ON THE CLASS OF $QS$-ALGEBRAS

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We consider some fundamental properties of $QS$-algebras and show that (1) the theory of $QS$-algebras is logically equivalent to the theory of Abelian groups, that is, each theorem of $QS$-algebras is provable in the theory of Abelian groups, and conversely, each theorem of Abelian groups is provable in the theory of $QS$-algebras; and (2) a $G$-part $G(X)$ of a $QS$-algebra $X$ is a normal subgroup generated by the class of all elements of order 2 of $X$ when it is considered as a group.

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1. Introduction. In [3], the notion of $Q$-algebras is introduced and some fundamental properties are established. The algebras are extensions of the BCK/BCI-algebras which were proposed by Y. Imai and K. Iséki in 1966. It is usually important to generalize the algebraic structures. Neggers and Kim [4] introduced a class of algebras which is related to several classes of algebras such as BCK/BCI/BCH-algebras. They call them $B$-algebras, and they proved that every group $(X; \circ, 0)$ determines a $B$-algebra $(X; *, 0)$, which is called the group-derived $B$-algebra. Conversely, in [2], we prove that every $B$-algebra is group-derived and hence that the class of $B$-algebras and the class of all groups are the same. Ahn and Kim [1] proposed the notion of $QS$-algebras which is also a generalization of BCK/BCI-algebras and obtained several results. Here, we consider some fundamental properties of $QS$-algebras and show that

(1) the theory of $QS$-algebras is logically equivalent to the theory of Abelian groups, that is, each theorem of $QS$-algebras is provable in the theory of Abelian groups and conversely each theorem of Abelian groups is provable in the theory of $QS$-algebras;

(2) a subset $G(X)$ called $G$-part of a $QS$-algebra $X$ is a normal subgroup which is generated by the class of all elements of order 2.

2. Preliminaries. A $QS$-algebra is a nonempty set $X$ with a constant 0 and a binary operation "*" satisfying the following axioms:

(QS1) $x * x = 0$,
(QS2) $x * 0 = x$,
(QS3) $(x * y) * z = (x * z) * y$,
(QS4) $(x * y) * (x * z) = y * z$,

for all $x, y, z$ in $X$. 

**Example 2.1.** (1) Let $X = \{0, 1, 2\}$ be a set with an operation $\ast$ defined as follows:

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(2.1)

Then $(X; \ast, 0)$ is a $QS$-algebra.

(2) Let $X$ be the set of all integers. Define a binary operation $\ast$ on $X$ by

$$x \ast y := x - y.$$  

(2.2)

Then $(X; \ast, 0)$ is a $QS$-algebra.

We note that these examples are both Abelian groups and the operation $\ast$ corresponds to the minus operation “−”. In the case of (1), $X$ can be considered as the set $\mathbb{Z}_3$ of integers of modulo 3 and the operation $\ast$ as a minus “−” modulo operation. It seems that any Abelian group gives an example of a $QS$-algebra. In fact, we can prove the fact.

**Theorem 2.2.** Let $(X; \cdot, -1, e)$ be an Abelian group. If $x \ast y = x \cdot y^{-1}$ is defined and $0 = e$, then $(X; \ast, 0)$ is a $QS$-algebra.

**Proof.** We only show that the conditions (QS3) and (QS4) of $QS$-algebras are satisfied. For the case of (QS3), since $X$ is an Abelian group,

$$(x \ast y) \ast z = (x \cdot y^{-1}) \cdot z^{-1}$$

$$= x \cdot (y^{-1} \cdot z^{-1})$$

$$= x \cdot (z^{-1} \cdot y^{-1})$$

$$= (x \cdot z^{-1}) \cdot y^{-1}$$

$$= (x \ast z) \ast y.$$  

(2.3)

For the case of (QS4), we also have

$$(x \ast y) \ast (x \ast z) = (x \cdot y^{-1}) \cdot (x \cdot z^{-1})^{-1}$$

$$= (x \cdot y^{-1}) \cdot (z \cdot x^{-1})$$

$$= x \cdot x^{-1} \cdot z \cdot y^{-1}$$

$$= z \cdot y^{-1}$$

$$= z \ast y.$$  

(2.4)

The theorem means that every Abelian group $(X; \cdot, -1, e)$ determines a $QS$-algebra $(X; \ast, 0)$; in other words, any Abelian group can be considered as a $QS$-algebra. Conversely, we will show in the next section that any $QS$-algebra determines an Abelian group, that is, every $QS$-algebra can be considered as an Abelian group. Hence, we are able to conclude that in this sense, the class of $QS$-algebras coincides with the class of Abelian groups.
3. Abelian groups can be derived from QS-algebras. We show that every QS-algebra determines an Abelian group. In order to do so, it is sufficient to construct an Abelian group from any QS-algebra. We need some lemmas to prove that.

Let \((X; *, 0)\) be a QS-algebra.

**Lemma 3.1.** For all \(x, y, z \in X\), if \(x * y = z\), then \(x * z = y\).

**Proof.** Suppose that \(x * y = z\). Then, since \(X\) is a QS-algebra, we have \(x * z = (x * 0) * (x * y) = y * 0 = y\).

It follows from the above that the condition \((QS4)'\) is established in any QS-algebra: \((QS4)' \ (x * z) * (y * z) = x * y\).

**Corollary 3.2.** If \(x * y = 0\), then \(x = y\).

Let \(B(X) = \{x \in X \mid 0 * x = 0\}\). A QS-algebra \(X\) is called \(p\)-semisimple if \(B(X) = \{0\}\) (cf. [1]). We can show that every QS-algebra is \(p\)-semisimple.

**Corollary 3.3.** Every QS-algebra is \(p\)-semisimple.

**Proof.** Suppose that \(X\) is a QS-algebra. For all elements \(x \in X\), since \(x \in B(X) \iff 0 * x = 0 \iff x = 0\) (by Corollary 3.2), we can conclude that \(X\) is \(p\)-semisimple.

**Remark 3.4.** It is proved in [1] that every associative QS-algebra is \(p\)-semisimple. The corollary above means that the assumption of associativity is superfluous.

**Lemma 3.5.** \(0 * (x * y) = y * x\).

**Proof.** \(0 * (x * y) = (x * x) * (x * y) = y * x\).

**Corollary 3.6.** \(0 * (0 * x) = x\).

**Lemma 3.7.** \(x * (0 * y) = y * (0 * x)\).

**Proof.** Since

\[
0 * (x * (y * (0 * x))) = (y * (0 * x)) * x \quad \text{(by Lemma 3.5)}
\]
\[
= (y * x) * (0 * x) \quad \text{(by (QS3))}
\]
\[
= y * 0 \quad \text{(by (QS4)')} \\
= y,
\]

we have \(0 * (0 * (x * (y * (0 * x)))) = 0 * y\). It follows from Corollary 3.6 that \(x * (y * (0 * x)) = 0 * y\) and hence \(x * (0 * y) = y * (0 * x)\) by Lemma 3.1.

These lemmas provide a proof that we can construct an Abelian group \((X; \cdot, e)\) from a QS-algebra \((X; *, 0)\).

**Theorem 3.8.** Let \((X; *, 0)\) be a QS-algebra. If \(x \cdot y = x * (0 \cdot y)\) is defined, \(x^{-1} = 0 \cdot x\), and \(e = 0\), then the structure \((X; \cdot, -1, e)\) is an Abelian group.
**Proof.** We only show that the structure \((X;\cdot,-1,e)\) satisfies the conditions of associativity and of commutativity with respect to the operation "\(\cdot\)".

For associativity, we have

\[
(x \cdot y) \cdot z = (x \ast (0 \ast y)) \ast (0 \ast z) \\
= (y \ast (0 \ast x)) \ast (0 \ast z) \quad \text{(by Lemma 3.7)} \\
= (y \ast (0 \ast z)) \ast (0 \ast x) \\
= x \ast (0 \ast (y \ast (0 \ast z))) \quad \text{(by Lemma 3.7)} \\
= x \cdot (y \cdot z).
\]

(3.3)

For commutativity, we also have

\[
x \cdot y = x \ast (0 \ast y) = y \ast (0 \ast x) = y \cdot x.
\]

Combining Theorems 2.2 and 3.8, we can conclude that the class of QS-algebras coincides with the class of Abelian groups.

In the following, we will describe our results in greater detail. We can show that each theorem of QS-algebras is translated to a formula of \(\mathcal{A}\) which is provable in the theory of Abelian groups and conversely each theorem of Abelian groups is proved in the theory of QS-algebras. To present our theorem precisely, we will develop the formal theories of QS-algebras and Abelian groups. Let \(\mathcal{D}\) and \(\mathcal{A}\) be the theories of QS-algebras and Abelian groups, respectively. Theories consist of languages and axioms. At first, we define languages of these theories which are needed to present statements formally in their theories. By \(L(\mathcal{D})\) (or \(L(\mathcal{A})\)), we mean a language of the theory \(\mathcal{D}\) (or \(\mathcal{A}\) of groups). We define them as follows.

The language of the theory of \(\mathcal{D}\) of QS-algebras consists of

(lq1) countable variables \(x,y,z,\ldots\),
(lq2) binary operation symbol \(\ast\),
(lq3) constant symbol 0;

and the language of the theory of \(\mathcal{A}\) of QS-algebras consists of

(lg1) countable variables \(x,y,z,\ldots\),
(lg2) binary operation symbol \(\circ\),
(lg3) unary operation symbol \(-1\),
(lg4) constant symbol \(e\).

Next we define terms which represent objects in the theory. By \(T(\mathcal{D})\) (or \(T(\mathcal{A})\)) we mean the set of terms of \(\mathcal{D}\) (or \(\mathcal{A}\)). Terms are defined as follows.

For terms of \(\mathcal{D}\),

(tb1) each variable is a term,
(tb2) the constant 0 is a term,
(tb3) if \(u\) and \(v\) are terms, then \(u \ast v\) is a term.

For terms of \(\mathcal{A}\),

(tg1) each variable is a term,
(tg2) the constant \(e\) is a term,
(tg3) if \(u\) and \(v\) are terms, then so are \(u \circ v\) and \(u^{-1}\).

We also define formulas which represent statements in each theory. Formulas of \(\mathcal{D}\) (or \(\mathcal{A}\)) are defined as the forms of \(s = t\), where \(s,t \in T(\mathcal{D})\) (or \(s,t \in T(\mathcal{A})\)).
As to the axioms of QS-algebras, we list the following:

(QS1) \( x \ast x = 0 \),
(QS2) \( x \ast 0 = x \),
(QS3) \( (x \ast y) \ast z = (x \ast z) \ast y \),
(QS4) \( (x \ast y) \ast (x \ast z) = y \ast z \).

For the axioms of Abelian groups, we use the following:

(G1) \( x \odot (y \odot z) = (x \odot y) \odot z \),
(G2) \( x \odot e = e \odot x = x \),
(G3) \( x \odot x^{-1} = x^{-1} \odot x = e \),
(G4) \( x \odot y = y \odot x \).

Two formal theories \( \mathcal{F}(\mathcal{A}) \) and \( \mathcal{F}(\mathcal{A}) \) have the same rules of inference concerning “equality,” for they have no predicate symbols.

**RULES OF INference.** For all terms \( s, t, w, s_1, s_2, \ldots \in \mathcal{F}(\mathcal{A}) \) (or \( \mathcal{F}(\mathcal{A}) \)),

\[
\begin{align*}
  s &= s, & t &= s, & s &= t, & s &= t, & s &= t, & s &= t, & s_1 = t_1, \ldots, s_n = t_n, & \phi(s_1, \ldots, s_n) = \phi(t_1, \ldots, t_n).
\end{align*}
\]

(3.4)

where \( \phi(x_1, \ldots, x_n) \) is a term of \( \mathcal{F}(\mathcal{A}) \) (or \( \mathcal{F}(\mathcal{A}) \)) whose variables are contained in \( \{x_1, \ldots, x_n\} \).

We are now ready to present a formal theory of Abelian groups and QS-algebras. Let \( \Gamma \) be a subset of formulas of \( \mathcal{F}(\mathcal{A}) \) (or \( \mathcal{F}(\mathcal{A}) \)). By

\[
\Gamma \vdash_{\mathcal{F}(\mathcal{A})} A \quad (\text{or } \Gamma \vdash_{\mathcal{F}(\mathcal{A})} A),
\]

we mean that there is a finite sequence of formulas \( A_1, A_2, \ldots, A_n \) of \( \mathcal{F}(\mathcal{A}) \) such that for each \( i \),

1. \( A_i \) is an axiom of \( \mathcal{F}(\mathcal{A}) \),
2. \( A_i \in \Gamma \),
3. there exists \( j_1, \ldots, j_k \) (\( j_1, \ldots, j_k < i \)) such that

\[
\frac{A_{j_1}, \ldots, A_{j_k}}{A}.
\]

(3.6)

We say that \( A \) is provable from \( \Gamma \) in \( \mathcal{F}(\mathcal{A}) \) when \( \Gamma \vdash_{\mathcal{F}(\mathcal{A})} A \). In particular, in case of \( \Gamma = \emptyset \), we say that \( A \) is a theorem of \( \mathcal{F}(\mathcal{A}) \) and simply denote it by \( \vdash_{\mathcal{F}(\mathcal{A})} A \).

As an example, we present the following which is called a cancelation rule in the theory of groups:

\[
x \odot y = z \odot x \vdash_{\mathcal{F}(\mathcal{A})} y = z.
\]

(3.7)
Indeed, we have the following finite sequence of formulas:

\[
\begin{align*}
  x \circ y &= z \circ x, \\
  z \circ x &= x \circ z, \\
  x \circ y &= x \circ z, \\
  x^{-1} \circ (x \circ y) &= x^{-1} \circ (x \circ z), \\
  x^{-1} \circ (x \circ y) &= (x^{-1} \circ x) \circ y, \\
  x^{-1} \circ (x \circ z) &= (x^{-1} \circ x) \circ z, \\
  (x^{-1} \circ x) \circ y &= (x^{-1} \circ x) \circ z, \\
  x^{-1} \circ x &= e, \\
  e \circ y &= e \circ z, \\
  e \circ z &= z, \\
  y &= z.
\end{align*}
\]

(3.8)

Next, we define two maps \(\xi\) from the theory \(\mathcal{A}\) of \(QS\)-algebras to the theory \(\mathcal{G}\) of Abelian groups and \(\eta\) from \(\mathcal{G}\) to \(\mathcal{A}\) as follows. For \(\mathcal{F}(\mathcal{A})\),

\[
\begin{align*}
  \xi(x) &\equiv x \quad \text{for each variable } x, \\
  \xi(0) &\equiv e, \\
  \xi(s \ast t) &\equiv \xi(s) \circ \xi(t^{-1}),
\end{align*}
\]

(3.9)

and for \(\mathcal{F}(\mathcal{G})\),

\[
\begin{align*}
  \xi(s = t) &\equiv \xi(s) = \xi(t), \\
  \xi(s = t \implies s' = t') &\equiv \xi(s = t) \implies \xi(s' = t'),
\end{align*}
\]

(3.10)

where \(s, s', t, t' \in \mathcal{F}(\mathcal{G})\).

Conversely, we define a map \(\eta: \mathcal{G} \to \mathcal{A}\) as follows. For \(\mathcal{F}(\mathcal{G})\),

\[
\begin{align*}
  \eta(x) &\equiv x \quad \text{for each variable } x, \\
  \eta(0) &\equiv 0, \\
  \eta(s^{-1}) &\equiv 0 \ast (\eta(s)), \\
  \eta(s \ast t) &\equiv \eta(s) \ast (0 \ast \eta(t)),
\end{align*}
\]

(3.11)

and for \(\mathcal{F}(\mathcal{A})\),

\[
\begin{align*}
  \eta(s = t) &\equiv \eta(s) = \eta(t), \\
  \eta(s = t \implies s' = t') &\equiv \eta(s = t) \implies \eta(s' = t'),
\end{align*}
\]

(3.12)

where \(s, s', t, t' \in \mathcal{F}(\mathcal{A})\).

We are now ready to state our theorem. It is as follows.

**Main theorem 3.9.** (1) For every \(\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{A})\), if \(\Gamma \vdash_{\mathcal{A}} A\), then \(\xi(\Gamma) \vdash_{\mathcal{G}} \xi(A)\); conversely,

(2) for every \(\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{A})\), if \(\Gamma \vdash_{\mathcal{G}} A\), then \(\eta(\Gamma) \vdash_{\mathcal{A}} \eta(A)\); moreover,

(3) \(\Gamma \vdash_{\mathcal{A}} \eta \xi(A)\) if and only if \(\Gamma \vdash_{\mathcal{A}} A\) for every \(\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{A})\);

(4) \(\Gamma \vdash_{\mathcal{A}} \xi \eta(A)\) if and only if \(\Gamma \vdash_{\mathcal{A}} A\) for every \(\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{A})\).

This means that each theorem of \(QS\)-algebras can be translated immediately to that of groups and conversely, every theorem of groups is applied to that of \(QS\)-algebras.

At first we will establish the former part, that is, if \(\Gamma \vdash_{\mathcal{A}} A\), then \(\xi(\Gamma) \vdash_{\mathcal{G}} \xi(A)\).
\textbf{Theorem 3.10.} For $\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{A})$, if $\Gamma \vdash_{\mathcal{A}} A$, then $\xi(\Gamma) \vdash_{\mathcal{A}^\mathcal{G}} \xi(A)$.

\textbf{Proof.} It is sufficient to show that for every axiom $A$ of $QS$-algebras, $\xi(A)$ is provable in the theory $\mathcal{A}^\mathcal{G}$ of Abelian groups. For the sake of simplicity, we treat only the case of axiom (QS3): $(x \ast y) \ast z = x \ast (z \ast (0 \ast y))$. Other cases can be proved similarly. Since

$$
\xi((x \ast y) \ast z) = (x \circ y^{-1}) \circ z^{-1},
$$

we have to show that

$$
(x \circ y^{-1}) \circ z^{-1} = x \circ \left\{ z \circ (y^{-1})^{-1} \right\}^{-1}.
$$

We have the following:

$$
x \circ \left\{ z \circ (y^{-1})^{-1} \right\}^{-1} = x \circ (z \circ y)^{-1}
= x \circ (y^{-1} \circ z^{-1})
= (x \circ y^{-1}) \circ z^{-1}.
$$

Hence, if $\Gamma \vdash_{\mathcal{A}} A$, then $\xi(\Gamma) \vdash_{\mathcal{A}^\mathcal{G}} \xi(A)$. $\square$

Conversely, we can show that if $\Gamma \vdash_{\mathcal{A}^\mathcal{G}} A$, then $\eta(\Gamma) \vdash_{\mathcal{A}} \eta(A)$.

\textbf{Theorem 3.11.} For $\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{A}^\mathcal{G})$, if $\Gamma \vdash_{\mathcal{A}^\mathcal{G}} A$, then $\eta(\Gamma) \vdash_{\mathcal{A}} \eta(A)$.

\textbf{Proof.} As above, it is sufficient to show that $\eta(A)$ is provable in the theory $\mathcal{A}^\mathcal{G}$ of $QS$-algebras for every axiom $A$ of the theory of Abelian groups.

For the case of (G1), we have to show that

$$
(x \ast (0 \ast y)) \ast (0 \ast z) = x \ast (0 \ast (y \ast (0 \ast z)))
$$

(3.16)

because $\eta((x \circ y) \circ z) = (x \ast (0 \ast y)) \ast (0 \ast z)$ and $\eta(x \circ (y \circ z)) = x \ast (0 \ast (y \ast (0 \ast z)))$.

By the proposition above, we have

$$
(x \ast (0 \ast y)) \ast (0 \ast z) = x \ast ((0 \ast z) \ast (0 \ast (0 \ast y)))
= x \ast ((0 \ast z) \ast y)
= x \ast (0 \ast (y \ast (0 \ast z))).
$$

(3.17)

Other cases are proved easily, so we omit their proofs.

The theorem can be proved completely. $\square$

Moreover, it follows from Lemma 3.5 and Corollary 3.6 that we have $\vdash_{\mathcal{A}} \eta \xi(t) = t$ for every term $t \in \mathcal{F}(\mathcal{A})$. For the case of Abelian groups, it is easy to prove that $\vdash_{\mathcal{A}^\mathcal{G}} \xi \eta(s) = s$ for every term $s \in \mathcal{F}(\mathcal{A}^\mathcal{G})$. Thus we have the following.

\textbf{Theorem 3.12.} For these maps,

1. $\Gamma \vdash_{\mathcal{A}} \eta \xi(A)$ if and only if $\Gamma \vdash_{\mathcal{A}} A$ for all $\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{A})$,
2. $\Gamma \vdash_{\mathcal{A}^\mathcal{G}} \xi \eta(A)$ if and only if $\Gamma \vdash_{\mathcal{A}^\mathcal{G}} A$ for all $\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{A}^\mathcal{G})$. 


4. Some properties. In this section, we prove other properties of QS-algebras, especially properties about the $G$-part and mediality. That is,

(1) the $G$-part $G(X)$ of a QS-algebra $X$ is a normal subgroup generated by the class of all elements of order 2 of $X$;

(2) every QS-algebra $X$ is medial, that is, it satisfies the condition

$$(x \ast y) \ast (z \ast u) = (x \ast z) \ast (y \ast u)$$

(4.1)

for all elements $x, y, z, u \in X$.

Let $X$ be a QS-algebra. A subset $G(X) = \{x \in X \mid 0 \ast x = x\}$ is called a $G$-part of $X$. For the $G$-part of $X$, we have the following results.

**Proposition 4.1.** If $x, y \in G(X)$, then $x \ast y = y \ast x$.

**Proof.** Suppose $x, y \in G(X)$. Since $0 \ast x = x$ and $0 \ast y = y$, we have

$$x \ast y = (0 \ast x) \ast (0 \ast y) = (0 \ast (0 \ast y)) \ast x \quad \text{(by \, QS3)}$$

$$= y \ast x. \quad \square$$

**Proposition 4.2.** If $x, y \in G(X)$, then $x \ast y \in G(X)$.

**Proof.** Suppose that $x, y \in G(X)$. It follows that $(0 \ast (x \ast y)) \ast (x \ast y) = (y \ast x) \ast (x \ast y) = 0$ by Lemma 3.5. Thus we have $0 \ast (x \ast y) = x \ast y$, that is, $x \ast y \in G(X)$. \hfill \square

Since any QS-algebra $X$ may be considered as an Abelian group, Proposition 4.2 implies that $G(X)$ is a (normal) subgroup of $X$. Moreover, since $x^2 = x \ast x = x \ast (0 \ast x) = x \ast x = 0$ for $x \in G(X)$, every nonunit element in $G(X)$ is of order 2. Hence, we can conclude that the $G$-part $G(X)$ is the normal subgroup generated by the class of all elements of order 2. It is easy to show that $G(X) = \{x \in X \mid x \text{ is of order } 2\} \cup \{0\}$.

For the statement (2) above, in [1, Theorem 3.6], it is proved that a QS-algebra $X$ is medial if and only if the condition $x \ast (y \ast z) = (x \ast y) \ast (0 \ast z)$ holds for all $x, y, z \in X$. On the other hand, by Lemma 3.7, we have $(x \ast y) \ast (0 \ast z) = z \ast (0 \ast (x \ast y)) = z \ast (y \ast x)$. Thus $X$ is medial if and only if the condition $x \ast (y \ast z) = z \ast (y \ast x)$ holds for all $x, y, z \in X$. By using this characterization of mediality, we will prove the following.

**Theorem 4.3.** Every QS-algebra is medial.

**Proof.** It is sufficient to show that $x \ast (y \ast z) = z \ast (y \ast x)$ holds for all $x, y, z \in X$. Since

$$x \ast (y \ast z) = 0 \ast (0 \ast (x \ast (y \ast z))) \quad \text{(by Corollary 3.6)}$$

$$= 0 \ast ((y \ast z) \ast x) \quad \text{(by Lemma 3.5)}$$

$$= 0 \ast ((y \ast x) \ast z) \quad \text{(by QS3)}$$

$$= z \ast (y \ast x) \quad \text{(by Lemma 3.5)},$$

it follows that $X$ is medial. \hfill \square
5. **Application.** Let \( V = \{x, y, z, \ldots \} \) be a set of variables and \( 0 \) a constant. We define a *term* and *equation* as follows:

1. \( 0 \) is a term;
2. each variable in \( V \) is a term;
3. if \( s, t \) are terms, then \( s \ast t \) is also a term;
4. if \( s, t \) are terms, then \( s = t \) is an equation.

Thus, for example, \( 0, 0 \ast x, x \ast (0 \ast y), x \ast y \) are terms and thus \( 0 = 0 \ast x, x \ast (0 \ast y) = x \ast y \) are equations. By \( t(x, y, \ldots) \) we mean a term whose variables are in \( \{x, y, \ldots\} \). We say that an equation \( s(x, y, \ldots) = t(x, y, \ldots) \) is satisfied in a QS-algebra \( X \) when for all elements \( a, b, \ldots \in X \), we have \( u^X(a, b, \ldots) = v^X(a, b, \ldots) \). In particular, an equation \( t(x, y, \ldots) = 0 \) is said to be satisfied in \( X \) if \( t^X(a, b, \ldots) = 0 \) for all elements \( a, b, \ldots \in X \). In the following, by \( t(a, b, \ldots) \) we mean an element \( t^X(a, b, \ldots) \) which is an interpretation of a term \( t(x, y, \ldots) \) in \( X \), that is, \( t(a, b, \ldots) \) is an abbreviation of \( t^X(a, b, \ldots) \).

We also define a condition \((C)\) which plays an important role to develop our theory:

\[(C) \text{ for all } x \text{ and for all } y, \text{ there exists } t(x, y) \text{ such that } (0 \ast (0 \ast x)) \ast (0 \ast (0 \ast y)) = 0 \ast (0 \ast t(x, y)), \quad t(x, x) = 0. \tag{5.1} \]

By using condition \((C)\), we have the following theorem which shows the relation between \( Q \)-algebras and QS-algebras.

**Theorem 5.1.** Let \((X; \ast, 0)\) be a \( Q \)-algebra. \((X; \ast, 0)\) satisfies condition \((C)\) if and only if \((X^*; \ast, 0)\) is a QS-algebra, where \( X^* = \{0 \ast (0 \ast a) \mid a \in X\} \).

**Proof.** If part. For all \( u, v \in X^* \), there exist \( a, b \in X \) such that \( u = 0 \ast (0 \ast a), v = 0 \ast (0 \ast b) \). It follows from condition \((C)\) that \( u \ast v \in X^* \) and that \( X^* \) is a subalgebra of \( X \). Hence, \( X^* \) is a \( Q \)-algebra. We define \( u \cdot v = u \ast (0 \ast v) \) and \( u^{-1} = 0 \ast u \). Since \( u, v, 0 \in X^* \), we have \( u \cdot v \in X^* \). Moreover for this operation, we can show that

\[
\begin{align*}
(5.2) \quad (i) \quad u \cdot v &= v \cdot u, \\
(ii) \quad u \cdot 0 &= 0 \cdot u = u, \\
(iii) \quad u \cdot (0 \ast u) &= (0 \ast u) \cdot u = 0, \\
(iv) \quad (u \cdot v) \cdot w &= u \cdot (v \cdot w).
\end{align*}
\]

For the sake of simplicity, we only prove the case of \((iv)\). Before doing so, we note the following result: \((u \cdot v) \cdot w = (u \cdot w) \cdot v\) for all \( u, v, w \in X^* \). Because

\[
\begin{align*}
(u \cdot v) \cdot w &= (u \cdot v) \ast (0 \ast w) \\
&= (u \ast (0 \ast v)) \ast (0 \ast w) \\
&= (u \ast (0 \ast w)) \ast (0 \ast v) \\
&= (u \cdot w) \cdot v,
\end{align*}
\]

it follows from the result that \((u \cdot v) \cdot w = (v \cdot u) \cdot w = (v \cdot w) \cdot u = u \cdot (v \cdot w)\).

Thus the above means that \((X^*; \cdot, -1, 0)\) is an Abelian group. For this group, if we define \( u \cdot v = u \cdot (0 \ast v) \), then \((X^*; \cdot, 0)\) is a QS-algebra. Clearly we have \( u \cdot v = u \cdot (0 \ast v) = u \ast (0 \ast (0 \ast v)) = u \ast v \) for all \( u, v \in X^* \). That is, \((X^*; \cdot, 0)\) is a QS-algebra.
Only if part. Conversely, we suppose that \((X^*;\ast,0)\) is a QS-algebra. For all \(u,v \in X^*\), there exist \(a,b \in X\) such that \(u = 0 \ast (0 \ast a)\), \(v = 0 \ast (0 \ast b)\). Since \(u \ast v \in X^*\), \(u \ast v\) has to have a form of \(0 \ast (0 \ast t(a,b))\). This means that \(X^*\) satisfies the condition

\[
\forall x \forall y \exists t(x,y) \quad (0 \ast (0 \ast x)) \ast (0 \ast (0 \ast y)) = 0 \ast (0 \ast t(x,y)). \tag{5.3}
\]

It is obvious that \(t(a,a) = 0\) for all \(a \in X\), that is, \(t(x,x) = 0\). Thus \(X^*\) satisfies condition (C).

We consider some cases of \(t(x,y)\) as corollaries to the theorem. First of all, let \(t(x,y)\) be a form of \(x \ast y\), that is, \(t(x,y) = x \ast y\). In this case, condition (C) has a form of

\[
(0 \ast (0 \ast x)) \ast (0 \ast (0 \ast y)) = 0 \ast (0 \ast (x \ast y)). \tag{5.4}
\]

For a map \(f : X \to X^*\) defined by \(f(x) = 0 \ast (0 \ast x)\), since \((x,y) \in \text{Ker } f\) if and only if \(0 \ast (0 \ast x) = 0 \ast (0 \ast y)\), we have that \(X/\text{Ker } f\), the quotient Q-algebra modulo \(\text{Ker } f\), is isomorphic to \(X^*\), that is, \(X/\text{Ker } f \cong X^*\). Hence, we have the following.

**COROLLARY 5.2.** If \(f : X \to X^*\) is a map defined by \(f(x) = 0 \ast (0 \ast x)\), then \(X/\text{Ker } f \cong X^*\).

We define a term \(t^n(x,y)\) for all nonnegative integers \(n\) as follows:

\[
t^0(x,y) = 0 \ast (x \ast y),
\[
t^n(x,y) = t^{n-1}(x,y) \ast (0 \ast (0 \ast (x \ast y))). \tag{5.5}
\]

In this case, the corresponding condition (C) is (C)

\[
(0 \ast (0 \ast x)) \ast (0 \ast (0 \ast y)) = 0 \ast (0 \ast t^n(x,y)). \tag{5.6}
\]

We now have the following result as to condition (C).

**COROLLARY 5.3.** Let \(X\) be a QS-algebra. If \(X\) satisfies condition (C), then \(X^*\) is an Abelian group in which every element has order at most \((n+2)\).

**PROOF.** For condition (C), if we take \(y = 0\), then we have \(0 \ast (0 \ast x) = 0 \ast (0 \ast t^n(x,0))\), that is,

\[
0 \ast (0 \ast x) = 0 \ast [0 \ast \{(0 \ast x) \ast (0 \ast (0 \ast x)) \ast (0 \ast (0 \ast x)) \ast \cdots \ast (0 \ast (0 \ast x))\}]. \tag{5.7}
\]

Since any element \(u \in X^*\) has a form of \(0 \ast (0 \ast a)\) for some element \(a \in X\), it follows from (C) that

\[
u = 0 \ast [0 \ast \{(0 \ast a) \ast (0 \ast (0 \ast a)) \ast (0 \ast (0 \ast a)) \ast \cdots \ast (0 \ast (0 \ast a))\}]
\[
= 0 \ast [0 \ast \{(\cdots ((0 \ast a) \ast u) \ast u) \ast \cdots \ast u\}]
\[
= ((\cdots ((0 \ast a) \ast u) \ast u) \ast \cdots \ast u \ast u). \tag{5.8}
\]
On the other hand, since \((0 \ast u) \ast u = (0 \ast u) \ast (0 \ast (0 \ast u)) = u^{-1} \cdot u^{-1}\) in the Abelian group \(X^*\), we have
\[ u = (u^{-1} \cdot u^{-1}) \cdot u^{-1} \cdots u^{-1} = (u^{-1})^{n+1} = u^{-(n+1)} \] (5.9)
and hence \(u^{n+2} = 0\). This means that each element of \(X^*\) has order at most \(n + 2\).

As the last case, we suppose \(t(x, y) = (x \ast y) \ast (y \ast x)\). Since condition \((C)\) is
\[ (0 \ast (0 \ast x)) \ast (0 \ast (0 \ast y)) = 0 \ast ((0 \ast (x \ast y) \ast (y \ast x))) \] (5.10)
in this case, if we take \(y = 0\), then we have the condition
\[ 0 \ast (0 \ast x) = 0 \ast (0 \ast (x \ast (0 \ast x))). \] (5.11)
This implies that
\[ 0 \ast (0 \ast a) = 0 \ast (0 \ast (a \ast (0 \ast a))) \] (5.12)
for all \(a \in X\). In particular, any element \(u \in X^*\) satisfies the condition. Hence, since \(u = 0 \ast (0 \ast u)\) for all elements \(u \in X^*\), we have in the Abelian group \(X^*\)
\[ u = u \ast (0 \ast u) = u \cdot u. \] (5.13)
This means that every element of \(X^*\) is an idempotent. But \(u \cdot u u\) means \(u = 0\) and \(X^* = \{0\}\) is trivial.

**Corollary 5.4.** Let \(X\) be a \(Q\)-algebra. If \(X\) satisfies the condition
\[ (0 \ast (0 \ast x)) \ast (0 \ast (0 \ast y)) = 0 \ast ((0 \ast (x \ast y) \ast (y \ast x))), \] (5.14)
then \(X^*\) is an Abelian group in which every element is an idempotent.

**References**


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