ON JORDAN IDEALS AND LEFT (θ, θ) -DERIVATIONS IN PRIME RINGS

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Let R be a ring and S a nonempty subset of R . Suppose that θ and ϕ are endomorphisms of R . An additive mapping $\delta : R \rightarrow R$ is called a left (θ, ϕ) -derivation (resp., Jordan left (θ, ϕ) -derivation) on S if $\delta(xy) = \theta(x)\delta(y) + \phi(y)\delta(x)$ (resp., $\delta(x^2) = \theta(x)\delta(x) + \phi(x)\delta(x)$) holds for all $x, y \in S$. Suppose that J is a Jordan ideal and a subring of a 2-torsion-free prime ring R . In the present paper, it is shown that if θ is an automorphism of R such that $\delta(x^2) = 2\theta(x)\delta(x)$ holds for all $x \in J$, then either $J \subseteq Z(R)$ or $\delta(J) = (0)$. Further, a study of left (θ, θ) -derivations of a prime ring R has been made which acts either as a homomorphism or as an antihomomorphism of the ring R .

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1. Introduction. Throughout the present paper, R will denote an associative ring with centre $Z(R)$. We will write for all $x, y \in R$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product, respectively. A ring R is said to be prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$. A ring R is said to be 2-torsion-free if whenever $2a = 0$, with $a \in R$, then $a = 0$. An additive subgroup J of R is said to be a Jordan ideal of R if $u \circ r \in J$, for all $u \in J$ and $r \in R$. An additive mapping $d : R \rightarrow R$ is called a derivation (resp., Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp., $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. Let θ, ϕ be endomorphisms of R . An additive mapping $f : R \rightarrow R$ is called a (θ, ϕ) -derivation (resp., Jordan (θ, ϕ) -derivation) if $f(xy) = f(x)\theta(y) + \phi(x)f(y)$ (resp., $f(x^2) = f(x)\theta(x) + \phi(x)f(x)$) holds, for all $x, y \in R$. Of course a $(1, 1)$ -derivation (resp., a Jordan $(1, 1)$ -derivation) is a derivation (resp., a Jordan derivation) on R , where 1 is the identity mapping on R . We will make use of the following basic commutator identities without any specific mention:

$$[xy, z] = x[y, z] + [x, z]y, \quad [x, yz] = y[x, z] + [x, y]z. \quad (1.1)$$

An additive mapping $\delta : R \rightarrow R$ is called a left derivation (resp., Jordan left derivation) if $\delta(xy) = x\delta(y) + y\delta(x)$ (resp., $\delta(x^2) = 2x\delta(x)$) holds for all $x, y \in R$. In view of the definition of a (θ, ϕ) -derivation, the notion of left (θ, ϕ) -derivation can be extended as follows: let θ, ϕ be endomorphisms of R and let S be a nonempty subset of R . An additive mapping $\delta : R \rightarrow R$ is called a left (θ, ϕ) -derivation (resp., Jordan left (θ, ϕ) -derivation) on S if $\delta(xy) = \theta(x)\delta(y) + \phi(y)\delta(x)$ (resp., $\delta(x^2) = \theta(x)\delta(x) + \phi(x)\delta(x)$) holds for all $x, y \in S$. Clearly, a left $(1, 1)$ -derivation (resp., a Jordan left $(1, 1)$ -derivation) is a left derivation (resp., a Jordan left derivation) on R ,

where 1 is the identity mapping on R . In [5], Brešar and Vukman have proved that the existence of a nonzero Jordan left derivation on a prime ring R of $\text{char } R \neq 2, 3$ forces R to be commutative. It should be mentioned that the result obtained in [5] concerning Jordan left derivation has been improved by Deng [7]. Some more related results can be seen in [1, 3, 5, 7, 9]. It is easy to see that every left derivation on a ring R is a Jordan left derivation. However, in general, a Jordan left derivation need not be a left derivation. The following example justifies this statement.

EXAMPLE 1.1. Let R be a commutative ring and let $a \in R$ such that $xax = 0$ for all $x \in R$ but $xay \neq 0$, for some x and y , $x \neq y$. Define a map $\delta : R \rightarrow R$ as follows:

$$\delta(x) = xa + ax. \quad (1.2)$$

Then δ is a Jordan left derivation but not a left derivation.

In the present paper, first it is shown that every Jordan left (θ, θ) -derivation on a Jordan ideal J of a 2-torsion-free prime ring is a left (θ, θ) -derivation on J . Finally, we will study the behaviour of left (θ, θ) -derivation on a prime ring which also acts either as a homomorphism or an antihomomorphism of the underlying ring.

2. Preliminary results. We begin with the following lemmas which are essential in developing the proof of our main result.

LEMMA 2.1 [6, Lemma 4]. *Let G and H be additive groups and let R be a 2-torsion-free ring. Let $f : G \times G \rightarrow H$ and $g : G \times G \rightarrow R$ be biadditive mappings. Suppose that for each pair $a, b \in G$ either $f(a, b) = 0$ or $g(a, b)^2 = 0$. In this case, either $f = 0$ or $g(a, b)^2 = 0$ for all $a, b \in G$.*

If J is assumed to be a Jordan ideal and a subring of a ring R , then using similar techniques as used in the proofs of Lemmas 2.2 and 2.3 of [1], one can easily obtain the following lemma.

LEMMA 2.2. *Let R be a 2-torsion-free ring, let J be a Jordan ideal and a subring of R . If θ is an endomorphism of R and $\delta : R \rightarrow R$ is an additive mapping satisfying $\delta(u^2) = 2\theta(u)\delta(u)$, for all $u \in J$, then*

- (i) $\delta(uv + vu) = 2\theta(u)\delta(v) + 2\theta(v)\delta(u)$, for all $u, v \in J$,
- (ii) $\delta(uvu) = \theta(u^2)\delta(v) + 3\theta(u)\theta(v)\delta(u) - \theta(v)\theta(u)\delta(u)$, for all $u, v \in J$,
- (iii) $\delta(uvw + wvu) = (\theta(u)\theta(w) + \theta(w)\theta(u))\delta(v) + 3\theta(u)\theta(v)\delta(w) + 3\theta(w)\theta(v)\delta(u) - \theta(v)\theta(u)\delta(w) - \theta(v)\theta(w)\delta(u)$, for all $u, v, w \in J$,
- (iv) $[\theta(u), \theta(v)]\theta(u)\delta(u) = \theta(u)[\theta(u), \theta(v)]\delta(u)$, for all $u, v \in J$,
- (v) $[\theta(u), \theta(v)](\delta(uv) - \theta(u)\delta(v) - \theta(v)\delta(u)) = 0$, for all $u, v \in J$.

LEMMA 2.3. *Let R be a 2-torsion-free ring, J a Jordan ideal and a subring of R . If θ is an endomorphism of R and $\delta : R \rightarrow R$ is an additive mapping satisfying $\delta(u^2) = 2\theta(u)\delta(u)$ for all $u \in J$, then*

- (i) $[\theta(u), \theta(v)]\delta([u, v]) = 0$, for all $u, v \in J$;
- (ii) $(\theta(u^2)\theta(v) - 2\theta(u)\theta(v)\theta(u) + \theta(v)\theta(u^2))\delta(v) = 0$, for all $u, v \in J$.

We begin with the following lemma.

LEMMA 2.4. *If R is a ring and J a nonzero Jordan ideal of R , then $2[R, R]J \subseteq J$ and $2J[R, R] \subseteq J$.*

PROOF. Let $x, y \in R$ and $u \in J$. Then $u \circ [x, y] - (u \circ x) \circ y + (u \circ y) \circ x \in J$. This implies that $uxy - uyx + xyu - yxu - uxy - xuy - yux - yxu + uyx + yux + xuy + xyu \in J$ and hence $2[x, y]u \in J$, for all $x, y \in R$, and $u \in J$, that is, $2[R, R]J \subseteq J$.

Similarly, it is easy to see that $2u[x, y] = (u \circ y) \circ x - u \circ [x, y] - (u \circ x) \circ y \in J$, for all $x, y \in R$ and $u \in J$, and hence $2J[R, R] \subseteq J$. $\square$

LEMMA 2.5. *Let R be a prime ring and J a nonzero Jordan ideal of R . If $a \in R$ and $aJ = (0)$ (or $Ja = (0)$), then $a = 0$.*

PROOF. Since J is a Jordan ideal of R , $u \circ x \in J$, for all $x \in R$ and $u \in J$. By hypotheses, we have $a(u \circ x) = 0$, for all $x \in R$, $u \in J$, and hence we get $ax u = 0$, for all $x \in R$, $u \in J$, that is, $aRJ = (0)$. Since J is a nonzero Jordan ideal and R is prime, the above relation yields that $a = 0$.

If $Ja = (0)$, then using similar arguments with necessary variations, we get the required result. $\square$

LEMMA 2.6. *Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal of R . If $aJb = (0)$, then $a = 0$ or $b = 0$.*

PROOF. By Lemma 2.4, we find that $2[R, R]J \subseteq J$. Thus, for any $x, y \in R$ and $u \in J$, we have $2a[x, y]ub = 0$. This implies that

$$a[x, y]ub = 0, \quad \forall x, y \in R, u \in J. \quad (2.1)$$

Replacing y by ya in the above expression, we get $a[x, ya]ub = 0$, for all $x, y \in R$ and $u \in J$ or $ay[x, a]ub + a[x, y]aub = 0$. Now, using the fact that $aJb = (0)$, we find that $ay[x, a]ub = 0$, for all $x, y \in R$ and $u \in J$ and hence $aR[x, a]ub = (0)$. Thus, primeness of R forces that either $a = 0$ or $[x, a]ub = 0$. If $[x, a]ub = 0$, for all $x \in R$, $u \in J$, then by our hypotheses we have $axub = 0$, for all $x \in R$, $u \in J$, that is, $aRub = (0)$. Again, primeness of R gives that either $a = 0$ or $ub = 0$. If $ub = 0$, for all $u \in J$, then by Lemma 2.5, we get $b = 0$. $\square$

LEMMA 2.7. *Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal of R . If J is a commutative Jordan ideal, then $J \subseteq Z(R)$.*

PROOF. By Lemma 2.4, we have $2[R, R]J \subseteq J$. Thus, for any $x, y \in R$ and $u, v \in J$, we find that $2[x, y]u, v = 0$ and hence $2[[x, y], v]u + 2[x, y][u, v] = 0$, for all $x, y \in R$ and $u, v \in J$. By hypotheses, we obtain $[[x, y], v]u = 0$, for all $x, y \in R$ and $u, v \in J$. Using Lemma 2.5, we get $[[x, y], v] = 0$, for all $x, y \in R$ and $v \in J$. Now, replace y by xy to get $[x, v][x, y] = 0$, for all $x, y \in R$ and $v \in J$. Further replacing y by yv , we have $[x, v]y[x, v] = 0$, for all $x, y \in R$ and $v \in J$. This implies that $[x, v]R[x, v] = (0)$, for all $x \in R$, $v \in J$. Now, primeness of R forces that $v \in Z(R)$, for all $v \in J$. Hence, $J \subseteq Z(R)$. $\square$

The next lemma can be regarded as a generalization of a lemma due to Smiley [8] for Jordan ideals of a prime ring.

LEMMA 2.8. *Let R be a 2-torsion-free prime ring and let J be a Jordan ideal and a subring of R such that $[u, v]^2 = 0$, for all $u, v \in J$. Then J is commutative and hence central.*

PROOF. By hypothesis, we have $[u, v]^2 = 0$, for all $u, v \in J$. On linearizing, we get $[u, v][u, w] + [u, w][u, v] = 0$, for all $u, v, w \in J$. Replacing v by vu in the above expression and using it, we obtain $[u, v][u, [u, w]] = 0$, for all $u, v, w \in J$. Again, replacing v by vv_1 in latter relation, we find that $[u, v]v_1[u, [u, w]] = 0$, that is, $[u, v]J[u, [u, w]] = (0)$, for all $u, v, w \in J$. Thus by Lemma 2.6, we have for each $u \in J$ either $[u, v] = 0$ or $[u, [u, w]] = 0$, for all $u, v, w \in J$. If $[u, [u, w]] = 0$, for all $u, w \in J$, then on replacing w by wv , we get $[u, w][u, v] = 0$, for all $u, v, w \in J$. Again, replacing v by vw , we have $[u, w]v[u, w] = 0$, for all $u, v, w \in J$ and hence $[u, w]J[u, w] = (0)$, for all $w \in J$. Again, by Lemma 2.6, we obtain $[u, w] = 0$. Thus in both cases we find that $[u, w] = 0$, for all $u, w \in J$. Thus, J is commutative, and by Lemma 2.7, J is central, that is, $J \subseteq Z(R)$. $\square$

LEMMA 2.9. *Let R be a 2-torsion-free ring, J a Jordan ideal and a subring of R . If $\delta : R \rightarrow R$ is an additive mapping satisfying $\delta(u^2) = 2\theta(u)\delta(u)$, for all $u \in U$, then*

- (i) $\delta(u^2v) = \theta(u^2)\delta(v) + (\theta(u)\theta(v) + \theta(v)\theta(u))\delta(u) + \theta(u)\delta([u, v])$, for all $u, v \in J$,
- (ii) $\delta(vu^2) = \theta(u^2)\delta(v) + (3\theta(v)\theta(u) - \theta(u)\theta(v))\delta(u) - \theta(u)\delta([u, v])$, for all $u, v \in J$.

PROOF. (i) Replacing v by vu and uv in Lemma 2.2(i), we find that

$$\delta(uvu + vu^2) = 2(\theta(u)\delta(vu) + \theta(v)\theta(u)\delta(u)), \quad \forall u, v \in J, \quad (2.2)$$

$$\delta(u^2v + uvu) = 2(\theta(u)\delta(uv) + \theta(u)\theta(v)\delta(u)), \quad \forall u, v \in J. \quad (2.3)$$

Now, subtracting (2.2) from (2.3), we get

$$\delta(u^2v - vu^2) = 2(\theta(u)\delta([u, v]) + [\theta(u), \theta(v)]\delta(u)), \quad \forall u, v \in J. \quad (2.4)$$

Replacing u by u^2 in Lemma 2.2(i), we have

$$\begin{aligned} \delta(u^2v + vu^2) &= 2\theta(u^2)\delta(v) + 2\theta(v)u\delta(u^2) \\ &= 2\theta(u^2)\delta(v) + 4\theta(v)\theta(u)\delta(u), \quad \forall u, v \in J. \end{aligned} \quad (2.5)$$

Hence adding (2.4), (2.5) and using the fact that $\text{char } R \neq 2$, we obtain

$$\begin{aligned} \delta(u^2v) &= \theta(u^2)\delta(v) + (\theta(u)\theta(v) + \theta(v)\theta(u))\delta(u) \\ &\quad + \theta(u)\delta([u, v]), \quad \forall u, v \in J. \end{aligned} \quad (2.6)$$

(ii) As in the proof of the case (i), subtracting (2.4) from (2.5), we find that

$$\begin{aligned} \delta(vu^2) &= \theta(u^2)\delta(v) + (3\theta(v)\theta(u) - \theta(u)\theta(v))\delta(u) \\ &\quad - \theta(u)\delta([u, v]), \quad \forall u, v \in J. \end{aligned} \quad (2.7) \quad \square$$

3. Left derivation on Jordan ideal of a prime ring. In [3], there is a more general result which implies that in a 2-torsion-free prime ring R , the existence of a nonzero Jordan left derivation on a Lie ideal U of R forces that either $U \subseteq Z(R)$ or $\delta(U) = (0)$. In the present section, we attempt to generalize the above-mentioned result for Jordan left (θ, θ) -derivation which acts on a Jordan ideal of the ring.

THEOREM 3.1. *Let R be a 2-torsion-free prime ring and let J be a Jordan ideal and a subring of R . If θ is an automorphism of R and $\delta : R \rightarrow R$ is an additive mapping satisfying $\delta(u^2) = 2\theta(u)\delta(u)$, for all $u \in J$, then either $J \subseteq Z(R)$ or $\delta(J) = (0)$.*

PROOF. Suppose that $J \not\subseteq Z(R)$. By Lemma 2.2(iv), we have

$$[\theta(u), \theta(v)]\theta(u)\delta(u) = \theta(u)[\theta(u), \theta(v)]\delta(u), \quad \forall u, v \in J. \quad (3.1)$$

This implies that

$$(\theta(u^2)\theta(v) - 2\theta(u)\theta(v)\theta(u) + \theta(v)\theta(u^2))\delta(u) = 0, \quad \forall u, v \in J. \quad (3.2)$$

Replacing u by $[u, w]$ in (3.2), we get

$$\begin{aligned} &\theta([u, w]^2)\theta(v)\delta([u, w]) - 2\theta([u, w])\theta(v)\theta([u, w])\delta([u, w]) \\ &\quad + \theta(v)\theta([u, w]^2)\delta([u, w]) = 0, \end{aligned} \quad (3.3)$$

for all $u, v, w \in J$. Now, application of Lemma 2.3(i) yields that $\theta([u, w]^2)\theta(v)\delta([u, w]) = (0)$, for all $u, v, w \in J$. Since θ is an automorphism of R , the latter expression gives $[u, w]^2 J \theta^{-1}(\delta([u, w])) = (0)$. Hence, by Lemma 2.6, we find that for each pair $u, w \in J$, either $[u, w]^2 = 0$ or $\theta^{-1}(\delta([u, w])) = 0$. Note that the mappings $(u, w) \mapsto [u, w]$ and $(u, w) \mapsto \theta^{-1}(\delta([u, w]))$ satisfy the requirements of Lemma 2.1. Hence, either $[u, w]^2 = 0$, for all $u, w \in J$, or $\theta^{-1}(\delta([u, w])) = 0$, for all $u, w \in J$. If $[u, w]^2 = 0$, for all $u, w \in J$, then by application of Lemma 2.8, J is commutative and hence central, that is, $J \subseteq Z(R)$, a contradiction. Now, we consider the case $\theta^{-1}(\delta([u, w])) = 0$, then $\delta([u, w]) = 0$, that is, $\delta(uw) = \delta(wu)$, for all $u, w \in J$. In view of Lemma 2.2(i), we have

$$\begin{aligned} 2\delta((wu)u) &= \delta((wu)u + u(wu)) \\ &= 2\theta(w)\theta(u)\delta(u) + 2\theta(u)\delta(wu + uw) \\ &= 2\{\theta(u^2)\delta(w) + \theta(u)\theta(w)\delta(u) + \theta(w)\theta(u)\delta(u)\}, \quad \forall u, w \in J. \end{aligned} \quad (3.4)$$

Since R is 2-torsion-free, we get $\delta((wu)u) = \theta(u^2)\delta(w) + \theta(u)\theta(w)\delta(u) + \theta(w)\theta(u)\delta(u)$, for all $u, w \in J$. By Lemma 2.9(ii), we obtain $[\theta(u), \theta(w)]\delta(u) = 0$, for all $u, w \in J$. Replacing w by wv in the latter expression, we get $[\theta(u), \theta(w)]\theta(v)\delta(u) = 0$, that is, $[u, w]J\theta^{-1}(\delta(u)) = (0)$. Thus, by Lemma 2.6, we find that for each $u \in J$ either

$[u, w] = 0$ or $\theta^{-1}(\delta(u)) = 0$. Since θ is an automorphism, we have either $[u, w] = 0$ or $\delta(u) = 0$, for all $w \in J$. Now let $J_1 = \{u \in J \mid [u, w] = 0, \text{ for all } w \in J\}$ and $J_2 = \{u \in J \mid \delta(u) = 0\}$. Clearly, J_1 and J_2 are additive subgroups of J whose union is J . Hence, by Brauer's trick, either $J = J_1$ or $J = J_2$. If $J = J_1$, then $[u, w] = 0$, for all $u, w \in J$, that is, J is commutative, and hence by Lemma 2.7, $J \subseteq Z(R)$, again a contradiction. Hence, we have the remaining possibility that $\delta(u) = 0$, for all $u \in J$, that is, $\delta(J) = (0)$. This completes the proof of the theorem. $\square$

REMARK 3.2. In the hypotheses of the above theorem, if we assume only that J is a subring of R , then neither J is central nor $\delta(J) = (0)$. This is shown by the following example.

EXAMPLE 3.3. Let S be a ring such that the square of each element in S is zero, but the product of some elements in S is nonzero. Further, suppose that $R = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in S\}$. Consider $J = \{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in S\}$, then J is a subring of R . Define mappings $\delta : R \rightarrow R$ and $\theta : R \rightarrow R$ as follows:

$$\delta \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}, \quad \theta \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & -y \\ 0 & 0 \end{pmatrix}. \quad (3.5)$$

It is easy to verify that δ is a Jordan left (θ, θ) -derivation, but neither $J \subseteq Z(R)$ nor $\delta(J) = (0)$.

COROLLARY 3.4. Let R be a 2-torsion-free prime ring. If $\delta : R \rightarrow R$ is a nonzero additive mapping satisfying $\delta(x^2) = 2x\delta(x)$, for all $x \in R$, then R is commutative.

The following example demonstrates that to have R prime is essential in the hypothesis of the above result.

EXAMPLE 3.5. Consider a ring R , as in Example 3.3, and define mappings $\delta : R \rightarrow R$ and $\theta : R \rightarrow R$ as follows:

$$\delta \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -x \\ 0 & 0 \end{pmatrix}, \quad \theta \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & -y \\ 0 & 0 \end{pmatrix}. \quad (3.6)$$

Then, with $J = R$, it can be easily seen that $\delta(x^2) = 2\theta(x)\delta(x)$, for all $x \in R$, but R is not commutative.

4. Left derivation as a homomorphism or as an antihomomorphism. Let S be a nonempty subset of a ring R and d a derivation of R . If $d(xy) = d(x)d(y)$ (resp., $d(xy) = d(y)d(x)$) holds for all $x, y \in S$, then we say that d acts as a homomorphism (resp., antihomomorphism) on S .

In 1989, Bell and Kappe [4] proved that if d is a derivation of a prime ring R which acts as a homomorphism or as an antihomomorphism on a nonzero right ideal I of R , then $d = 0$ on R . Further, this result was extended for (θ, ϕ) -derivation in [2] as follows.

THEOREM 4.1. *Let R be a prime ring, I a nonzero right ideal of R , and let θ, ϕ be automorphisms of R . Suppose that $\delta : R \rightarrow R$ is a (θ, ϕ) -derivation of R .*

- (i) *If δ acts as a homomorphism on I , then $\delta = 0$ on R .*
- (ii) *If δ acts as an antihomomorphism on I , then $\delta = 0$ on R .*

In the present section, our objective is to extend the above result for left (θ, θ) -derivation of a prime ring R which acts as a homomorphism or as an antihomomorphism on a Jordan ideal J of R . In fact, we prove the following theorem.

THEOREM 4.2. *Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal and a subring of R . Suppose that θ is an automorphism of R and $\delta : R \rightarrow R$ is a left (θ, θ) -derivation of R .*

- (i) *If δ acts as a homomorphism on J , then $\delta = 0$ on R .*
- (ii) *If δ acts as an antihomomorphism on J , then $\delta = 0$ on R .*

PROOF. (i) By our hypotheses, we have

$$\delta(u)\delta(v) = \delta(uv) = \theta(u)\delta(v) + \theta(v)\delta(u), \quad \forall u, v \in J. \quad (4.1)$$

Replacing u by uv in (4.1), we find that

$$\delta(uv)\delta(v) = \theta(uv)\delta(v) + \theta(v)\delta(uv), \quad \forall u, v \in J. \quad (4.2)$$

Now, application of (4.1) yields that $\theta(u)\delta(v)\delta(v) = \theta(uv)\delta(v)$, for all $u, v \in J$. This implies that

$$\theta(u)(\delta(v) - \theta(v))\delta(v) = 0, \quad \forall u, v \in J. \quad (4.3)$$

Thus, $\theta(J)(\delta(v) - \theta(v))\delta(v) = (0)$, for all $v \in J$. Since θ is an automorphism and J is a nonzero Jordan ideal of R , $\theta(J)$ is also a nonzero Jordan ideal of R . Application of Lemma 2.6 yields that $(\delta(v) - \theta(v))\delta(v) = 0$, for all $v \in J$ and hence $\delta(v^2) = \theta(v)\delta(v)$, for all $v \in J$. Since δ is a left (θ, θ) -derivation, we have $\theta(v)\delta(v) = 0$, for all $v \in J$. On linearizing the latter relation, we find that

$$\theta(v)\delta(u) + \theta(u)\delta(v) = 0, \quad \forall u, v \in J. \quad (4.4)$$

Again, replacing u by vu in (4.4), we get $\theta(v)\theta(u)\delta(v) = 0$, for all $u, v \in J$, that is, $vJ\theta^{-1}(\delta(v)) = (0)$, for all $v \in J$. Application of Lemma 2.6 yields that either $v = 0$ or $\theta^{-1}(\delta(v)) = 0$. But $v = 0$ also gives that $\theta^{-1}(\delta(v)) = 0$, that is, $\delta(v) = 0$, for all $v \in J$. Further, replace v by $v \circ r$ to get $2\theta(v)\delta(r) = 0$, for all $v \in J$ and $r \in R$. Since R is 2-torsion-free and $\theta(J)$ is a nonzero Jordan ideal of R , application of Lemma 2.6 yields the required result.

(ii) If δ acts as an antihomomorphism on J , then

$$\begin{aligned} \delta(u)\delta(v) &= \delta(vu) = \theta(v)\delta(u) + \theta(u)\delta(v) \\ &= \theta(u)\delta(v) + \theta(v)\delta(u) = \delta(uv) = \delta(v)\delta(u), \end{aligned} \quad (4.5)$$

and hence δ also acts as a homomorphism on J . Therefore, in view of (i) we get the required result. $\square$

REMARK 4.3. We feel that [Theorem 3.1](#) (resp., [Theorem 4.2](#)) could be proved for Jordan left (θ, ϕ) -derivation (resp., left (θ, ϕ) -derivation) of a prime ring. However, we did not succeed to settle it.

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