CALCULATIONS ON SOME SEQUENCE SPACES

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We deal with space of sequences generalizing the well-known spaces $w_\infty^p(\lambda)$, $c_\infty(\lambda, \mu)$, replacing the operators $C(\lambda)$ and $\Delta(\mu)$ by their transposes. We get generalizations of results concerning the strong matrix domain of an infinite matrix A .

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1. Notations and preliminary results. For a given infinite matrix $A = (a_{nm})_{n,m \geq 1}$, the operators A_n are defined, for any integer $n \geq 1$, by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm} x_m, \quad (1.1)$$

where $X = (x_n)_{n \geq 1}$, the series intervening in the second member being convergent. So we are led to the study of the infinite linear system

$$A_n(X) = b_n, \quad n = 1, 2, \dots, \quad (1.2)$$

where $B = (b_n)_{n \geq 1}$ is a one-column matrix and X the unknown, see [1, 2, 3, 4, 5, 6, 7, 8, 10]. Equation (1.2) can be written in the form $AX = B$, where $AX = (A_n(X))_{n \geq 1}$. In this paper, we will also consider A an operator from a sequence space into another sequence space.

A Banach space E of complex sequences with the norm $\|\cdot\|_E$ is a BK space if each projection $P_n X = x_n$ is continuous for all $X \in E$. A BK space E is said to have AK, (see [12, 13]), if $B = \sum_{m=1}^{\infty} b_m e_m$, for every $B = (b_n)_{n \geq 1} \in E$, (with $e_n = (0, \dots, 1, \dots)$, 1 being in the n th position), that is,

$$\left\| \sum_{m=N+1}^{\infty} b_m e_m \right\|_E \rightarrow 0 \quad (n \rightarrow \infty). \quad (1.3)$$

We will write s for the set of all complex sequences, l_∞ , c , c_0 for the sets of bounded, convergent, and null sequences, respectively. We will denote by cs and l_1 the sets of convergent and absolutely convergent series, respectively.

In all that follows we will use the set

$$U^{+*} = \{(u_n)_{n \geq 1} \in s \mid u_n > 0 \ \forall n\}. \quad (1.4)$$

From Wilansky's notations [15], we define for any sequence

$$\alpha = (\alpha_n)_{n \geq 1} \in U^{+*}, \quad (1.5)$$

and for any set of sequences E , the set

$$\left(\frac{1}{\alpha}\right)^{-1} * E = \left\{ (x_n)_{n \geq 1} \in s \mid \left(\frac{x_n}{\alpha_n}\right)_n \in E \right\}. \quad (1.6)$$

We will write $\alpha * E$ instead of $(1/\alpha)^{-1} * E$ for short. So we get

$$\alpha * E = \begin{cases} s_\alpha^\circ & \text{if } E = c_0, \\ s_\alpha^{(c)} & \text{if } E = c, \\ s_\alpha & \text{if } E = l_\infty. \end{cases} \quad (1.7)$$

We have for instance

$$\alpha * c_0 = s_\alpha^\circ = \{(x_n)_{n \geq 1} \in s \mid x_n = o(\alpha_n) \text{ } n \rightarrow \infty\}. \quad (1.8)$$

Each of the spaces $\alpha * E$, where $E \in \{c_0, c, l_\infty\}$, is a BK space normed by

$$\|X\|_{s_\alpha} = \sup_{n \geq 1} \left(\frac{|x_n|}{\alpha_n} \right), \quad (1.9)$$

and s_α° has AK.

Now let $\alpha = (\alpha_n)_{n \geq 1}$ and $\beta = (\beta_n)_{n \geq 1} \in U^{+*}$. $S_{\alpha, \beta}$ is the set of infinite matrices $A = (a_{nm})_{n, m \geq 1}$ such that

$$(a_{nm} \alpha_m)_{m \geq 1} \in l^1 \quad \forall n \geq 1, \quad \sum_{m=1}^{\infty} (|a_{nm}| \alpha_m) = O(\beta_n) \quad (n \rightarrow \infty). \quad (1.10)$$

$S_{\alpha, \beta}$ is a Banach space with the norm

$$\|A\|_{S_{\alpha, \beta}} = \sup_{n \geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| \frac{\alpha_m}{\beta_n} \right). \quad (1.11)$$

Let E and F be any subsets of s . When A maps E into F , we will write $A \in (E, F)$, see [11]. So for every $X \in E$, $AX \in F$, ($AX \in F$ will mean that for each $n \geq 1$ the series defined by $y_n = \sum_{m=1}^{\infty} a_{nm} x_m$ is convergent and $(y_n)_{n \geq 1} \in F$). It has been proved in [9] that $A \in (s_\alpha, s_\beta)$ if and only if $A \in S_{\alpha, \beta}$. So we can write that $(s_\alpha, s_\beta) = S_{\alpha, \beta}$.

When $s_\alpha = s_\beta$, we obtain the unital Banach algebra $S_{\alpha, \beta} = S_\alpha$, (see [1, 2, 3, 5, 6, 10]) normed by $\|A\|_{S_\alpha} = \|A\|_{S_{\alpha, \alpha}}$.

We also have $A \in (s_\alpha, s_\alpha)$ if and only if $A \in S_\alpha$. If $\|I - A\|_{S_\alpha} < 1$, we will say that $A \in \Gamma_\alpha$. Since the set S_α is a unital algebra, we have the useful result that if $A \in \Gamma_\alpha$, A is bijective from s_α into itself.

If $\alpha = (r^n)_{n \geq 1}$, Γ_α , S_α , s_α , s_α° , and $s_\alpha^{(c)}$ are replaced by Γ_r , S_r , s_r , s_r° , and $s_r^{(c)}$, respectively, (see [1, 2, 3, 5, 6, 10]). When $r = 1$, we obtain $s_1 = l_\infty$, $s_1^\circ = c_0$, and $s_1^{(c)} = c$, and putting $e = (1, 1, \dots)$, we have $S_1 = S_e$. It is well known, see [11], that

$$(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1. \quad (1.12)$$

For any subset E of s , we put

$$AE = \{Y \in s \mid \exists X \in E, Y = AX\}. \quad (1.13)$$

If F is a subset of s , we will denote

$$F(A) = F_A = \{X \in s \mid Y = AX \in F\}. \quad (1.14)$$

We can see that $F(A) = A^{-1}F$.

2. Some properties of the operators Δ^+ and Σ^+ . Here we will deal with the operators represented by $C^+(\lambda)$ and $\Delta^+(\lambda)$.

Let

$$U = \{(u_n)_{n \geq 1} \in s \mid u_n \neq 0 \ \forall n\}. \quad (2.1)$$

We define $C(\lambda) = (c_{nm})_{n,m \geq 1}$, for $\lambda = (\lambda_n)_{n \geq 1} \in U$, by

$$c_{nm} = \begin{cases} \frac{1}{\lambda_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2)$$

So, we put $C^+(\lambda) = C(\lambda)^t$. It can be proved that the matrix $\Delta(\lambda) = (c'_{nm})_{n,m \geq 1}$ with

$$c'_{nm} = \begin{cases} \lambda_n & \text{if } m = n, \\ -\lambda_{n-1} & \text{if } m = n-1, \ n \geq 2, \\ 0 & \text{otherwise,} \end{cases} \quad (2.3)$$

is the inverse of $C(\lambda)$, see [12, 14]. Similarly, we put $\Delta^+(\lambda) = \Delta(\lambda)^t$. If $\lambda = e$, we get the well-known operator of first difference represented by $\Delta(e) = \Delta$ and it is usually written as $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$ and Σ belong to any given space S_R with $R > 1$. Writing $D_\lambda = (\lambda_n \delta_{nm})_{n,m \geq 1}$, (where $\delta_{nm} = 0$ for $n \neq m$ and $\delta_{nn} = 1$ otherwise), we have $\Delta^+(\lambda) = D_\lambda \Delta^+$. So for any given $\alpha \in U^{+*}$, we see that if $(\alpha_{n-1}/\alpha_n) |\lambda_n/\lambda_{n-1}| = O(1)$, then $\Delta^+(\lambda) \in (s_{(\alpha/|\lambda|)}, s_\alpha)$. Since $\text{Ker } \Delta^+(\lambda) \neq 0$, we are lead to define the set

$$s_\alpha^*(\Delta^+(\lambda)) = s_\alpha(\Delta^+(\lambda)) \bigcap s_{(\alpha/|\lambda|)} = \{X = (x_n)_{n \geq 1} \in s_{(\alpha/|\lambda|)} \mid \Delta^+(\lambda)X \in s_\alpha\}. \quad (2.4)$$

It can easily be seen that

$$s_{(\alpha/|\lambda|)}^*(\Delta^+(e)) = s_{(\alpha/|\lambda|)}^*(\Delta^+) = s_{\alpha}^*(\Delta^+(\lambda)). \quad (2.5)$$

2.1. Properties of the sequence $C(\alpha)\alpha$. We will use the following sets:

$$\begin{aligned} \widehat{C}_1 &= \left\{ \alpha \in U^{+*} \mid \frac{1}{\alpha_n} \left(\sum_{k=1}^n \alpha_k \right) = O(1) \ (n \rightarrow \infty) \right\}, \\ \widehat{C} &= \left\{ \alpha \in U^{+*} \mid \frac{1}{\alpha_n} \left(\sum_{k=1}^n \alpha_k \right) \in c \right\}, \\ \widehat{C}_1^+ &= \left\{ \alpha \in U^{+*} \cap cS \mid \frac{1}{\alpha_n} \left(\sum_{k=n}^{\infty} \alpha_k \right) = O(1) \ (n \rightarrow \infty) \right\}, \\ \Gamma &= \left\{ \alpha \in U^{+*} \mid \overline{\lim}_{n \rightarrow \infty} \left(\frac{\alpha_{n-1}}{\alpha_n} \right) < 1 \right\}, \\ \Gamma^+ &= \left\{ \alpha \in U^{+*} \mid \overline{\lim}_{n \rightarrow \infty} \left(\frac{\alpha_{n+1}}{\alpha_n} \right) < 1 \right\}. \end{aligned} \quad (2.6)$$

Note that $\alpha \in \Gamma^+$ if and only if $1/\alpha \in \Gamma$. We will see in [Proposition 2.1](#) that if $\alpha \in \widehat{C}_1$, α tends to infinity. On the other hand, we see that $\Delta \in \Gamma_{\alpha}$ implies $\alpha \in \Gamma$ and $\alpha \in \Gamma$ if and only if there is an integer $q \geq 1$ such that

$$y_q(\alpha) = \sup_{n \geq q+1} \left(\frac{\alpha_{n-1}}{\alpha_n} \right) < 1. \quad (2.7)$$

We obtain the following results in which we put $[C(\alpha)\alpha]_n = (\sum_{k=1}^n \alpha_k)/\alpha_n$.

PROPOSITION 2.1. *Let $\alpha \in U^{+*}$. Then*

- (i) $\alpha_{n-1}/\alpha_n \rightarrow 0$ if and only if $[C(\alpha)\alpha]_n \rightarrow 1$,
- (ii) (a) $\alpha \in \widehat{C}$ implies that $(\alpha_{n-1}/\alpha_n)_{n \geq 1} \in c$,
 (b) $[C(\alpha)\alpha]_n \rightarrow l$ implies that $\alpha_{n-1}/\alpha_n \rightarrow 1 - 1/l$,
- (iii) if $\alpha \in \widehat{C}_1$, there are $K > 0$ and $\gamma > 1$ such that

$$\alpha_n \geq K\gamma^n \quad \forall n, \quad (2.8)$$

- (iv) the condition $\alpha \in \Gamma$ implies that $\alpha \in \widehat{C}_1$ and there exists a real $b > 0$ such that

$$[C(\alpha)\alpha]_n \leq \frac{1}{1-\chi} + b\chi^n \quad \text{for } n \geq q+1, \chi = y_q(\alpha) \in]0, 1[, \quad (2.9)$$

- (v) the condition $\alpha \in \Gamma^+$ implies that $\alpha \in \widehat{C}_1^+$.

PROOF. Assume that $\alpha_{n-1}/\alpha_n \rightarrow 0$. Then there is an integer N such that

$$n \geq N+1 \Rightarrow \frac{\alpha_{n-1}}{\alpha_n} \leq \frac{1}{2}. \quad (2.10)$$

So there exists a real $K > 0$ such that $\alpha_n \geq K2^n$ for all n and

$$\frac{\alpha_k}{\alpha_n} = \frac{\alpha_k}{\alpha_{k+1}} \cdots \frac{\alpha_{n-1}}{\alpha_n} \leq \left(\frac{1}{2}\right)^{n-k} \quad \text{for } N \leq k \leq n-1. \quad (2.11)$$

Then

$$\frac{1}{\alpha_n} \left(\sum_{k=1}^{n-1} \alpha_k \right) = \frac{1}{\alpha_n} \left(\sum_{k=1}^{N-1} \alpha_k \right) + \sum_{k=N}^{n-1} \frac{\alpha_k}{\alpha_n} \leq \frac{1}{K2^n} \left(\sum_{k=1}^{N-1} \alpha_k \right) + \sum_{k=N}^{n-1} \left(\frac{1}{2} \right)^{n-k}, \quad (2.12)$$

and since $\sum_{k=N}^{n-1} (1/2)^{n-k} = 1 - (1/2)^{n-N} \rightarrow 1$, ($n \rightarrow \infty$), we deduce that

$$\frac{1}{\alpha_n} \left(\sum_{k=1}^{n-1} \alpha_k \right) = O(1) \quad (2.13)$$

and $([C(\alpha)\alpha]_n) \in l_\infty$. Using the identity

$$[C(\alpha)\alpha]_n = \frac{\alpha_1 + \cdots + \alpha_{n-1}}{\alpha_{n-1}} \frac{\alpha_{n-1}}{\alpha_n} + 1 = [C(\alpha)\alpha]_{n-1} \left(\frac{\alpha_{n-1}}{\alpha_n} \right) + 1, \quad (2.14)$$

we get $[C(\alpha)\alpha]_n \rightarrow 1$. This proves the necessity.

Conversely, if $[C(\alpha)\alpha]_n \rightarrow 1$, then

$$\frac{\alpha_{n-1}}{\alpha_n} = \frac{[C(\alpha)\alpha]_n - 1}{[C(\alpha)\alpha]_{n-1}} \rightarrow 0. \quad (2.15)$$

(ii) is a direct consequence of the identity (2.14).

(iii) We put $\Sigma_n = \sum_{k=1}^n \alpha_k$. Then for a real $M > 1$,

$$[C(\alpha)\alpha]_n = \frac{\Sigma_n}{\Sigma_n - \Sigma_{n-1}} \leq M \quad \forall n. \quad (2.16)$$

So $\Sigma_n \geq (M/(M-1))\Sigma_{n-1}$ and $\Sigma_n \geq (M/(M-1))^{n-1} \alpha_1 \forall n$. Therefore, from

$$\frac{\alpha_1}{\alpha_n} \left(\frac{M}{M-1} \right)^{n-1} \leq [C(\alpha)\alpha]_n = \frac{\Sigma_n}{\alpha_n} \leq M, \quad (2.17)$$

we conclude that $\alpha_n \geq K\gamma^n$ for all n , with $K = (M-1)\alpha_1/M^2$ and $\gamma = M/(M-1) > 1$.

(iv) If $\alpha \in \Gamma$, there is an integer $q \geq 1$ for which

$$k \geq q+1 \text{ implies } \frac{\alpha_{k-1}}{\alpha_k} \leq \chi < 1, \text{ with } \chi = \gamma_q(\alpha). \quad (2.18)$$

So there is a real $M' > 0$ for which

$$\alpha_n \geq \frac{M'}{\chi^n} \quad \forall n \geq q+1. \quad (2.19)$$

Writing $\sigma_{nq} = (1/\alpha_n)(\sum_{k=1}^q \alpha_k)$ and $d_n = [C(\alpha)\alpha]_n - \sigma_{nq}$, we get

$$d_n = \frac{1}{\alpha_n} \left(\sum_{k=q+1}^n \alpha_k \right) = 1 + \sum_{j=q+1}^{n-1} \left(\prod_{k=1}^{n-j} \frac{\alpha_{n-k}}{\alpha_{n-k+1}} \right) \leq \sum_{j=q+1}^n \chi^{n-j} \leq \frac{1}{1-\chi}. \quad (2.20)$$

Using (2.19), we get $\sigma_{nq} \leq (1/M')\chi^n(\sum_{k=1}^q \alpha_k)$. So

$$[C(\alpha)\alpha]_n \leq a + b\chi^n \quad (2.21)$$

with $a = 1/(1-\chi)$ and $b = (1/M')(\sum_{k=1}^q \alpha_k)$.

(v) If $\alpha \in \Gamma^+$, there are $\chi' \in]0, 1[$ and an integer $q' \geq 1$ such that

$$\frac{\alpha_k}{\alpha_{k-1}} \leq \chi' \quad \text{for } k \geq q'. \quad (2.22)$$

Then for every $n \geq q'$, we have

$$\frac{1}{\alpha_n} \left(\sum_{k=n}^{\infty} \alpha_k \right) = \sum_{k=n}^{\infty} \left(\frac{\alpha_k}{\alpha_n} \right) \leq 1 + \sum_{k=n+1}^{\infty} \prod_{i=0}^{k-n-1} \left(\frac{\alpha_{k-i}}{\alpha_{k-i-1}} \right) \leq \sum_{k=n}^{\infty} \chi'^{k-n} = O(1). \quad (2.23)$$

This gives the conclusion. $\square$

REMARK 2.2. Note that as a direct consequence of [Proposition 2.1](#), we have $\widehat{C}_1 \cap \widehat{C}_1^+ = \Gamma \cap \Gamma^+ = \phi$.

REMARK 2.3. The condition $\alpha \in \widehat{C}_1$ does not imply that $\alpha \in \Gamma$, see [\[8\]](#).

2.2. Some new properties of the operators Δ and Δ^+ . In the following we will use some lemmas, the next one is well known, see [\[15\]](#).

LEMMA 2.4. *The condition $A \in (c_0, c_0)$ is equivalent to*

$$A \in S_1, \quad \lim_n a_{nm} = 0 \quad \text{for each } m \geq 1. \quad (2.24)$$

LEMMA 2.5. *If Δ^+ is bijective from s_α into itself, then $\alpha \in cs$.*

PROOF. Assume that $\alpha \notin cs$, that is, $\sum_n \alpha_n = \infty$. Two cases are possible.

(1) $e \in \text{Ker } \Delta^+ \cap s_\alpha$. Then Δ^+ cannot be bijective from s_α into itself.

(2) $e \notin \text{Ker } \Delta^+ \cap s_\alpha$. Then $1/\alpha \notin s_1$ and there is a sequence of integers $(n_i)_i$ strictly increasing such that $1/\alpha_{n_i} \rightarrow \infty$. Assume that the equation $\Delta^+ X = \alpha$ has a solution

$X = (x_{n,0})_{n \geq 1}$ in s_α . Then there is a unique scalar x_1 such that

$$x_{n,0} = x_1 - \sum_{k=1}^{n-1} \alpha_k. \quad (2.25)$$

So

$$\left| \frac{x_{n_i,0}}{\alpha_{n_i}} \right| = \left| \frac{1}{\alpha_{n_i}} \left(x_1 - \sum_{k=1}^{n_i-1} \alpha_k \right) \right| \rightarrow \infty \quad \text{as } i \rightarrow \infty, \quad (2.26)$$

and $X \notin s_\alpha$, which is contradictory.

We conclude that each of the properties $e \in \text{Ker } \Delta^+ \cap s_\alpha$ and $e \notin \text{Ker } \Delta^+ \cap s_\alpha$ is impossible and Δ^+ is not bijective from s_α into itself. This proves the lemma. $\square$

LEMMA 2.6. *For every $X \in c_0$, $\Sigma^+(\Delta^+X) = X$ and for every $X \in cs$, $\Delta^+(\Sigma^+X) = X$.*

PROOF. It can easily be seen that

$$\begin{aligned} [\Sigma^+(\Delta^+X)]_n &= \sum_{m=n}^{\infty} (x_m - x_{m+1}) = x_n \quad \forall X \in c_0, \\ [\Delta^+(\Sigma^+X)]_n &= \sum_{m=n}^{\infty} x_m - \sum_{m=n+1}^{\infty} x_m = x_n \quad \forall X \in cs. \end{aligned} \quad (2.27)$$

We can assert the following result, in which we put $\alpha^+ = (\alpha_{n+1})_{n \geq 1}$ and $s_\alpha^{\circ*}(\Delta^+) = s_\alpha^\circ(\Delta^+) \cap s_\alpha^\circ$. Note that from (2.5) we have

$$s_\alpha^*(\Delta^+(e)) = s_\alpha^*(\Delta^+) = s_\alpha(\Delta^+) \cap s_\alpha. \quad (2.28)$$

$\square$

- THEOREM 2.7.** (i) (a) $s_\alpha(\Delta) = s_\alpha$ if and only if $\alpha \in \widehat{C}_1$,
 (b) $s_\alpha^\circ(\Delta) = s_\alpha^\circ$ if and only if $\alpha \in \widehat{C}_1$,
 (c) $s_\alpha^{(c)}(\Delta) = s_\alpha^{(c)}$ if and only if $\alpha \in \widehat{C}$.
 (ii) (a) $\alpha \in \widehat{C}_1^+$ if and only if $s_{\alpha^+}(\Delta^+) = s_\alpha$ and Δ^+ is surjective from s_α into s_{α^+} ,
 (b) $\alpha \in \widehat{C}_1^+$ if and only if $s_\alpha^*(\Delta^+) = s_\alpha$ and Δ^+ is bijective from s_α into s_α ,
 (c) $\alpha \in \widehat{C}_1^+$ implies that $s_\alpha^{\circ*}(\Delta^+) = s_\alpha^\circ$ and Δ^+ is bijective from s_α° into s_α° .
 (iii) $\alpha \in \widehat{C}_1^+$ if and only if $s_\alpha(\Sigma^+) = s_\alpha$ and $s_\alpha(\Sigma^+) = s_\alpha$ implies $s_\alpha^\circ(\Sigma^+) = s_\alpha^\circ$.

PROOF. (i) has been proved in [8].

(ii)(a) Sufficiency. If Δ^+ is surjective from s_α into s_{α^+} , then for every $B \in s_{\alpha^+}$ the solutions of $\Delta^+X = B$ in s_α are given by

$$x_{n+1} = x_1 - \sum_{k=1}^n b_k \quad n = 1, 2, \dots, \quad (2.29)$$

where x_1 is arbitrary. If we take $B = \alpha^+$, we get $x_n = x_1 - \sum_{k=2}^n \alpha_k$. So

$$\frac{x_n}{\alpha_n} = \frac{x_1}{\alpha_n} - \frac{1}{\alpha_n} \left(\sum_{k=2}^n \alpha_k \right) = O(1). \quad (2.30)$$

Taking $x_1 = -\alpha_1$, we conclude that $(\sum_{k=1}^{n-1} \alpha_k)/\alpha_n = O(1)$ and $\alpha \in \widehat{C}_1$.

Conversely, assume that $\alpha \in \widehat{C}_1$. From the inequality

$$\frac{\alpha_{n-1}}{\alpha_n} \leq \frac{1}{\alpha_n} \left(\sum_{k=1}^n \alpha_k \right) = O(1), \quad (2.31)$$

we deduce that $\alpha_{n-1}/\alpha_n = O(1)$ and $\Delta^+ \in (s_\alpha, s_{\alpha^+})$. Then for any given $B \in s_{\alpha^+}$, the solutions of the equation $\Delta^+ X = B$ are given by $x_1 = -u$ and

$$-x_n = u + \sum_{k=1}^{n-1} b_k \quad \text{for } n \geq 2, \quad (2.32)$$

where u is an arbitrary scalar. So there exists a real $K > 0$ such that

$$\frac{|x_n|}{\alpha_n} = \frac{|u + \sum_{k=1}^{n-1} b_k|}{\alpha_n} \leq \frac{|u| + K(\sum_{k=2}^n \alpha_k)}{\alpha_n} = O(1) \quad (2.33)$$

and $X \in s_\alpha$. We conclude that Δ^+ is surjective from s_α into s_{α^+} .

(ii)(b) Necessity. Assume that $\alpha \in \widehat{C}_1^+$. Then $\Delta^+ \in (s_\alpha, s_\alpha)$, since

$$\frac{\alpha_{n+1}}{\alpha_n} \leq \frac{1}{\alpha_n} \left(\sum_{k=n}^{\infty} \alpha_k \right) = O(1) \quad (n \rightarrow \infty). \quad (2.34)$$

Further, from $s_\alpha \subset cs$, we deduce, using [Lemma 2.4](#), that for any given $B \in s_\alpha$,

$$\Delta^+(\Sigma^+ B) = B. \quad (2.35)$$

On the other hand, $\Sigma^+ B = (\sum_{k=n}^{\infty} b_k)_{n \geq 1} \in s_\alpha$, since $\alpha \in \widehat{C}_1^+$. So Δ^+ is surjective from s_α into s_α . Finally, Δ^+ is injective because the equation

$$\Delta^+ X = O \quad (2.36)$$

admits the unique solution $X = O$ in s_α , since

$$\text{Ker } \Delta^+ = \{ue^t \mid u \in C\} \quad (2.37)$$

and $e^t \notin s_\alpha$.

Sufficiency. For every $B \in s_\alpha$, the equation $\Delta^+ X = B$ admits a unique solution in s_α . Then from [Lemma 2.5](#), $\alpha \in cs$ and since $s_\alpha \subset cs$, we deduce from [Lemma 2.6](#) that $X = \Sigma^+ B \in s_\alpha$ is the unique solution of $\Delta^+ X = B$. Taking $B = \alpha$, we get $\Sigma^+ \alpha \in s_\alpha$, that is, $\alpha \in \widehat{C}_1^+$.

(ii)(c) If $\alpha \in \widehat{C}_1^+$, Δ^+ is bijective from s_α° into itself. Indeed, we have $D_{1/\alpha} \Delta^+ D_\alpha \in (c_0, c_0)$ from (2.34) and [Lemma 2.4](#). Furthermore, since $\alpha \in \widehat{C}_1^+$ we have $s_\alpha^\circ \subset cs$ and for every $B \in s_\alpha^\circ$,

$$\Delta^+(\Sigma^+ B) = B. \quad (2.38)$$

From [Lemma 2.4](#), we have $\Sigma^+ \in (s_\alpha^\circ, s_\alpha^\circ)$, so the equation $\Delta^+ X = B$ admits the solution $X_0 = \Sigma^+ B$ in s_α° and we have proved that Δ^+ is surjective from s_α° into itself. Finally, $\alpha \in \widehat{C}_1^+$ implies that $e^t \notin s_\alpha^\circ$, so $\text{Ker } \Delta^+ \cap s_\alpha^\circ = \{0\}$ and we conclude that Δ^+ is bijective from s_α° into itself.

(iii) comes from (ii), since $\alpha \in \widehat{C}_1^+$ if and only if Δ^+ is bijective from s_α into itself and

$$\Sigma^+(\Delta^+ X) = \Delta^+(\Sigma^+ X) = X \quad \forall X \in s_\alpha. \quad (2.39)$$

□

As a direct consequence of [Theorem 2.7](#) we obtain the following results.

COROLLARY 2.8. *Let R be any real > 0 . Then*

$$R > 1 \iff s_R(\Delta) = s_R \iff s_R^\circ(\Delta) = s_R^\circ \iff s_R(\Delta^+) = s_R. \quad (2.40)$$

PROOF. From (i) and (ii) in [Theorem 2.7](#), we see that it is enough to prove that $\alpha = (R^n)_{n \geq 1} \in \widehat{C}_1$ if and only if $R > 1$. We have $(R^n)_{n \geq 1} \in \widehat{C}_1$ if and only if $R \neq 1$ and

$$R^{-n} \left(\sum_{k=1}^n R^k \right) = \frac{1}{1-R} R^{-n+1} - \frac{R}{1-R} = O(1) \quad \text{as } n \rightarrow \infty. \quad (2.41)$$

This means that $R > 1$ and the corollary is proved. □

Using the notation $\alpha^- = (1, \alpha_1, \alpha_2, \dots, \alpha_{n-1}, \dots)$ we get the next result.

COROLLARY 2.9. *Let $\alpha \in U^{+*}$ and $\mu \in U$. Then*

(i) $\alpha/|\mu| \in \widehat{C}_1$ if and only if

$$s_\alpha(\Delta^+(\mu)) = s_{(\alpha/|\mu|)^-}, \quad (2.42)$$

(ii) $\alpha/|\mu| \in \widehat{C}_1^+$ if and only if

$$s_\alpha^*(\Delta^+(\mu)) = s_{(\alpha/|\mu|)}. \quad (2.43)$$

PROOF. First we have

$$s_\alpha(\Delta^+(\mu)) = s_{(\alpha/|\mu|)}(\Delta^+). \quad (2.44)$$

Indeed,

$$X \in s_\alpha(\Delta^+(\mu)) \iff D_\mu \Delta^+ X \in s_\alpha \iff \Delta^+ X \in s_{(\alpha/|\mu|)} \iff X \in s_{(\alpha/|\mu|)}(\Delta^+). \quad (2.45)$$

Now, if $\alpha/|\mu| \in \widehat{C}_1$, from (i) in [Theorem 2.7](#), we have $s_{(\alpha/|\mu|)}(\Delta^+) = s_{(\alpha/|\mu|)^-}$ and $s_\alpha(\Delta^+(\mu)) = s_{(\alpha/|\mu|)^-}$. Conversely, assume $s_\alpha(\Delta^+(\mu)) = s_{(\alpha/|\mu|)^-}$. Reasoning as above, we get $s_{(\alpha/|\mu|)}(\Delta^+) = s_{(\alpha/|\mu|)^-}$, and using (i) in [Theorem 2.7](#) we conclude that $\alpha/|\mu| \in \widehat{C}_1$ and (i) holds.

(ii) $\alpha/|\mu| \in \widehat{C}_1^+$ implies that Δ^+ is bijective from $s_{(\alpha/|\mu|)}$ into itself. Thus

$$s_\alpha^*(\Delta^+(\mu)) = s_{(\alpha/|\mu|)}^*(\Delta^+) = s_{(\alpha/|\mu|)}. \quad (2.46)$$

This proves the necessity. Conversely, assume that $s_\alpha^*(\Delta^+(\mu)) = s_{(\alpha/|\mu|)}$. Then $s_{(\alpha/|\mu|)}^*(\Delta^+) = s_{(\alpha/|\mu|)}$ and from [Theorem 2.7\(ii\)\(b\)](#), $\alpha/|\mu| \in \widehat{C}_1^+$ and (ii) holds. $\square$

2.3. Spaces $w_\alpha^p(\lambda)$ and $w_\alpha^{+p}(\lambda)$ for given real $p > 0$. Here we will define sets generalizing the well-known sets

$$\begin{aligned} w_\infty^p(\lambda) &= \{X \in s \mid C(\lambda)(|X|^p) \in l_\infty\}, \\ w_0^p(\lambda) &= \{X \in s \mid C(\lambda)(|X|^p) \in c_0\}, \end{aligned} \quad (2.47)$$

see [\[9, 12, 13, 14, 15\]](#). It is proved that each of the sets $w_0^p = w_0^p((n)_n)$ and $w_\infty^p = w_\infty^p((n)_n)$ is a p -normed FK space for $0 < p < 1$ (i.e., a complete linear metric space for which each projection P_n is continuous) and a BK space for $1 \leq p < \infty$ with respect to the norm

$$\|X\| = \begin{cases} \sup_{v \geq 1} \left(\frac{1}{2^v} \left(\sum_{n=2^v}^{2^{v+1}-1} |x_n|^p \right) \right) & \text{if } 0 < p < 1, \\ \sup_{v \geq 1} \left(\frac{1}{2^v} \left(\sum_{n=2^v}^{2^{v+1}-1} |x_n|^p \right) \right)^{1/p} & \text{if } 1 \leq p < \infty. \end{cases} \quad (2.48)$$

The set w_0^p has the property AK, (i.e., every $X = (x_n)_{n \geq 1} \in w_0^p$ has a unique representation $X = \sum_{n=1}^\infty x_n e_n^t$) and every sequence $X = (x_n)_{n \geq 1} \in w^p$ has a unique representation

$$X = l e^t + \sum_{n=1}^\infty (x_n - l) e_n^t, \quad (2.49)$$

where $l \in C$ is such that $X - l e^t \in w_0^p$, (see [\[4\]](#)). Now, let $\alpha \in U^{+*}$ and $\lambda \in U^{+*}$. We have

$$\begin{aligned} w_\alpha^p(\lambda) &= \{X \in s \mid C(\lambda)(|X|^p) \in s_\alpha\}, \\ w_\alpha^{+p}(\lambda) &= \{X \in s \mid C^+(\lambda)(|X|^p) \in s_\alpha\}, \\ w_\alpha^{\circ p}(\lambda) &= \{X \in s \mid C(\lambda)(|X|^p) \in s_\alpha^\circ\}, \\ w_\alpha^{\circ +p}(\lambda) &= \{X \in s \mid C^+(\lambda)(|X|^p) \in s_\alpha^\circ\}. \end{aligned} \quad (2.50)$$

We deduce from the previous section the following theorem.

THEOREM 2.10. (i) (a) *The condition $\alpha \in \widehat{C_1^+}$ is equivalent to*

$$w_{\alpha}^{+p}(\lambda) = s_{(\alpha\lambda)^{1/p}}. \quad (2.51)$$

(b) *If $\alpha \in \widehat{C_1^+}$, then*

$$w_{\alpha}^{\circ p}(\lambda) = s_{(\alpha\lambda)^{1/p}}^{\circ}. \quad (2.52)$$

(ii) (a) *The condition $\alpha\lambda \in \widehat{C_1}$ is equivalent to*

$$w_{\alpha}^p(\lambda) = s_{(\alpha\lambda)^{1/p}}. \quad (2.53)$$

(b) *If $\alpha\lambda \in \widehat{C_1}$, then*

$$w_{\alpha}^{\circ p}(\lambda) = s_{(\alpha\lambda)^{1/p}}^{\circ}. \quad (2.54)$$

PROOF. Assume that $\alpha \in \widehat{C_1^+}$. Since $C^+(\lambda) = \Sigma^+ D_{1/\lambda}$, we have

$$w_{\alpha}^{+p}(\lambda) = \{X \mid (\Sigma^+ D_{1/\lambda})(|X|^p) \in s_{\alpha}\} = \{X \mid D_{1/\lambda}(|X|^p) \in s_{\alpha}(\Sigma^+)\}, \quad (2.55)$$

and since $\alpha \in \widehat{C_1^+}$ implies $s_{\alpha}(\Sigma^+) = s_{\alpha}$, we conclude that

$$w_{\alpha}^{+p}(\lambda) = \{X \mid |X|^p \in D_{\lambda} s_{\alpha} = s_{\alpha\lambda}\} = s_{(\alpha\lambda)^{1/p}}. \quad (2.56)$$

Conversely, we have $(\alpha\lambda)^{1/p} \in s_{(\alpha\lambda)^{1/p}} = w_{\alpha}^{+p}(\lambda)$. So

$$C^+(\lambda)[(\alpha\lambda)^{1/p}]^p = \left(\sum_{k=n}^{\infty} \frac{\alpha_k \lambda_k}{\lambda_k} \right)_{n \geq 1} \in s_{\alpha}, \quad (2.57)$$

that is, $\alpha \in \widehat{C_1^+}$ and we have proved (i). We obtain (i)(b) by reasoning as above.

(ii) Assume that $\alpha\lambda \in \widehat{C_1}$. Then

$$w_{\alpha}^p(\lambda) = \{X \mid |X|^p \in \Delta(\lambda) s_{\alpha}\}. \quad (2.58)$$

Since $\Delta(\lambda) = \Delta D_\lambda$, we get $\Delta(\lambda)s_\alpha = \Delta s_{\alpha\lambda}$. Now, from $\alpha\lambda \in \widehat{C}_1$ we deduce that Δ is bijective from $s_{\alpha\lambda}$ into itself and $w_\alpha^p(\lambda) = s_{(\alpha\lambda)^{1/p}}$. Conversely, assume that $w_\alpha^p(\lambda) = s_{(\alpha\lambda)^{1/p}}$. Then $(\alpha\lambda)^{1/p} \in s_{(\alpha\lambda)^{1/p}}$ implies that

$$C(\lambda)(\alpha\lambda) \in s_\alpha, \quad (2.59)$$

and since $D_{1/\alpha}C(\lambda)(\alpha\lambda) \in s_1 = l_\infty$, we conclude that $C(\alpha\lambda)(\alpha\lambda) \in l_\infty$. The proof of (ii)(b) follows the same lines as in the proof of the necessity in (ii) replacing $s_{\alpha\lambda}$ by $s_{\alpha\lambda}^\circ$. $\square$

3. New sets of sequences of the form $[A_1, A_2]$. In this section, we will deal with the sets

$$[A_1(\lambda), A_2(\mu)] = \{X \in s \mid A_1(\lambda)(|A_2(\mu)X|) \in s_\alpha\}, \quad (3.1)$$

where A_1 and A_2 are of the form $C(\xi)$, $C^+(\xi)$, $\Delta(\xi)$, or $\Delta^+(\xi)$ and we give necessary conditions to get $[A_1(\lambda), A_2(\mu)]$ in the form s_γ .

Let λ and $\mu \in U^{+*}$. For simplification, we will write throughout this section

$$[A_1, A_2] = [A_1(\lambda), A_2(\mu)] = \{X \in s \mid A_1(\lambda)(|A_2(\mu)X|) \in s_\alpha\} \quad (3.2)$$

for any matrices

$$\begin{aligned} A_1(\lambda) &\in \{\Delta(\lambda), \Delta^+(\lambda), C(\lambda), C^+(\lambda)\}, \\ A_2(\mu) &\in \{\Delta(\mu), \Delta^+(\mu), C(\mu), C^+(\mu)\}. \end{aligned} \quad (3.3)$$

So we have for instance

$$[C, \Delta] = \{X \in s \mid C(\lambda)(|\Delta(\mu)X|) \in s_\alpha\} = (w_\alpha(\lambda))_{\Delta(\mu)}, \dots \quad (3.4)$$

In all that follows, the conditions $\xi \in \Gamma$, or $1/\eta \in \Gamma$ for any given sequences ξ and η can be replaced by the conditions $\xi \in \widehat{C}_1$ and $\eta \in \widehat{C}_1^+$.

3.1. Spaces $[C, C]$, $[C, \Delta]$, $[\Delta, C]$, and $[\Delta, \Delta]$. For the convenience of the reader we will write the following identities, where $A_1(\lambda)$ and $A_2(\mu)$ are lower triangles and we will use the convention $\mu_0 = 0$:

$$\begin{aligned} [C, C] &= \left\{ X \in s \mid \frac{1}{\lambda_n} \left(\sum_{m=1}^n \left| \frac{1}{\mu_m} \left(\sum_{k=1}^m x_k \right) \right| \right) = \alpha_n O(1) \right\}, \\ [C, \Delta] &= \left\{ X \in s \mid \frac{1}{\lambda_n} \left(\sum_{k=1}^n |\mu_k x_k - \mu_{k-1} x_{k-1}| \right) = \alpha_n O(1) \right\}, \\ [\Delta, C] &= \left\{ X \in s \mid -\lambda_{n-1} \left| \frac{1}{\mu_{n-1}} \left(\sum_{k=1}^{n-1} x_i \right) \right| + \lambda_n \left| \frac{1}{\mu_n} \left(\sum_{k=1}^n x_i \right) \right| = \alpha_n O(1) \right\}, \\ [\Delta, \Delta] &= \{X \in s \mid -\lambda_{n-1} |\mu_{n-1} x_{n-1} - \mu_{n-2} x_{n-2}| + \lambda_n |\mu_n x_n - \mu_{n-1} x_{n-1}| = \alpha_n O(1)\}. \end{aligned} \quad (3.5)$$

Note that for $\alpha = e$ and $\lambda = \mu$, $[C, \Delta]$ is the well-known set of sequences that are strongly bounded, denoted by $c_\infty(\lambda)$, see [9, 12, 13, 14, 15]. We get the following result.

THEOREM 3.1. (i) If $\alpha\lambda$ and $\alpha\lambda\mu \in \Gamma$, then

$$[C, C] = s_{(\alpha\lambda\mu)}, \quad (3.6)$$

(ii) if $\alpha\lambda \in \Gamma$, then

$$[C, \Delta] = s_{(\alpha(\lambda/\mu))}, \quad (3.7)$$

(iii) if α and $\alpha\mu/\lambda \in \Gamma$, then

$$[\Delta, C] = s_{(\alpha(\mu/\lambda))}, \quad (3.8)$$

(iv) if α and $\alpha/\lambda \in \Gamma$, then

$$[\Delta, \Delta] = s_{(\alpha(\mu/\lambda))}. \quad (3.9)$$

PROOF. We have for any given X

$$C(\lambda)(|C(\mu)X|) \in s_\alpha \quad (3.10)$$

if and only if $C(\mu)X \in s_\alpha(C(\lambda)) = s_{(\alpha\lambda)}$, since $\alpha\lambda \in \Gamma$. So we get

$$X \in \Delta(\mu)s_{\alpha\lambda} \quad (3.11)$$

and the condition $\alpha\lambda\mu \in \Gamma$ implies $\Delta(\mu)s_{\alpha\lambda} = s_{(\alpha\lambda\mu)}$, which permits us to conclude (i).

(ii) Now, for any given X , the condition $C(\lambda)(|\Delta(\mu)X|) \in s_\alpha$ is equivalent to

$$|\Delta(\mu)X| \in \Delta(\lambda)s_\alpha = \Delta s_{\alpha\lambda} = s_{\alpha\lambda}, \quad (3.12)$$

since $\alpha\lambda \in \Gamma$. Thus

$$X \in C(\mu)s_{\alpha\lambda} = D_{1/\mu}\Sigma s_{\alpha\lambda} = s_{(\alpha(\lambda/\mu))}. \quad (3.13)$$

(iii) Similarly, $\Delta(\lambda)(|C(\mu)X|) \in s_\alpha$ if and only if

$$|C(\mu)X| \in s_\alpha(\Delta(\lambda)) = C(\lambda)s_\alpha = D_{1/\lambda}\Sigma s_\alpha = s_{(\alpha/\lambda)}, \quad (3.14)$$

since $\alpha \in \Gamma$. So

$$X \in \Delta(\mu)s_{(\alpha/\lambda)} = \Delta s_{(\alpha\mu/\lambda)}. \quad (3.15)$$

We conclude since $\alpha\mu/\lambda \in \Gamma$ implies that $\Delta s_{(\alpha\mu/\lambda)} = s_{(\alpha\mu/\lambda)}$. (iv) Here,

$$\Delta(\lambda)(|\Delta(\mu)X|) \in s_\alpha \quad \text{if and only if} \quad \Delta(\mu)X \in C(\lambda)s_\alpha = s_{(\alpha/\lambda)}, \quad (3.16)$$

if $\alpha \in \Gamma$. Thus we have

$$X \in C(\mu)s_{(\alpha/\lambda)} = s_{(\alpha/\lambda\mu)} \quad (3.17)$$

since $\alpha/\lambda \in \Gamma$. So (iv) holds. $\square$

REMARK 3.2. If we define

$$[A_1, A_2]_0 = \{X \in s \mid A_1(\lambda)(|A_2(\mu)X|) \in s_\alpha^\circ\}, \quad (3.18)$$

we get the same results as in [Theorem 3.1](#), replacing in each case (i), (ii), (iii), and (iv) s_ξ by s_ξ° .

3.2. Sets $[\Delta, \Delta^+]$, $[\Delta, C^+]$, $[C, \Delta^+]$, $[\Delta^+ \Delta]$, $[\Delta^+, C]$, $[\Delta^+ \Delta^+]$, $[C^+, C]$, $[C^+, \Delta]$, $[C^+, \Delta^+]$, and $[C^+, C^+]$. We get immediately from the definitions of the operators $\Delta(\xi)$, $\Delta^+(\eta)$, $C(\xi)$, and $C^+(\eta)$, the following:

$$\begin{aligned} [\Delta, \Delta^+] &= \{X \mid \lambda_n |\mu_n x_n - \mu_{n+1} x_{n+1}| - \lambda_{n-1} |\mu_{n-1} x_{n-1} - \mu_n x_n| = \alpha_n O(1)\}, \\ [\Delta, C^+] &= \left\{X \mid \lambda_n \left| \sum_{i=n}^{\infty} \frac{x_i}{\mu_i} \right| - \lambda_{n-1} \left| \sum_{i=n-1}^{\infty} \frac{x_i}{\mu_i} \right| = \alpha_n O(1) \right\}, \\ [C, \Delta^+] &= \left\{X \mid \frac{1}{\lambda_n} \left(\sum_{k=1}^n |\mu_k x_k - \mu_{k+1} x_{k+1}| \right) = \alpha_n O(1) \right\}, \\ [\Delta^+, \Delta] &= \{X \mid \lambda_n |\mu_n x_n - \mu_{n+1} x_{n+1}| - \lambda_{n+1} |\mu_{n+1} x_{n+1} - \mu_{n+2} x_{n+2}| = \alpha_n O(1)\}, \\ [\Delta^+, C] &= \left\{X \mid \frac{\lambda_n}{\mu_n} \left| \sum_{i=1}^n x_i \right| - \frac{\lambda_{n+1}}{\mu_{n+1}} \left| \sum_{i=1}^{n+1} x_i \right| = \alpha_n O(1) \right\}, \\ [\Delta^+, \Delta^+] &= \{X \mid \lambda_n |\mu_n x_n - \mu_{n+1} x_{n+1}| - \lambda_{n+1} |\mu_{n+1} x_{n+1} - \mu_{n+2} x_{n+2}| = \alpha_n O(1)\}, \\ [C^+, C] &= \left\{X \mid \sum_{k=n}^{\infty} \left(\frac{1}{\lambda_k} \left| \frac{1}{\mu_k} \sum_{i=1}^k x_i \right| \right) = \alpha_n O(1) \right\}, \\ [C^+, \Delta] &= \left\{X \mid \sum_{k=n}^{\infty} \left(\frac{1}{\lambda_k} |\mu_k x_k - \mu_{k-1} x_{k-1}| \right) = \alpha_n O(1) \right\}, \\ [C^+, \Delta^+] &= \left\{X \mid \sum_{k=n}^{\infty} \left(\frac{1}{\lambda_k} |\mu_k x_k - \mu_{k+1} x_{k+1}| \right) = \alpha_n O(1) \right\}, \\ [C^+, C^+] &= \left\{X \mid \sum_{k=n}^{\infty} \left(\frac{1}{\lambda_k} \left| \sum_{i=k}^{\infty} \frac{x_i}{\mu_i} \right| \right) = \alpha_n O(1) \right\}. \end{aligned} \quad (3.19)$$

We can assert the following result, in which we do the convention $\alpha_n = 1$ for $n \leq 0$.

THEOREM 3.3. (i) Assume that $\alpha \in \Gamma$. Then

$$\begin{aligned} [\Delta, \Delta^+] &= S_{(\alpha/\lambda\mu)^-} \quad \text{if } \frac{\alpha}{\lambda\mu} \in \Gamma, \\ [\Delta, C^+] &= S_{(\alpha(\mu/\lambda))} \quad \text{if } \frac{\lambda}{\alpha} \in \Gamma. \end{aligned} \quad (3.20)$$

(ii) The conditions $\alpha\lambda \in \Gamma$ and $\alpha\lambda/\mu \in \Gamma$ together imply

$$[C, \Delta^+] = S_{(\alpha(\lambda/\mu))^-}. \quad (3.21)$$

(iii) The condition $\alpha/\lambda \in \Gamma$ implies

$$[\Delta^+, \Delta] = s_{(\alpha_{n-1}/\mu_n \lambda_{n-1})_n} = s_{(1/\mu(\alpha/\lambda)^-)} \quad (3.22)$$

(iv) If α/λ and $\mu(\alpha/\lambda)^- = (\mu_n(\alpha_{n-1}/\lambda_{n-1}))_n \in \Gamma$, then

$$[\Delta^+, C] = s_{\mu(\alpha/\lambda)^-} \quad (3.23)$$

(v) If α/λ and $1/\mu(\alpha/\lambda)^- = (\alpha_{n-1}/\mu_n \lambda_{n-1})_n \in \Gamma$, then

$$[\Delta^+, \Delta^+] = s_{((\alpha/\lambda)^-/\mu)^-} = s_{(\alpha_{n-2}/\lambda_{n-2} \mu_{n-1})_n} \quad (3.24)$$

(vi) If $1/\alpha$ and $\alpha\lambda\mu \in \Gamma$, then

$$[C^+, C] = s_{(\alpha\lambda\mu)} \quad (3.25)$$

(vii) If $1/\alpha$ and $\alpha\lambda \in \Gamma$, then

$$[C^+, \Delta] = s_{(\alpha(\lambda/\mu))} \quad (3.26)$$

(viii) If $1/\alpha$ and $\alpha(\lambda/\mu) \in \Gamma$, then

$$[C^+, \Delta^+] = s_{(\alpha(\lambda/\mu))^-} \quad (3.27)$$

(ix) If $1/\alpha$ and $1/\alpha\lambda \in \Gamma$, then

$$[C^+, C^+] = s_{(\alpha\lambda\mu)} \quad (3.28)$$

PROOF. (i) First, for any given X , the condition $\Delta(\lambda)(|\Delta^+(\mu)X|) \in s_\alpha$ is equivalent to

$$|\Delta^+(\mu)X| \in s_\alpha(\Delta(\lambda)) = s_{(\alpha/\lambda)}, \quad (3.29)$$

since $\alpha \in \Gamma$. So $X \in s_{\alpha\lambda}(\Delta^+(\mu))$ and applying [Corollary 2.9](#), we conclude the first part of the proof of (i).

We have $\Delta(\lambda)(|C^+(\mu)X|) \in s_\alpha$ if and only if

$$|C^+(\mu)X| \in C(\lambda)s_\alpha = D_{1/\lambda}\Sigma s_\alpha. \quad (3.30)$$

Since $\alpha \in \Gamma$, we have $\Sigma s_\alpha = s_\alpha$ and $D_{1/\lambda}\Sigma s_\alpha = s_{(\alpha/\lambda)}$. Then, for $\alpha/\lambda \in \Gamma^+$, $X \in [\Delta, C^+]$ if and only if

$$X \in w_{(\alpha/\lambda)}^{+1}(\mu) = s_{(\alpha(\mu/\lambda))}. \quad (3.31)$$

(ii) For any given X , $C(\lambda)(|\Delta^+(\mu)X|) \in s_\alpha$ is equivalent to

$$\Delta^+(\mu)X \in w_\alpha^1(\lambda), \quad (3.32)$$

and since $\alpha\lambda \in \Gamma$ we have $w_\alpha^1(\lambda) = s_{\alpha\lambda}$. So

$$X \in s_{\alpha\lambda}(\Delta^+(\mu)) = s_{(\alpha(\lambda/\mu))^-} \quad (3.33)$$

if $\alpha\lambda/\mu \in \Gamma$. Then (ii) is proved.

(iii) Here, $\Delta^+(\lambda)(|\Delta(\mu)X|) \in s_\alpha$ if and only if

$$|\Delta(\mu)X| \in s_\alpha(\Delta^+(\lambda)) = s_{(\alpha/\lambda)^-}, \quad (3.34)$$

since $\alpha/\lambda \in \Gamma$. Thus

$$X \in C(\mu)s_{(\alpha/\lambda)^-} = D_{1/\mu}\Sigma s_{(\alpha/\lambda)^-} = s_{(\alpha_{n-1}/\lambda_{n-1}\mu_n)} \quad (3.35)$$

if $(\alpha/\lambda)^- \in \Gamma$, that is, $\alpha/\lambda \in \Gamma$.

(iv) If $\alpha/\lambda \in \Gamma$, we get

$$\begin{aligned} \Delta^+(\lambda)(|C(\mu)X|) \in s_\alpha &\iff |C(\mu)X| \in s_\alpha(\Delta^+(\lambda)) \\ &= s_{(\alpha/\lambda)^-} \iff X \in \Delta(\mu)s_{(\alpha/\lambda)^-}. \end{aligned} \quad (3.36)$$

Since $\mu(\alpha/\lambda)^- \in \Gamma$, we conclude that $[\Delta^+, C] = s_{(\mu(\alpha/\lambda)^-)}$.

(v) One has

$$[\Delta^+, \Delta^+] = \{X \mid \Delta^+(\mu)X \in s_\alpha(\Delta^+(\lambda))\}, \quad (3.37)$$

and since $\alpha/\lambda \in \Gamma$, we get

$$s_\alpha(\Delta^+(\lambda)) = s_{(\alpha/\lambda)^-}. \quad (3.38)$$

We deduce that if $\alpha/\lambda \in \Gamma$,

$$[\Delta^+, \Delta^+] = s_{(\alpha/\lambda)^-}(\Delta^+(\mu)). \quad (3.39)$$

Then, from [Corollary 2.9](#), if $\alpha/\lambda \in \Gamma$ and $(\alpha/\lambda)^-/\mu = (\alpha_{n-1}/\lambda_{n-1}\mu_n)_n \in \Gamma$,

$$s_{(\alpha/\lambda)^-}(\Delta^+(\mu)) = s_{((\alpha/\lambda)^-/\mu)^-} = s_{(\alpha_{n-2}/\lambda_{n-2}\mu_{n-1})_n}. \quad (3.40)$$

(vi) We have

$$C^+(\lambda)(|C(\mu)X|) \in s_\alpha \iff C(\mu)X \in w_\alpha^{+1}(\lambda), \quad (3.41)$$

and since $\alpha \in \Gamma^+$, we have $w_\alpha^{+1}(\lambda) = s_{\alpha\lambda}$. Then for $\alpha\lambda\mu \in \Gamma$, $X \in [C^+, C]$ if and only if

$$X \in \Delta(\mu)s_{\alpha\lambda} = s_{(\alpha\lambda\mu)}. \quad (3.42)$$

(vii) The condition $C^+(\lambda)(|\Delta(\mu)X|) \in s_\alpha$ is equivalent to

$$\Delta(\mu)X \in w_\alpha^{+1}(\lambda), \quad (3.43)$$

and since $\alpha \in \Gamma^+$, we have $w_\alpha^{+1}(\lambda) = s_{\alpha\lambda}$. Thus

$$X \in s_{\alpha\lambda}(\Delta(\mu)) = D_{1/\mu}\Sigma s_{\alpha\lambda} = s_{(\alpha(\lambda/\mu))}, \quad (3.44)$$

since $\alpha\lambda \in \Gamma$. So (vii) holds.

(viii) First, we have

$$[C^+, \Delta^+] = \{X \mid \Delta^+(\mu)X \in w_{\alpha}^{+1}(\lambda)\}, \quad (3.45)$$

and the condition $\alpha \in \Gamma^+$ implies that $w_{\alpha}^{+1}(\lambda) = s_{\alpha\lambda}$. Thus

$$[C^+, \Delta^+] = \{X \mid \Delta^+(\mu)X \in s_{\alpha\lambda}\} = s_{\alpha\lambda}(\Delta^+(\mu)), \quad (3.46)$$

and we conclude since

$$s_{\alpha\lambda}(\Delta^+(\mu)) = s_{(\alpha\lambda/\mu)^-} \quad \text{for } \frac{\alpha\lambda}{\mu} \in \Gamma. \quad (3.47)$$

(ix) If $\alpha \in \Gamma^+$,

$$[C^+, C^+] = \{X \mid C^+(\mu)X \in w_{\alpha}^{+1}(\lambda) = s_{\alpha\lambda}\} = w_{\alpha\lambda}^{+1}(\mu). \quad (3.48)$$

We conclude that $w_{\alpha\lambda}^{+1}(\mu) = s_{(\alpha\lambda\mu)}$, since $\alpha\lambda \in \Gamma^+$. $\square$

REMARK 3.4. Note that in [Theorem 3.3](#), we have $[A_1, A_2] = s_{\alpha}(A_1 A_2) = (s_{\alpha}(A_1))_{A_2}$ for $A_1 \in \{\Delta(\lambda), \Delta^+(\lambda), C(\lambda), C^+(\lambda)\}$ and $A_2 \in \{\Delta(\mu), \Delta^+(\mu), C(\mu), C^+(\mu)\}$. For instance, we have

$$[\Delta, C] = \left\{ X \mid \left(\frac{\lambda_n}{\mu_n} - \frac{\lambda_{n-1}}{\mu_{n-1}} \right) \sum_{i=1}^{n-1} x_i + \frac{\lambda_n}{\mu_n} x_n = \alpha_n O(1) \right\} \quad \text{for } \frac{\alpha\mu}{\lambda} \in \Gamma. \quad (3.49)$$

Similarly, under the corresponding conditions given in [Theorems 3.1](#) and [3.3](#), we get

$$\begin{aligned} [\Delta, \Delta] &= \{X \mid -\lambda_{n-1}\mu_{n-2}x_{n-2} + \mu_{n-1}(\lambda_n + \lambda_{n-1})x_{n-1} - \lambda_n\mu_n x_n = \alpha_n O(1)\}, \\ [\Delta, C^+] &= \left\{ X \mid \frac{\lambda_n}{\mu_n} x_n + (\lambda_n - \lambda_{n-1}) \sum_{m=n-1}^{\infty} \frac{x_m}{\mu_m} = \alpha_n O(1) \right\}, \\ [\Delta, \Delta^+] &= \{X \mid -\lambda_{n-1}\mu_{n-1}x_{n-1} + \mu_n(\lambda_n + \lambda_{n-1})x_n - \lambda_n\mu_{n+1}x_{n+1} = \alpha_n O(1)\}, \\ [\Delta^+, \Delta] &= \{X \mid -\lambda_n\mu_{n-1}x_{n-1} + (\lambda_n + \lambda_{n+1})\mu_n x_n - \lambda_{n+1}\mu_{n+1}x_{n+1} = \alpha_n O(1)\}. \end{aligned} \quad (3.50)$$

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