MULTIVALENT FUNCTIONS AND Q_K SPACES

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We give a criterion for q -valent analytic functions in the unit disk to belong to Q_K , a Möbius-invariant space of functions analytic in the unit disk in the plane for a nondecreasing function $K : [0, \infty) \rightarrow [0, \infty)$, and we show by an example that our condition is sharp. As corollaries, classical results on univalent functions, the Bloch space, BMOA, and Q_p spaces are obtained.

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1. Introduction. For analytic univalent function f in the unit disk Δ , Pommerenke [8] proved that $f \in \mathcal{B}$ if and only if $f \in \text{BMOA}$, which easily implies a result of Baernstein II [4] about univalent Bloch functions: if $g(z) \neq 0$ is an analytic univalent function in Δ , then $\log g \in \text{BMOA}$. We know that Pommerenke's result mentioned above was generalized to Q_p spaces for all p , $0 < p < \infty$, by Aulaskari et al. (cf. [2, Theorem 6.1]). Their result can be stated as follows.

THEOREM 1.1. *Let f be an analytic function in Δ such that*

$$\iint_{|w-w_0|<1} n(w, f) dA(w) \leq A < \infty, \quad (1.1)$$

for all $w_0 \in \mathbb{C}$, where $n(w, f)$ denotes the number of roots of the equation $f(z) = w$ in Δ counted according to their multiplicity and $dA(z)$ is the Euclidean area element on Δ . Then $f \in \mathcal{B}(\mathcal{B}_0)$ if and only if $f \in Q_p(Q_{p,0})$ for all $p \in (0, \infty)$.

Here, Q_p and its subspace $Q_{p,0}$, $0 < p < \infty$, denote the spaces of analytic functions f in Δ defined, respectively, as follows (cf. [1, 3]):

$$\begin{aligned} Q_p &= \left\{ f : f \text{ analytic in } \Delta, \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^2 (g(z, a))^p dA(z) < \infty \right\}, \\ Q_{p,0} &= \left\{ f \in Q_p : \lim_{|a| \rightarrow 1} \iint_{\Delta} |f'(z)|^2 (g(z, a))^p dA(z) = 0 \right\}, \end{aligned} \quad (1.2)$$

where $g(z, a) = \log 1/|\varphi_a(z)|$ is a Green's function in Δ with pole at $a \in \Delta$, and $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ is a Möbius transformation of Δ .

We know that $Q_1 = \text{BMOA}$, the space of all analytic functions of bounded mean oscillation (cf. [5]), and for each $p \in (1, \infty)$, the space Q_p is the Bloch space $\mathcal{B}$ (cf. [1]), which

is defined as follows:

$$\mathcal{B} = \left\{ f : f \text{ analytic in } \Delta, \|f\|_{\mathcal{B}} = \sup_{z \in \Delta} (1 - |z|^2) |f'(z)| < \infty \right\}. \quad (1.3)$$

Similar to the above we have $Q_{1,0} = \text{VMOA}$, the space of all analytic functions of vanishing mean oscillation (cf. [5]), and $Q_{p,0} = \mathcal{B}_0$ for all $p \in (1, \infty)$, where $\mathcal{B}_0$ denotes the little Bloch space defined by

$$\mathcal{B}_0 = \left\{ f \in \mathcal{B} : \lim_{|z| \rightarrow 1} (1 - |z|^2) |f'(z)| = 0 \right\}. \quad (1.4)$$

In the present paper, we consider a more general space Q_K (see below) and show that all the above-mentioned results are true for space Q_K . Our contribution gives an extended version of Pommerenke's theorem, which is also a slight improvement of all the above results, and the proof presented here is independently developed.

Let $K : [0, \infty) \rightarrow [0, \infty)$ be a right-continuous and nondecreasing function. Recall that the space Q_K consists of analytic functions f in Δ for which

$$\|f\|_{Q_K}^2 = \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^2 K(g(z, a)) dA(z) < \infty; \quad (1.5)$$

$f \in Q_K$ belongs to the space $Q_{K,0}$ if

$$\iint_{\Delta} |f'(z)|^2 K(g(z, a)) dA(z) \rightarrow 0, \quad |a| \rightarrow 1. \quad (1.6)$$

Modulo constants, Q_K is a Banach space under the norm defined in (1.5). It is clear that Q_K is Möbius-invariant and a subspace of the Bloch space $\mathcal{B}$ (cf. [6]). For $0 < p < \infty$, $K(t) = t^p$ gives the space Q_p . Choosing $K(t) = 1$, we get the Dirichlet space $\mathcal{D}$.

By [6, Proposition 2.1] we know that if the integral

$$\int_0^{1/e} K\left(\log \frac{1}{\rho}\right) \rho d\rho = \int_1^\infty K(t) e^{-2t} dt \quad (1.7)$$

is divergent, then the space Q_K is trivial; that is, the space Q_K contains only constant functions. From now on, we assume that the function $K : [0, \infty) \rightarrow [0, \infty)$ is right-continuous and nondecreasing and that the integral (1.7) is convergent. Without loss of generality, we can assume that $K(1) > 0$. For a general theory for Q_K spaces, see [6, 11].

2. Main results. A function f analytic in the unit disk is said to be q -valent if the equation $f(z) = w$ has never more than q solutions. Let

$$p(\rho) = \frac{1}{2\pi} \int_0^{2\pi} n(\rho e^{i\phi}, f) d\phi. \quad (2.1)$$

If

$$\int_0^R p(\rho) d(\rho^2) \leq qR^2, \quad R > 0, \quad (2.2)$$

or

$$p(R) \leq q, \quad R > 0, \quad (2.3)$$

where q is a positive number, we say that f is areally mean q -valent or circumferentially mean q -valent, respectively (cf. [7, pages 38 and 144]). It is clear that if f is circumferentially mean q -valent, then f is areally mean q -valent.

Note that if (1.1) holds, f will be areally mean q -valent in Δ for some $q > 0$. We know that if f is univalent, then f must be areally and circumferentially mean 1-valent. Thus, it is natural to conjecture that Pommerenke's result and Theorem 1.1 are also true for the areally and circumferentially mean q -valent functions.

We know that the space Q_K can be nontrivial if K is not too big at infinity (see condition (1.7)). For such functions K , the properties of Q_K depend essentially on the behavior of K near the origin. From [6, Theorems 2.3 and 2.5], we know that $Q_K = \mathcal{B}(Q_{K,0} = \mathcal{B}_0)$ if and only if

$$\int_0^1 (1-r^2)^{-2} K\left(\log \frac{1}{r}\right) r \, dr < \infty. \quad (2.4)$$

A natural idea is to look for an integral condition which is weaker than that given by (2.4) such that $f \in \mathcal{B}(\mathcal{B}_0)$ if and only if $f \in Q_K(Q_{K,0})$ for some special f . For the areally mean q -valent case, we present the main result in this paper as follows.

THEOREM 2.1. *Let f be an areally mean q -valent function in Δ . If*

$$\int_0^1 \left(\log \frac{1}{1-r}\right)^2 (1-r)^{-1} K\left(\log \frac{1}{r}\right) r \, dr < \infty, \quad (2.5)$$

then

- (i) $f \in \mathcal{B}$ if and only if $f \in Q_K$;
- (ii) $f \in \mathcal{B}_0$ if and only if $f \in Q_{K,0}$.

Note that (2.4) implies (2.5) since $(\log 1/(1-r))^2 \leq 4e^{-2}/(1-r)$ for $0 < r < 1$, but the converse is not true. For example, $K(t) = t$ gives that (2.5) holds but (2.4) fails. By [6, Theorems 2.3 and 2.5], (2.5) is also necessary for Theorem 2.1(i) and (ii) in case f is an areally mean q -valent function in Δ .

In the light of the following example it is impossible to drop the assumption of areally mean q -valence of the functions f in Theorem 2.1. Indeed, choose $K_1(t) = t^{2\alpha-1}$ and

$$f_1(z) = \sum_{j=1}^{\infty} 2^{-j(1-\alpha)} z^{2^j}, \quad \frac{1}{2} < \alpha < 1. \quad (2.6)$$

It is easy to see that $f_1 \in \mathcal{B}$ and (2.5) holds for K_1 . Since f_1 has a gap series representation, f_1 is not an areally mean q -valent in Δ . The following argument shows that $f \notin Q_{K_1}$.

For $r \in [3/4, 1)$, we find k so that $1/2 \leq 2^k(1-r) < 1$. Using the inequality $\log r \geq 2(r-1)$, $1/2 < r < 1$, we see that

$$\begin{aligned}
 \int_0^{2\pi} |f'_1(re^{i\theta})|^2 d\theta &= 2\pi \sum_{j=1}^{\infty} 2^{j2\alpha} r^{2^{j+1}-2} \\
 &\geq 2\pi(1-r)^{-2\alpha} \sum_{j=1}^{\infty} (2^j(1-r))^{2\alpha} \exp(-2^{j+2}(1-r)) \\
 &\geq 2^{-2\alpha+1}\pi(1-r)^{-2\alpha} \sum_{j=1}^{\infty} 2^{(j-k)(2\alpha)} \exp(-2^{j-k+2}) \\
 &\geq 2^{-2\alpha+1}\pi(1-r)^{-2\alpha} \sum_{j=0}^{\infty} (2^{j2\alpha} \exp(-2^{j+2})) \\
 &= C(\alpha)(1-r)^{-2\alpha}.
 \end{aligned} \tag{2.7}$$

Hence

$$\begin{aligned}
 &\sup_{a \in \Delta} \iint_{\Delta} |f'_1(z)|^2 K_1(g(z, a)) dA(z) \\
 &\geq \iint_{\Delta} |f'_1(z)|^2 K_1\left(\log \frac{1}{|z|}\right) dA(z) \\
 &= \int_0^1 K\left(\log \frac{1}{r}\right) r dr \int_0^{2\pi} |f'_1(re^{i\theta})|^2 d\theta \\
 &\geq C(\alpha) \int_{3/4}^1 (1-r)^{-2\alpha} \left(\log \frac{1}{r}\right)^{2\alpha-1} r dr.
 \end{aligned} \tag{2.8}$$

Since the last integral is divergent, we conclude that $f_1 \notin Q_K$.

THEOREM 2.2. *Let f be a circumferentially mean q -valent and nonvanishing function in Δ . If (2.5) holds, then $\log f \in Q_K$.*

It is clear that the integral in (2.5) is convergent for $K(t) = t^p$, $p > 0$. Thus, we have the following result which extends Theorem 1.1.

COROLLARY 2.3. *Let f be an areally mean q -valent function in Δ , $0 < p < \infty$. Then*

- (i) $f \in \mathcal{B}$ if and only if $f \in Q_p$;
- (ii) $f \in \mathcal{B}_0$ if and only if $f \in Q_{p,0}$.

3. Proofs. In the proofs of Theorems 2.1 and 2.2, we need two lemmas, the first one can be considered as a generalization of a result of Pommerenke (cf. [9, page 174]).

LEMMA 3.1. *Let f be areally mean q -valent in Δ . Then*

$$\int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \leq \frac{4q\pi(M(\sqrt{r}, f))^2}{1-r}, \quad \frac{1}{2} < r < 1, \tag{3.1}$$

where $M(r, f) = \sup_{|z|=r} |f(z)|$, $0 < r < 1$.

PROOF. If $1/2 < r < 1$, we obtain

$$\begin{aligned} \iint_{|z| < \sqrt{r}} |f'(z)|^2 dA(z) &= \int_0^{\sqrt{r}} \rho \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 d\theta d\rho \\ &\geq \frac{1}{4}(1-r) \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta. \end{aligned} \quad (3.2)$$

Since f is areally mean q -valent, we deduce that

$$\begin{aligned} \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta &\leq \frac{4}{1-r} \iint_{|z| < \sqrt{r}} |f'(z)|^2 dA(z) \\ &\leq \frac{4}{1-r} \iint_{|w| < M(\sqrt{r}, f)} n(w, f) dA(w) \\ &\leq \frac{4q\pi(M(\sqrt{r}, f))^2}{1-r}, \end{aligned} \quad (3.3)$$

which proves [Lemma 3.1](#). □

LEMMA 3.2. Let K be defined as in [Section 1](#). Then

- (i) $Q_{K,0} \subset \mathcal{B}_0$;
- (ii) an analytic function f belongs to $\mathcal{B}_0$ if and only if there exists an $r \in (0, 1)$ such that

$$\lim_{|a| \rightarrow 1} \iint_{\Delta(a, r)} |f'(z)|^2 K(g(z, a)) dA(z) = 0, \quad (3.4)$$

where $\Delta(a, r) = \{z \in \Delta : |\varphi_a(z)| < r\}$.

PROOF. See [\[6, Theorem 2.4\]](#). □

Now we turn to give the proofs of our main theorems.

PROOF OF THEOREM 2.1. We first prove (i). Since $Q_K \subset \mathcal{B}$, it suffices to prove that if a Bloch function f is areally mean q -valent in Δ , then $f \in Q_K$. We use the change of variable $w = \varphi_a(z)$ to deduce that

$$\begin{aligned} &\iint_{\Delta \setminus \Delta(a, 1/2)} |f'(z)|^2 K(g(z, a)) dA(z) \\ &= \iint_{\Delta \setminus \Delta(a, 1/2)} |(f(z) - f(a))'|^2 K\left(\log \frac{1}{|\varphi_a(z)|}\right) dA(z) \\ &= \iint_{1/2 < |w| < 1} |(f \circ \varphi_a(w) - f(a))'|^2 K\left(\log \frac{1}{|w|}\right) dA(w) \\ &= \int_{1/2}^1 K\left(\log \frac{1}{r}\right) r \int_0^{2\pi} |(f \circ \varphi_a(re^{i\theta}) - f(a))'|^2 d\theta dr. \end{aligned} \quad (3.5)$$

It is known that if $g \in \mathcal{B}$, then

$$|g(z) - g(0)| \leq \frac{1}{2} \|g\|_{\mathcal{B}} \log \frac{1+|z|}{1-|z|}. \quad (3.6)$$

Choosing $g = f \circ \varphi_a - f(a)$ and observing that $\|g\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$, we obtain

$$M(r, f \circ \varphi_a - f(a)) \leq \frac{1}{2} \|f\|_{\mathcal{B}} \log \frac{1+r}{1-r}. \quad (3.7)$$

It follows from (3.5) and Lemma 3.1 that

$$\begin{aligned} & \iint_{\Delta \setminus \Delta(a, 1/2)} |f'(z)|^2 K(g(z, a)) dA(z) \\ &= \int_{1/2}^1 K\left(\log \frac{1}{r}\right) r \int_0^{2\pi} |(f \circ \varphi_a(re^{i\theta}) - f(a))'|^2 d\theta dr \\ &\leq 4q\pi \int_{1/2}^1 K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr \\ &\leq q\pi C \|f\|_{\mathcal{B}}^2 \int_{1/2}^1 K\left(\log \frac{1}{r}\right) \left(\log \frac{1}{1-r}\right)^2 (1-r)^{-1} r dr. \end{aligned} \quad (3.8)$$

On the other hand, we have

$$\begin{aligned} & \iint_{\Delta(a, 1/2)} |f'(z)|^2 K(g(z, a)) dA(z) \\ &\leq \|f\|_{\mathcal{B}}^2 \iint_{\Delta(a, 1/2)} (1-|z|^2)^{-2} K(g(z, a)) dA(z) \\ &= \|f\|_{\mathcal{B}}^2 \iint_{\Delta(0, 1/2)} (1-|w|^2)^{-2} K\left(\log \frac{1}{|w|}\right) dA(w) \\ &\leq 4\pi \|f\|_{\mathcal{B}}^2 \int_0^{1/2} K\left(\log \frac{1}{r}\right) r dr. \end{aligned} \quad (3.9)$$

Combining the upper bounds given by (3.8), (3.9), and (2.5), we see that $f \in Q_K$, which proves part (i) of Theorem 2.1.

To prove (ii), we assume that f is an areally mean q -valent function in Δ which is also in $\mathcal{B}_0$. By Lemma 3.2(i), it suffices to prove that $f \in Q_{K,0}$. By Lemma 3.2(ii), there exists an r_0 , $1/2 < r_0 < 1$, such that

$$\lim_{|a| \rightarrow 1} \iint_{\Delta(a, r_0)} |f'(z)|^2 K(g(z, a)) dA(z) = 0. \quad (3.10)$$

Now we show that

$$\lim_{|a| \rightarrow 1} \iint_{\Delta \setminus \Delta(a, r_0)} |f'(z)|^2 K(g(z, a)) dA(z) = 0. \quad (3.11)$$

By the proof of part (i) and assumption (2.5), we see that

$$\begin{aligned}
 & \iint_{\Delta \setminus \Delta(a, r_0)} |f'(z)|^2 K(g(z, a)) dA(z) \\
 &= \int_{r_0}^1 K\left(\log \frac{1}{r}\right) r \int_0^{2\pi} |(f \circ \varphi_a(re^{i\theta}) - f(a))'|^2 d\theta dr \\
 &\leq 4q\pi \int_{r_0}^1 K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr \\
 &\leq q\pi \|f\|_{\mathfrak{B}}^2 \int_{r_0}^1 K\left(\log \frac{1}{r}\right) \left(\log \frac{1+r}{1-r}\right)^2 (1-r)^{-1} r dr < \infty
 \end{aligned} \tag{3.12}$$

for all $a \in \Delta$. Thus, for any given $\varepsilon > 0$, there exists an r_1 , $r_0 < r_1 < 1$, such that

$$\int_{r_1}^1 K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr < \varepsilon \tag{3.13}$$

for all $a \in \Delta$. Hence, what we need to prove is that

$$\lim_{|a| \rightarrow 1} \int_{r_0}^{r_1} K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr = 0. \tag{3.14}$$

In fact, we have

$$\begin{aligned}
 & \int_{r_0}^{r_1} K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr \\
 &\leq C(r_0, r_1) K\left(\log \frac{1}{r_0}\right) (M(r_2, f \circ \varphi_a - f(a)))^2,
 \end{aligned} \tag{3.15}$$

where $r_2 = \sqrt{r_1}$ and $C(r_0, r_1)$ is a constant depending on r_0 and r_1 . Define $f_t(z) = f(tz)$ for $0 < t < 1$ and then

$$\begin{aligned}
 & (M(r_2, f \circ \varphi_a - f(a)))^2 \\
 &\leq 2 \left(\frac{1}{4} \|f - f_t\|_{\mathfrak{B}}^2 \left(\log \frac{1+r_2}{1-r_2} \right)^2 + (M(r_2, f_t \circ \varphi_a - f_t(a)))^2 \right).
 \end{aligned} \tag{3.16}$$

Since $f \in \mathfrak{B}_0$, $\|f - f_t\|_{\mathfrak{B}} \rightarrow 0$, $t \rightarrow 1$. Also,

$$\max_{|z| \leq r_2} |f_t \circ \varphi_a(z) - f_t(a)| \leq \frac{1 - |a|^2}{(1 - r_2)^2} \max_{|w| \leq t} |f'(w)|, \tag{3.17}$$

which implies that

$$\lim_{|a| \rightarrow 1} M(r_2, f_t \circ \varphi_a - f_t(a)) = 0. \tag{3.18}$$

Thus we have (3.14). Hence

$$\lim_{|a| \rightarrow 1} \iint_{\Delta} |f'(z)|^2 K(g(z, a)) dA(z) = 0, \quad (3.19)$$

which shows that $f \in Q_{K,0}$. The proof of Theorem 2.1 is complete. $\square$

PROOF OF THEOREM 2.2. Assume that f is a nonvanishing circumferentially mean q -valent function in Δ . According to [7, Theorem 5.1], we have $\log f \in \mathcal{B}$. From [7, Lemma 5.2] and the argument in the beginning of the proof of [7, Theorem 5.1], we see that we can define a single-valued branch of $f(z)^{1/q}$ which is circumferentially mean 1-valent in Δ and such that on each circle $\{|w| = R\}$ there exists a point which is not assumed by $f(z)^{1/q}$. It follows that

$$\begin{aligned} \int_{-\infty}^{\infty} n\left(\log \rho + i\phi, \frac{1}{q} \log f\right) d\phi &= \int_0^{2\pi} n(\rho e^{i\phi}, f^{1/q}) d\phi \leq 2\pi, \\ \iint_{|w| < R} n(w, \log f) dA(w) &\leq 4\pi Rq, \end{aligned} \quad (3.20)$$

which means that $\log f$ is areally mean q_1 -valued in Δ for some $q_1 > 0$. It follows from Theorem 2.1 that $\log f \in Q_K$. $\square$

4. Further discussion. In [10] we studied the conditions for analytic univalent Bloch function f to belong to Q_K spaces. The log-order of the function $K(r)$ is defined as

$$\rho = \overline{\lim}_{r \rightarrow \infty} \frac{\log^+ \log^+ K(r)}{\log r}, \quad (4.1)$$

where $\log^+ x = \max\{\log x, 0\}$, and if $0 < \rho < \infty$, the log-type of the function $K(r)$ is defined as

$$\sigma = \overline{\lim}_{r \rightarrow \infty} \frac{\log^+ K(r)}{r^\rho}. \quad (4.2)$$

THEOREM 4.1. *Let f be an analytic univalent function in Δ and let $K: [0, \infty) \rightarrow [0, \infty)$ satisfy that $K(t) = O((t \log 1/t)^p)$ as $t \rightarrow 0$ for some $p > 0$. If the log-order ρ and the log-type σ of K satisfy one of the conditions*

- (i) $0 \leq \rho < 1$,
- (ii) $\rho = 1$ and $\sigma < 2$,

then $f \in \mathcal{B}$ if and only if $f \in Q_K$.

We note that Theorem 4.1 can be viewed as a consequence of Theorem 2.1. In fact, conditions (i) and (ii) of Theorem 4.1 show that the space Q_K is not trivial. That is, the integral (1.7) is convergent in this case. Suppose that $K(t) = O((t \log 1/t)^p)$, $t \rightarrow 0$. There exist an $r_0 \in (1/2, 1)$ and a constant $C > 0$ such that both $\log 1/r \leq 2(1-r)$ and

$$K\left(\log \frac{1}{r}\right) \leq C \left(\log \frac{1}{r} \log \left(\log \frac{1}{r}\right)^{-1}\right)^p \quad (4.3)$$

hold for $r_0 < r < 1$. Thus

$$\begin{aligned}
 & \int_0^1 \left(\log \frac{1}{1-r} \right)^2 (1-r)^{-1} K \left(\log \frac{1}{r} \right) r \, dr \\
 &= \int_0^{r_0} + \int_{r_0}^1 \left(\log \frac{1}{1-r} \right)^2 (1-r)^{-1} K \left(\log \frac{1}{r} \right) r \, dr \\
 &\leq \left(\log \frac{1}{1-r_0} \right)^2 (1-r_0)^{-1} \int_0^{r_0} K \left(\log \frac{1}{r} \right) r \, dr \\
 &\quad + C \int_{r_0}^1 \left(\log \frac{1}{1-r} \right)^2 (1-r)^{-1} \left(\log \frac{1}{r} \log \left(\log \frac{1}{r} \right)^{-1} \right)^p r \, dr \quad (4.4) \\
 &\leq C_1 + C_2 \int_{r_0}^1 \left(\log \frac{1}{1-r} \right)^{2+p} (1-r)^{p-1} r \, dr \\
 &\leq C_1 + C_2 \int_{R_0}^\infty e^{-ps} s^{2+p} \, ds \\
 &\leq C_1 + C_2 p^{-3-p} \Gamma(3+p) < \infty.
 \end{aligned}$$

For a general analytic function f , we have the following theorem.

THEOREM 4.2. *Suppose that (2.5) holds. If*

$$\sup_{a \in \Delta} \iint_{|z| < r} |(f \circ \varphi_a(z))'|^2 dA(z) = O \left(\left(\log \frac{1}{1-r} \right)^2 \right), \quad (4.5)$$

then

- (i) $f \in \mathcal{B}$ if and only if $f \in Q_K$;
- (ii) $f \in \mathcal{B}_0$ if and only if $f \in Q_{K,0}$.

PROOF. We know that

$$\begin{aligned}
 \int_0^{2\pi} |(f \circ \varphi_a(re^{i\theta}))'|^2 d\theta &\leq \frac{4}{1-r} \iint_{|z| < \sqrt{r}} |(f \circ \varphi_a(z))'|^2 dA(z) \\
 &\leq \frac{1}{1-r} O \left(\left(\log \frac{1}{1-\sqrt{r}} \right)^2 \right) \\
 &\leq \frac{C}{1-r} \left(\log \frac{1}{1-r} \right)^2. \quad (4.6)
 \end{aligned}$$

The proof can be completed by an argument similar to that used in the proof of Theorem 2.1. $\square$

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REFERENCES

- [1] R. Aulaskari and P. Lappan, *Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal*, Complex Analysis and Its Applications (Hong Kong, 1993), Pitman Res. Notes Math. Ser., vol. 305, Longman Scientific & Technical, Harlow, 1994, pp. 136-146.
- [2] R. Aulaskari, P. Lappan, J. Xiao, and R. Zhao, *On α -Bloch spaces and multipliers of Dirichlet spaces*, J. Math. Anal. Appl. **209** (1997), no. 1, 103-121.
- [3] R. Aulaskari, J. Xiao, and R. Zhao, *On subspaces and subsets of BMOA and UBC*, Analysis **15** (1995), no. 2, 101-121.
- [4] A. Baernstein II, *Univalence and bounded mean oscillation*, Michigan Math. J. **23** (1976), no. 3, 217-223.
- [5] ———, *Analytic functions of bounded mean oscillation*, Aspects of Contemporary Complex Analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979), Academic Press, London, 1980, pp. 3-36.
- [6] M. Essén and H. Wulan, *On analytic and meromorphic functions and spaces of Q_K -type*, Illinois J. Math. **46** (2002), no. 4, 1233-1258.
- [7] W. K. Hayman, *Multivalent Functions*, 2nd ed., Cambridge Tracts in Mathematics, vol. 110, Cambridge University Press, Cambridge, 1994.
- [8] Ch. Pommerenke, *Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation*, Comment. Math. Helv. **52** (1977), no. 4, 591-602 (German).
- [9] ———, *Boundary Behaviour of Conformal Maps*, Grundlehren der mathematischen Wissenschaften, vol. 299, Springer-Verlag, Berlin, 1992.
- [10] H. Wulan, *On univalent Bloch functions*, Acta Math. Sci. Ser. B Engl. Ed. **21** (2001), no. 1, 37-40.
- [11] H. Wulan and P. Wu, *Characterizations of Q_T spaces*, J. Math. Anal. Appl. **254** (2001), no. 2, 484-497.

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