpha))_\alpha \). We first prove

\[
\left( \bigcup_{k=1}^{m} S_k(\alpha) \right)_\alpha = \bigcup_{k=1}^{m} S_k(\alpha)_\alpha. \tag{3.9}
\]

We have

\[
\begin{align*}
\mu_{\bigcup_{k=1}^{m} S_k(\alpha)_\alpha} (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \\
= \bigvee_{k=1}^{m} \mu_{S_k(\alpha)_\alpha} (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \\
= \alpha, \quad \text{if } (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in S_k(\alpha) \text{ for some } k \in \{1, 2, \ldots, m\}, \\
= \alpha, \quad \text{if } (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in \bigcup_{k=1}^{m} S_k(\alpha), \\
= \mu_{(\bigcup_{k=1}^{m} S_k(\alpha))_\alpha} (z^{(1)}, z^{(2)}, \ldots, z^{(n)}), \quad \forall (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in \mathbb{R}^n.
\end{align*}
\] (3.10)

Therefore, \( (\bigcup_{k=1}^{m} S_k(\alpha))_\alpha = \bigcup_{k=1}^{m} S_k(\alpha)_\alpha \). Hence

\[
\left( \bigcup_{k=1}^{m} (D_k(\alpha) + E_k(\alpha)) \right)_\alpha = \bigcup_{k=1}^{m} (D_k(\alpha) + E_k(\alpha))_\alpha. \tag{3.11}
\]

For each \( \alpha \in [0,1] \) and each \( k \), (1°) holds. Therefore,

\[
\bigcup_{k=1}^{m} (D_k(\alpha) + E_k(\alpha))_\alpha = \bigcup_{k=1}^{m} (D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha). \tag{3.12}
\]

Finally, we will prove

\[
\bigcup_{k=1}^{m} (D_k(\alpha)_\alpha \oplus E_k(\alpha)_\alpha) = \bigcup_{k=1}^{m} (D_k(\alpha)_\alpha) \oplus \bigcup_{k=1}^{m} (E_k(\alpha)_\alpha). \tag{3.13}
\]
We have

\[
\begin{align*}
\mu_{\bigcup_{k=1}^{m} (D_k(\alpha) \oplus E_k(\alpha))_\alpha} (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \\
= \bigvee_{k=1}^{m} \mu_{D_k(\alpha) \oplus E_k(\alpha)_\alpha} (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \\
= \bigvee_{k=1}^{m} \sup_{\prod_{j=1}^{n} z^{(j)} = X^{(j)} + Y^{(j)}} \mu_{D_k(\alpha)_\alpha} (X^{(1)}, X^{(2)}, \ldots, X^{(n)}) \land \mu_{E_k(\alpha)_\alpha} (Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)}) \\
= \bigvee_{k=1}^{m} \prod_{j=1}^{n} \left[ \mu_{D_k(\alpha)_\alpha} (z^{(1)} - y^{(1)}, z^{(2)} - y^{(2)}, \ldots, z^{(n)} - y^{(n)}) \land \mu_{E_k(\alpha)_\alpha} (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \right] \\
= \bigvee_{\prod_{j=1}^{n} (y^{(1)}, y^{(2)}, \ldots, y^{(n)})} \left[ \bigvee_{k=1}^{m} \left( \bigvee_{\alpha \in [0,1]} \left( \bigcup_{k=1}^{m} D_k(\alpha)_\alpha \bigoplus \bigcup_{k=1}^{m} E_k(\alpha)_\alpha \right) \right) \right].
\end{align*}
\]

(3.14)

(3*2) By decomposition theorem and (3*-1), we have

\[
\bigcup_{k=1}^{m} (\tilde{D}_k \oplus \tilde{E}_k) = \bigcup_{k=1}^{m} \bigcup_{\alpha \in [0,1]} (D_k(\alpha)(+) E_k(\alpha))_\alpha \\
= \bigcup_{\alpha \in [0,1]} \bigcup_{k=1}^{m} (D_k(\alpha)(+) E_k(\alpha))_\alpha \\
= \bigcup_{\alpha \in [0,1]} \left( \bigcup_{k=1}^{m} D_k(\alpha)_\alpha \bigoplus \bigcup_{k=1}^{m} E_k(\alpha)_\alpha \right).
\]

(3.15)

Let \( \tilde{A} = \bigcup_{k=1}^{m} \tilde{D}_k, \tilde{B} = \bigcup_{k=1}^{m} \tilde{E}_k. \) From (3.9),

\[
A(\alpha)_\alpha = \bigcup_{k=1}^{m} \tilde{D}_k(\alpha)_\alpha, \quad B(\alpha)_\alpha = \bigcup_{k=1}^{m} \tilde{E}_k(\alpha)_\alpha, \quad \forall \alpha \in [0,1],
\]

\[
\tilde{A} \oplus \tilde{B} = \bigcup_{\alpha \in [0,1]} \left[ A(\alpha)(+) B(\alpha) \right]_\alpha = \bigcup_{\alpha \in [0,1]} \left[ A(\alpha)_\alpha \oplus B(\alpha)_\alpha \right]
\]

(3.16)

(3.17)
From (3.15), (3.17), we have
\[
\bigcup_{k=1}^{m} (\tilde{D}_k \oplus \tilde{E}_k) = \bigcup_{\alpha \in [0,1]} \left[ \left( \bigcup_{k=1}^{m} D_k(\alpha) \right) \oplus \left( \bigcup_{k=1}^{m} E_k(\alpha) \right) \right] = \left( \bigcup_{k=1}^{m} \tilde{D}_k \right) \oplus \left( \bigcup_{k=1}^{m} \tilde{E}_k \right).
\]
(3.18)

Properties (4\(^*\)), (4\(^*\)-1), and (4\(^*\)-2) can be proved similarly as (3\(^*\)), (3\(^*\)-1), and (3\(^*\)-2).

**Property 3.12.** \(\tilde{D}_k \in F_c, k = 1, 2, \ldots, m, \) and \(q \neq 0;\) then
(1\(^*\)) the \(\alpha\)-cut of \(\bigcup_{k=1}^{m} (q \odot \tilde{D}_k)\) is \(\bigcup_{k=1}^{m} (q(\cdot)D_k(\alpha))\),
(2\(^*\)) \(\bigcup_{k=1}^{m} (q(\odot)D_k(\alpha)) = q_1 \odot (\bigcup_{k=1}^{m} \tilde{D}_k(\alpha))\),
(3\(^*\)) \(\bigcup_{k=1}^{m} (q \odot \tilde{D}_k) = q_1 \odot (\bigcup_{k=1}^{m} \tilde{D}_k)\).

**Proof.** The proof goes on the lines of the proof of Property 3.11.

**Property 3.13.** \(\tilde{D}_m, \tilde{E}_m, \tilde{D}, \tilde{E} \in F_c, m = 1, 2, \ldots, \) and \(\lim_{m \to \infty} \tilde{D}_m = \tilde{D}, \lim_{m \to \infty} \tilde{E}_m = \tilde{E},\) then
(1\(^*\)) \(\lim_{m \to \infty} (\tilde{D}_m \oplus \tilde{E}_m) = \tilde{D} \oplus \tilde{E} = \lim_{m \to \infty} (\tilde{D}_m) \oplus \lim_{m \to \infty} (\tilde{E}_m),\)
(2\(^*\)) \(\lim_{m \to \infty} (\tilde{D}_m \oplus \tilde{E}_m) = \tilde{D} \oplus \tilde{E} = \lim_{m \to \infty} (\tilde{D}_m) \oplus \lim_{m \to \infty} (\tilde{E}_m),\)
(3\(^*\)) \(\lim_{m \to \infty} (k_1 \odot \tilde{D}_m) = k_1 \odot \tilde{D} = k_1 \odot \lim_{m \to \infty} (\tilde{D}_m), k \neq 0.\)

**Proof.** (1\(^*\)) Since \(\lim_{m \to \infty} \tilde{D}_m = \tilde{D}, \lim_{m \to \infty} \tilde{E}_m = \tilde{E},\) by Property 3.10, for each \(\alpha \in [0,1] \) and every neighborhood \(O((a^{(1,1)}, a^{(1,2)}), \ldots, (a^{(n,1)}, a^{(n,2)}))\) of \(D(\alpha)\), there exists a natural number \(m^{(1)}\) such that when \(k \geq m^{(1)}\), \(D_k(\alpha) \subset O((a^{(1,1)}, a^{(1,2)}), \ldots, (a^{(n,1)}, a^{(n,2)})).\) Also, for every neighborhood \(O((b^{(1,1)}, b^{(1,2)}), \ldots, (b^{(n,1)}, b^{(n,2)}))\) of \(E(\alpha)\), there exists a natural number \(m^{(2)}\) such that when \(k \geq m^{(2)}\), \(E_k(\alpha) \subset O((b^{(1,1)}, b^{(1,2)}), \ldots, (b^{(n,1)}, b^{(n,2)})).\)

Let \(m = \max(m^{(1)}, m^{(2)}).\) Then, for each \(\alpha \in [0,1],\) when \(k \geq m,\) by (3.2), we have \(D_k(\alpha)(+)E_k(\alpha) \subset O((a^{(1,1)} + b^{(1,1)}, a^{(1,2)} + b^{(1,2)}), \ldots, (a^{(n,1)} + b^{(1,1)}, a^{(n,2)} + b^{(1,2)})) \in T_F,\) and \(O((a^{(1,1)} + b^{(1,1)}, a^{(1,2)} + b^{(1,2)}), \ldots, (a^{(n,1)} + b^{(1,1)}, a^{(n,2)} + b^{(1,2)}))\) is the neighborhood of \(D(\alpha)(+)E(\alpha).\) By decomposition theorem,
\[
\tilde{D}_k \oplus \tilde{E}_k = \bigcup_{\alpha \in [0,1]} [D_k(\alpha) + E_k(\alpha)]_{\alpha'},
\]
\[
\tilde{D} \oplus \tilde{E} = \bigcup_{\alpha \in [0,1]} [D(\alpha) + E(\alpha)]_{\alpha'}.
\]
(3.19)

Hence, by Property 3.10, we have \(\lim_{m \to \infty} \tilde{D}_m \oplus \tilde{E}_m = \tilde{D} \oplus \tilde{E}.\)

Properties (2\(^*\)) and (3\(^*\)) can be proved the same way as (1\(^*\)).

**Property 3.14.** \(\tilde{D}_k, \tilde{E}_k, \tilde{D}, \tilde{E} \in F_c, k = 1, 2, \ldots, \) and
\[
\lim_{m \to \infty} \mu_{\bigcup_{k=1}^{m} \tilde{D}_k}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = \mu_\tilde{D}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}),
\]
\[
\lim_{m \to \infty} \mu_{\bigcup_{k=1}^{m} \tilde{E}_k}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = \mu_\tilde{E}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \quad \forall (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n,
\]
\[
\mu_{\bigcup_{k=1}^{m} \tilde{D}_k} \subset \tilde{D}, \quad \mu_{\bigcup_{k=1}^{m} \tilde{E}_k} \subset \tilde{E}, \quad \forall m = 1, 2, \ldots,
\]
(3.20)
then
\[
\lim_{m \to \infty} \bigcup_{k=1}^{m} (\tilde{D}_k \oplus \tilde{E}_k) = \tilde{D} \oplus \tilde{E} = (\lim_{m \to \infty} \bigcup_{k=1}^{m} \tilde{D}_k) \oplus (\lim_{m \to \infty} \bigcup_{k=1}^{m} \tilde{E}_k),
\]
(\ref{property1}) $\lim_{m \to \infty} \sum_{k=1}^{m} (\tilde{D}_k \oplus \tilde{E}_k) = \tilde{D} \oplus \tilde{E} = (\lim_{m \to \infty} \sum_{k=1}^{m} \tilde{D}_k) \oplus (\lim_{m \to \infty} \sum_{k=1}^{m} \tilde{E}_k),$
(\ref{property2}) $\lim_{m \to \infty} \sum_{k=1}^{m} (\tilde{D}_k \oplus \tilde{E}_k) = \tilde{D} \oplus \tilde{E} = (\lim_{m \to \infty} \sum_{k=1}^{m} \tilde{D}_k) \oplus (\lim_{m \to \infty} \sum_{k=1}^{m} \tilde{E}_k),$
(\ref{property3}) when $q \neq 0$, $\lim_{m \to \infty} \sum_{k=1}^{m} (q_1 \oplus \tilde{D}_k) = q_1 \oplus \tilde{D}.$

**Proof.** (\ref{property1}) Since $\tilde{D}_1 \subset \tilde{D}_1 \cup \tilde{D}_2 \subset \cdots \subset \bigcup_{k=1}^{m} \tilde{D}_k \subset \cdots \subset \tilde{D}$ and
\[
\lim_{m \to \infty} \sum_{k=1}^{m} (\tilde{D}_k \oplus \tilde{E}_k) = \left( \lim_{m \to \infty} \sum_{k=1}^{m} (\tilde{D}_k) \right) \oplus \left( \lim_{m \to \infty} \sum_{k=1}^{m} (\tilde{E}_k) \right) = \tilde{D} \oplus \tilde{E},
\]
for all $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n$, hence, by Property 3.7, we have $\lim_{m \to \infty} \sum_{k=1}^{m} \tilde{D}_k = \tilde{D}$.
Similarly, $\lim_{m \to \infty} \sum_{k=1}^{m} \tilde{E}_k = \tilde{E}$. By Property 3.11(\ref{property3}-2),
\[
\left( \bigcup_{k=1}^{m} (\tilde{D}_k \oplus \tilde{E}_k) \right) = \left( \bigcup_{k=1}^{m} (\tilde{D}_k) \right) \oplus \left( \bigcup_{k=1}^{m} (\tilde{E}_k) \right) = \tilde{D} \oplus \tilde{E},
\]
from Property 3.13(\ref{property1}),
\[
\lim_{m \to \infty} \sum_{k=1}^{m} (\tilde{D}_k \oplus \tilde{E}_k) = \left( \lim_{m \to \infty} \sum_{k=1}^{m} (\tilde{D}_k) \right) \oplus \left( \lim_{m \to \infty} \sum_{k=1}^{m} (\tilde{E}_k) \right) = \tilde{D} \oplus \tilde{E},
\]
and (\ref{property2}), (\ref{property3}) can be proved as (\ref{property1}).

Next, we will discuss the convergency of the fuzzy vectors in SFR.

**Property 3.15.** For $\tilde{D}_m, \tilde{E}_m, \tilde{D}, \tilde{E} \in F_C$, $m = 1, 2, \ldots$, $\lim_{m \to \infty} \tilde{D}_m = \tilde{D}$, $\lim_{m \to \infty} \tilde{E}_m = \tilde{E}$, then the fuzzy vectors $\tilde{E}_m \tilde{D}_m$, $m = 1, 2, \ldots$, converge to the fuzzy vectors $\tilde{E} \tilde{D}$.

**Proof.** Since $\tilde{E}_m \tilde{D}_m = \tilde{D}_m \oplus \tilde{E}_m$, $\tilde{E} \tilde{D} = \tilde{D} \oplus \tilde{E}$, then, by Property 3.13(\ref{property2}),
\[
\lim_{m \to \infty} \tilde{E}_m \tilde{D}_m = \tilde{D} \oplus \tilde{E} = \tilde{E} \tilde{D}.
\]

**Property 3.16.** $\tilde{D}_k, \tilde{E}_k, \tilde{D}, \tilde{E} \in F_C$, $k = 1, 2, \ldots$; let $\tilde{Q}_m = \bigcup_{k=1}^{m} \tilde{D}_k$, $\tilde{S}_m = \bigcup_{k=1}^{m} \tilde{E}_k$, and let $\lim_{m \to \infty} \mu_{\tilde{Q}_m} (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = \mu_{\tilde{D}} (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ and $\lim_{m \to \infty} \mu_{\tilde{S}_m} (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = \mu_{\tilde{E}} (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ for all $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n$, and $\tilde{Q}_m \subset \tilde{D}$, $\tilde{S}_m \subset \tilde{E}$.

Then, the sequence of fuzzy vectors $\tilde{S}_m \tilde{Q}_m$, $m = 1, 2, \ldots$, converges to the fuzzy vector $\tilde{E} \tilde{D}$.

**Proof.** Similar to Property 3.14, $\lim_{m \to \infty} \bigcup_{k=1}^{m} \tilde{D}_k = \tilde{D}$ and $\lim_{m \to \infty} \bigcup_{k=1}^{m} \tilde{E}_k = \tilde{E}$. By Property 3.13(\ref{property2}), $\lim_{m \to \infty} \tilde{S}_m \tilde{Q}_m = (\lim_{m \to \infty} \bigcup_{k=1}^{m} \tilde{D}_k) \oplus (\lim_{m \to \infty} \bigcup_{k=1}^{m} \tilde{E}_k) = \tilde{D} \oplus \tilde{E} = \tilde{E} \tilde{D}.$

For convenience, we denote $(q_1^{(1)} \oplus \tilde{E}_1 \tilde{D}_1) \oplus (q_1^{(2)} \oplus \tilde{E}_2 \tilde{D}_2) \oplus \cdots \oplus (q_1^{(r)} \oplus \tilde{E}_r \tilde{D}_r)$ by $\sum_{k=1}^{r} \oplus (q_1^{(k)} \oplus \tilde{E}_k \tilde{D}_k)$.

**Property 3.17.** $\tilde{D}_m,k, \tilde{E}_m,k, \tilde{D}_k, \tilde{E}_k \in F_C$, $m = 1, 2, \ldots$, $k = 1, 2, \ldots, r$, and for each $k \in \{1, 2, \ldots, r\}$, $\lim_{m \to \infty} \tilde{D}_m,k = \tilde{D}_k$, $\lim_{m \to \infty} \tilde{E}_m,k = \tilde{E}_k$, $q^k \neq 0$. The sequence of the fuzzy vectors $\sum_{k=1}^{r} \oplus (q_1^{(k)} \oplus \tilde{E}_m,k \tilde{D}_m,k), m = 1, 2, \ldots$, converges to the fuzzy vector $\sum_{k=1}^{r} \oplus (q_1^{(k)} \oplus \tilde{E}_k \tilde{D}_k)$. 
**Proof.** Since $\sum_{k=1}^{r} \oplus (q_1^{(k)} \odot \tilde{D}_{m,k} \tilde{E}_{m,k}) = \sum_{k=1}^{r} \oplus (q_1^{(k)} \odot (\tilde{D}_{m,k} \cup \tilde{E}_{m,k}))$, $m = 1, 2, \ldots$, for each $k$, by Property 3.13(2°), $\lim_{m \to \infty} \tilde{D}_{m,k} \cup \tilde{E}_{m,k} = \tilde{D}_k \cup \tilde{E}_k$. By Property 3.13(1°), (3°), we have

$$\lim_{m \to \infty} \sum_{k=1}^{r} \oplus (q_1^{(k)} \odot (\tilde{D}_{m,k} \cup \tilde{E}_{m,k})) = \sum_{k=1}^{r} \oplus (q_1^{(k)} \odot (\tilde{D}_k \cup \tilde{E}_k)) \quad (3.25)$$

**Example 3.18.** Consider the fuzzy vectors $\lim_{m \to \infty} \tilde{Z}_m$ in Example 2.11. Let

$$\mu_{\tilde{Z}}(x, y) = \begin{cases} 1 - (x - 10)^2 - (y - 30)^2, & \text{if } (x - 10)^2 + (y - 30)^2 \leq 1, \\ 0, & \text{elsewhere}. \end{cases} \quad (3.26)$$

We will prove $\lim_{m \to \infty} \tilde{Z}_m = \tilde{Z}$. Since $C((10, 30), 1 + 1/m) \subset C((10, 30), 1 + 1/(m - 1))$ and for any $(x, y) \in \mathbb{R}^2$, the following holds:

$$\frac{1}{(1 + 1/m)^2} \left[ \left(1 + \frac{1}{m}\right)^2 - (x - 10)^2 - (y - 30)^2 \right] \leq \frac{1}{(1 + 1/(m - 1))^2} \left[ \left(1 + \frac{1}{m - 1}\right)^2 - (x - 10)^2 - (y - 30)^2 \right], \quad (3.27)$$

therefore, $\mu_{\tilde{Z}_m}(x, y) \leq \mu_{\tilde{Z}_{m-1}}(x, y)$ for all $(x, y) \in \mathbb{R}^2$, and hence $\tilde{Z}_1 \supset \tilde{Z}_2 \supset \cdots \supset \tilde{Z}_m \supset \cdots \supset \tilde{Z}$, and obviously, $\lim_{m \to \infty} \mu_{\tilde{Z}_m}(x, y) = \mu_{\tilde{Z}}(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Let $\tilde{Z}_m', \tilde{Z}'$ be the compliment fuzzy sets of $\tilde{Z}_m, \tilde{Z}$, respectively. We have $\lim_{m \to \infty} \mu_{\tilde{Z}_m'}(x, y) = \mu_{\tilde{Z}'}(x, y)$ for all $(x, y) \in \mathbb{R}^2$ and $\tilde{Z}_m' \subset \tilde{Z}_m \subset \cdots \subset \tilde{Z}_2 \subset \tilde{Z}_1 \subset \cdots \subset \tilde{Z}'. \quad$ By Property 3.7, $\lim_{m \to \infty} \tilde{Z} + m' = \tilde{Z}'$. Thus, $\lim_{m \to \infty} \tilde{Z}_m = \tilde{Z}$. Therefore, from Property 3.15, $\lim_{m \to \infty} \mu_{\tilde{Z}_m} = \mu_{\tilde{Z}}$. Thus, the membership function of $\tilde{Q}\tilde{Z}$ is

$$\mu_{\tilde{Q}\tilde{Z}}(x, y) = \mu_{\tilde{Q}\tilde{Z}}(x', y') = \sup_{x' = x(1)^{i(1)} - y(1)^{i(1)} \atop y' = x(2)^{i(2)} - y(2)^{i(2)}} \mu_{\tilde{Q}}(x(1)^{i(1)}, x(2)^{i(2)}) \wedge \mu_{\tilde{Q}}(y(1)^{i(1)}, y(2)^{i(2)})$$

$$= \mu_{\tilde{Q}}(x + 1, y + 2) = \begin{cases} 1 - (x - 9)^2 - (y - 28)^2, & \text{if } (x - 9)^2 + (y - 28)^2 \leq 1, \\ 0, & \text{elsewhere}. \end{cases} \quad (3.28)$$

In the crisp case, starting from $Q = (1, 2)$, aiming at $Z = (10, 30)$, we could have the vector $\tilde{Q}\tilde{Z} = (9, 28)$. The grade of membership of $\tilde{Q}\tilde{Z}$ which belongs to the fuzzy vector $\tilde{Q}\tilde{Z}$ is $\mu_{\tilde{Q}\tilde{Z}}(9, 28) = 1$, that is, the grade of membership function of the fuzzy vector $\tilde{P}\tilde{Z}$ for the crisp vector $\tilde{P}\tilde{S}$ is 1, and the point $R = (9.5, 29.5)$ is in the circle of center $(9, 28)$ and radius 1. The crisp vector of $Q$ to $\mathbb{R}$ is $\tilde{Q}R = (8.5, 27.5)$. The grade of membership
function of $\overrightarrow{QZ}$ is $\mu_{\overrightarrow{QZ}}(8.5, 27.5) = 0.5$, that is, the grade of membership function of the fuzzy vector $\overrightarrow{PZ}$ for the crisp vector $QR$ is 0.5.

References


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