UNIFORMLY SUMMING SETS OF OPERATORS ON SPACES OF CONTINUOUS FUNCTIONS

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Received 30 March 2004

Let \( X \) and \( Y \) be Banach spaces. A set \( \mathcal{M} \) of 1-summing operators from \( X \) into \( Y \) is said to be \( \textit{uniformly summing} \) if the following holds: given a weakly 1-summing sequence \( (x_n) \) in \( X \), the series \( \sum_n \|Tx_n\| \) is uniformly convergent in \( T \in \mathcal{M} \). We study some general properties and obtain a characterization of these sets when \( \mathcal{M} \) is a set of operators defined on spaces of continuous functions.

2000 Mathematics Subject Classification: 47B38, 47B10.

1. Introduction. Throughout this paper, \( X \) and \( Y \) will be Banach spaces. If \( X \) is a Banach space, \( B_X = \{ x \in X : \|x\| \leq 1 \} \) will denote its closed unit ball and \( X^* \) will be the topological dual of \( X \). Given a real number \( p \in [1, \infty) \), a (linear) operator \( T : X \to Y \) is said to be \( p \)-\textit{summing} if there exists a constant \( C > 0 \) such that

\[
\left( \sum_{i=1}^{n} \|Tx_i\|^p \right)^{1/p} \leq C \cdot \sup \left\{ \left( \sum_{i=1}^{n} |\langle x^*, x_i \rangle|^p \right)^{1/p} : x^* \in B_{X^*} \right\},
\]

for every finite set \( \{x_1, \ldots, x_n\} \subset X \). The least \( C \) for which the above inequality always holds is denoted by \( \pi_p(T) \) (the \( p \)-summing norm of \( T \)). The linear space of all \( p \)-summing operators from \( X \) into \( Y \) is denoted by \( \Pi_p(X,Y) \), which is a Banach space endowed with the \( p \)-summing norm.

As usual, \( \ell^p_w(X) \) will be the Banach space of weakly \( p \)-summable sequences in \( X \), that is, the sequences \( (x_n) \subset X \) satisfying \( \sum_n |\langle x^*, x_n \rangle|^p \) is convergent for all \( x^* \in X^* \); the norm in \( \ell^p_w(X) \) is \( \epsilon_p(x_n) = \sup \left\{ (\sum_n |\langle x^*, x_n \rangle|^p)^{1/p} : x^* \in B_{X^*} \right\} \). The set of all strongly \( p \)-summable sequences in \( X \) is denoted by \( \ell^p_s(X) \); the norm in this space is \( \pi_p(x_n) = \left( \sum_n \|x_n\|^p \right)^{1/p} \). If \( T \in \Pi_p(X,Y) \), the correspondence \( \hat{T} : (x_n) \mapsto (Tx_n) \) always induces a bounded operator from \( \ell^p_w(X) \) into \( \ell^p_s(Y) \) with \( \|\hat{T}\| = \pi_p(T) \) [5, Proposition 2.1].

Families of operators arise in different applications: equations containing a parameter, homotopies of operators, and so forth. In these applications, it may be very interesting to know that, given a set \( \mathcal{M} \subset \Pi_p(X,Y) \) and \( (x_n) \in \ell^p_w(X) \), the series \( \sum_n \|Tx_n\|^p \) is uniformly convergent in \( T \in \mathcal{M} \). The main purpose of this paper is to study \textit{uniformly \( p \)-summing} sets, that is, those sets \( \mathcal{M} \subset \Pi_p(X,Y) \) for which, given \( (x_n) \in \ell^p_w(X) \), the series \( \sum_n \|Tx_n\|^p \) is uniformly convergent in \( T \in \mathcal{M} \). These sets also enjoy some properties that justify their study; the next proposition lists some of them.
**Proposition 1.1.** (a) Let \((T_k)\) be a sequence in \(\Pi_p(X,Y)\). Then, \(\tilde{T}_k\to 0\) pointwise if and only if \(T_k\to 0\) pointwise and \((T_k)\) is uniformly \(p\)-summing.

(b) Let \(\mathcal{M} \subset \Pi_p(X,Y)\) be a uniformly \(p\)-summing set. If \(\mathcal{M}\) is endowed with the strong operator topology, then the map \(T \in \mathcal{M} \rightarrow \sum_n \|Tx_n\|^p \in \mathbb{R}\) is continuous for every \((x_n) \in \ell^p_w(X)\).

A basic argument shows that uniformly \(p\)-summing sets are bounded for the \(p\)-summing norm. In fact, if \(X\) does not contain any copy of \(c_0\), bounded sets and uniformly 1-summing sets are the same. That is the reason for which we only consider operators defined on a \(\mathcal{C}(\Omega)\)-space, \(\Omega\) being a compact Hausdorff space. We recall that every weakly compact operator \(T : \mathcal{C}(\Omega) \rightarrow Y\) has a representing measure \(m_T : \Sigma \rightarrow Y\) defined by \(m_T(B) = T^{**}(\chi_B)\) for all \(B \in \Sigma\), where \(\Sigma\) denotes the Borel \(\sigma\)-field of subsets of \(\Omega\) and \(\chi_B\) denotes the characteristic function of \(B\). The vector measure \(m_T\) is regular and countably additive [6, Theorem VI.2.5 and Corollary VI.2.14]. If we denote by \(\tilde{T}\) the operator \(T^{**}\) restricted to \(B(\Sigma)\) (the space of all bounded Borel-measurable scalar-valued functions defined on \(\Omega\)), then

\[
\tilde{T}\varphi = \int_{\Omega} \varphi \, dm_T,
\]

for all \(\varphi \in B(\Sigma)\) (the integral is the elementary Bartle integral [6, Definition I.1.12]).

It is well known that every \(p\)-summing operator defined on a Banach space \(X\) is weakly compact. In Section 2, we consider 1-summing operators \(T\) defined on \(\mathcal{C}(\Omega)\); these operators are characterized as those with representing measure \(m_T\) having finite variation and \(\pi_1(T) = |m_T| (\Omega)\) [6, Theorem VI.3.3]. We show that a set \(\mathcal{M} \subset \Pi_1(\mathcal{C}(\Omega), Y)\) is uniformly 1-summing if and only if the family of all variation measures \(|m_T| : T \in \mathcal{M}\) is uniformly bounded and there is a countably additive measure \(\mu : \Sigma \rightarrow [0, \infty)\) such that \(|m_T| : T \in \mathcal{M}\) is uniformly \(\mu\)-continuous.

In Section 3, we mention a special class of uniformly \(p\)-summing operators: uniformly dominated sets. The relationship between uniformly summing sets and relatively weak compactness is also studied. Finally, we give some examples and open problems.

### 2. Uniformly 1-summing sets in \(\Pi_1(\mathcal{C}(\Omega), Y)\).

Before facing our main theorem, we include three results which correspond to the vector measure theory. These results will be usually invoked along the following lines.

**Proposition 2.1** [6, Proposition I.1.17]. The following statements about a collection \(\{m_i : i \in I\}\) of \(Y\)-valued measures defined on a \(\sigma\)-field \(\Sigma\) are equivalent:

(a) the set \(\{m_i : i \in I\}\) is uniformly countably additive, that is, if \((E_n)\) is a sequence of pairwise disjoint members of \(\Sigma\), then \(\lim_n \|\sum_{k \geq n} m_i(E_k)\| = 0\) uniformly in \(i \in I\),

(b) the set \(\{\gamma^* \circ m_i : i \in I, \gamma^* \in B_Y^{**}\}\) is uniformly countably additive,

(c) if \((E_n)\) is a sequence of pairwise disjoint members of \(\Sigma\), then \(\lim_n \|m_i(E_n)\| = 0\) uniformly in \(i \in I\),

(d) if \((E_n)\) is a sequence of pairwise disjoint members of \(\Sigma\), then \(\lim_n \|m_i\|(E_n) = 0\) uniformly in \(i \in I\), where \(\|m_i\|\) denotes the semivariation of \(m_i\),

(e) the set \(\{||\gamma^* \circ m_i| : i \in I, \gamma^* \in B_Y^{**}\}\) is uniformly countably additive.
**Theorem 2.2** [6, Theorem I.2.4]. Let \( \{m_i : \Sigma \to Y : i \in I\} \) be a uniformly bounded (with respect to the semivariation) family of countably additive vector measures on a \( \sigma \)-field \( \Sigma \). The family \( \{m_i : i \in I\} \) is uniformly countably additive if and only if there exists a positive real-valued countably additive measure \( \mu \) on \( \Sigma \) such that \( \{m_i : i \in I\} \) is uniformly \( \mu \)-continuous, that is,

\[
\lim_{\mu(E) \to 0} \|m_i(E)\| = 0
\]

uniformly in \( i \in I \).

If \( \Omega \) is a compact Hausdorff space and \( \Sigma \) denotes the \( \sigma \)-field of the Borel subsets of \( \Omega \), a vector measure \( m \) on \( \Sigma \) is regular if for each Borel set \( E \) and \( \varepsilon > 0 \) there exists a compact set \( K \) and an open set \( O \) such that \( K \subset E \subset O \) and \( \|m|| (O \setminus K) < \varepsilon \).

**Proposition 2.3** [6, Lemma VI.2.13]. Let \( \mathcal{K} \) be a family of regular (countably additive) scalar measures defined on \( \Sigma \). Each of the following statements implies all the others:

(a) for each pairwise disjoint sequence \( (O_n) \) of open subsets of \( \Omega \), \( \lim_{n} \mu(O_n) = 0 \) uniformly in \( \mu \in \mathcal{K} \),

(b) for each pairwise disjoint sequence \( (O_n) \) of open subsets of \( \Omega \), \( \lim_{n} |\mu|(O_n) = 0 \) uniformly in \( \mu \in \mathcal{K} \),

(c) \( \mathcal{K} \) is uniformly countably additive,

(d) \( \mathcal{K} \) is uniformly regular, that is, if \( E \in \Sigma \) and \( \varepsilon > 0 \), then there exists a compact set \( K \) and an open set \( O \) such that \( K \subset E \subset O \) and \( \sup_{\mu \in \mathcal{K}} |\mu|(O \setminus K) < \varepsilon \).

Now, we are able to show our main result. In the proof, we will use the fact that \(|m_T|\) is regular when \( T : \mathcal{E}(\Omega) \to Y \) is 1-summing [7, Proposition 15.21].

**Theorem 2.4.** Let \( \mathcal{M} \subset \Pi_1 (\mathcal{E}(\Omega), Y) \) be a bounded set. The following statements are equivalent:

(a) \( \mathcal{M} \) is uniformly 1-summing,

(b) the family of nonnegative measures \( \{|m_T| : T \in \mathcal{M}\} \) is uniformly countably additive,

(c) given \( \varepsilon > 0 \) and a disjoint sequence \( (E_n) \) of Borel subsets of \( \Omega \), there exists \( n_0 \in \mathbb{N} \) such that

\[
\sum_{n \geq n_0} ||m_T(E_n)|| < \varepsilon,
\]

for all \( T \in \mathcal{M} \).

**Proof.** (a) \( \Rightarrow \) (b). According to [6, Lemma VI.2.13], it suffices to show that \( \lim_{n \to \infty} |m_T|(O_n) = 0 \) uniformly in \( T \in \mathcal{M} \), for all disjoint sequences \( (O_n) \) of open subsets of \( \Omega \). By contradiction, suppose that there exists \( \varepsilon > 0 \), a sequence \( (T_n) \) in \( \mathcal{M} \), and a strictly increasing sequence \( (k_n) \) of natural numbers such that

\[
|m_{T_n}|(O_{k_n}) > 2 \varepsilon, \quad \forall n \in \mathbb{N}.
\]
Now we consider the operators $S_n : \mathcal{C}(\Omega, O_{k_n}) \to Y$ defined by

$$S_n \varphi = \int_{O_{k_n}} \varphi \, dm_{T_n},$$

for all $\varphi \in \mathcal{C}(\Omega, O_{k_n})$, where $\mathcal{C}(\Omega, O_{k_n})$ is the closed subspace of $\mathcal{C}(\Omega)$ formed by all continuous functions $\varphi$ on $\Omega$ such that $\varphi$ vanishes in $\Omega \setminus O_{k_n}$. It is known that $\pi_1(S_n) = |m_{T_n}|(O_{k_n})$, for all $n \in \mathbb{N}$ [7, Theorem 19.3]. For each $n \in \mathbb{N}$, we can choose a finite set $\{\varphi_1^n, \ldots, \varphi_{p_n}^n\} \subset \mathcal{C}(\Omega, O_{k_n})$ satisfying $\sum_{i=1}^{p_n} \|S_n \varphi_i^n\| > \pi_1(S_n) - \varepsilon$. (2.5)

Since the open sets $O_{k_n}$ are disjoint, it follows that the sequence $(\varphi_1^1, \ldots, \varphi_{p_1}^1, \varphi_2^1, \ldots, \varphi_{p_2}^1, \ldots)$ belongs to $\ell^1_w(\mathcal{C}(\Omega))$. Nevertheless, for all $n \in \mathbb{N}$, we have

$$\sum_{m \geq n} \sum_{i=1}^{p_m} \|T_n \varphi_i^m\| \geq \sum_{i=1}^{p_n} \|T_n \varphi_i^n\| = \sum_{i=1}^{p_n} \|S_n \varphi_i^n\| > \pi_1(S_n) - \varepsilon = |m_{T_n}|(O_{k_n}) - \varepsilon > \varepsilon. \quad (2.6)$$

This denies (a) and proves that (a) implies (b).

(b)⇒(c). Again we proceed by contradiction. Suppose $(E_n)$ is a disjoint sequence of Borel subsets of $\Omega$ for which there exists $\varepsilon > 0$, a sequence $(T_n)$ in $\mathcal{M}$, and a strictly increasing sequence $(k_n)$ of natural numbers so that

$$\sum_{i=k_n+1}^{k_{n+1}} \|m_{T_n}(E_i)\| > \varepsilon, \quad \forall n \in \mathbb{N}. \quad (2.7)$$

If we put $B_n = \bigsqcup_{i=k_n+1}^{k_{n+1}} E_i$, the above inequality yields $|m_{T_n}|(B_n) > \varepsilon$. So, in view of [6, Proposition I.1.17], the family $\{|m_T| : T \in \mathcal{M}\}$ is not uniformly countably additive.

(c)⇒(b). We need to prove

$$\lim_{n \to \infty} |m_T|(E_n) = 0 \quad \text{uniformly in } T \in \mathcal{M}, \quad (2.8)$$

for all disjoint sequences $(E_n)$ of Borel subsets of $\Omega$. Suppose (b) fails. Then, there exists $\varepsilon > 0$, a sequence $(T_n)$ in $\mathcal{M}$, and a strictly increasing sequence $(k_n)$ of natural numbers satisfying

$$|m_{T_n}|(E_{k_n}) > \varepsilon, \quad \forall n \in \mathbb{N}. \quad (2.9)$$

For each $n \in \mathbb{N}$, we choose a finite partition $\{E_1^n, \ldots, E_{p_n}^n\}$ of $E_{k_n}$ for which

$$\sum_{i=1}^{p_n} \|m_{T_n}(E_i^n)\| > \varepsilon. \quad (2.10)$$

Then, the disjoint sequence $(E_1^1, E_2^1, E_1^2, \ldots, E_{p_2}^2, \ldots)$ does not satisfy (c).
(b)⇒(a). According to [6, Theorem I.2.4] there exists a countably additive measure $\mu : \Sigma \rightarrow [0, \infty)$ so that

$$\lim_{\mu(E) \to 0} |m_T|(E) = 0 \quad \text{uniformly in } T \in \mathcal{M}. \quad (2.11)$$

Hence, given $\varepsilon > 0$, there exists $\delta > 0$ such that, if $E \in \Sigma$ verifies $\mu(E) < \delta$, then $|m_T|(E) < \varepsilon/2$, for all $T \in \mathcal{M}$.

Next, given $(\varphi_n) \in \ell^1_w(\mathcal{E}(\Omega))$ with $\varepsilon_1(\varphi_n) \leq 1$, notice that the series $\sum_{n=1}^{\infty} |\varphi_n(t)|$ is convergent for all $t \in \Omega$. Put $f_n(t) = \sum_{k=1}^{n} |\varphi_k(t)|$ and $f(t) = \lim_{n \to \infty} f_n(t)$. By Egorov’s theorem, the sequence $(f_n)$ is quasi-uniformly convergent to $f$. Then, there exists $E \in \Sigma$ such that $\mu(E) < \delta$ and

$$f_n|_{\Omega \setminus E} \rightharpoonup f|_{\Omega \setminus E} \quad (2.12)$$

uniformly. If $C = \sup \{|m_T|(\Omega) : T \in \mathcal{M}\}$, there exists $n_0 \in \mathbb{N}$ so that

$$\sum_{n \geq n_0} |\varphi_n(t)| < \frac{\varepsilon}{2C}, \quad \forall t \in \Omega \setminus E. \quad (2.13)$$

Now,

$$\begin{align*}
\sum_{n \geq n_0} \|T \varphi_n\| &= \sum_{n \geq n_0} \left\| \int_{\Omega} \varphi_n(t) \, dm_T \right\| \\
&\leq \sum_{n \geq n_0} \left\| \int_{E} \varphi_n(t) \, dm_T \right\| + \sum_{n \geq n_0} \left\| \int_{\Omega \setminus E} \varphi_n(t) \, dm_T \right\| \\
&\leq \sum_{n \geq n_0} \int_{E} \| \varphi_n(t) \| \, dm_T + \sum_{n \geq n_0} \int_{\Omega \setminus E} \| \varphi_n(t) \| \, dm_T \\
&= \int_{E} \left( \sum_{n \geq n_0} |\varphi_n(t)| \right) \, dm_T + \int_{\Omega \setminus E} \left( \sum_{n \geq n_0} |\varphi_n(t)| \right) \, dm_T \\
&\leq |m_T|(E) + \frac{\varepsilon}{2C} |m_T|(\Omega \setminus E) \\
&< \varepsilon. \quad \square
\end{align*} \quad (2.14)$$

We denote by $\mathcal{V}(X,Y)$ the class of completely continuous operators from $X$ into $Y$, that is, the class of operators which map weakly convergent sequences in $X$ into norm-convergent sequences in $Y$. A set $\mathcal{M} \subset \mathcal{V}(X,Y)$ is said to be uniformly completely continuous if, given a weakly convergent sequence $(x_n)$ in $X$, $(Tx_n)$ is norm convergent uniformly in $T \in \mathcal{M}$. The following result gives some characterizations of uniformly completely continuous sets in $\mathcal{V}(\mathcal{E}(\Omega), Y)$. Recall that an operator $T$ defined on $\mathcal{E}(\Omega)$ is completely continuous if and only if $T$ is weakly compact [6, Corollary VI.2.17], so $m_T$ is countably additive and regular, too.

**Theorem 2.5.** Let $\mathcal{M} \subset \mathcal{V}(\mathcal{E}(\Omega), Y)$ be a bounded set for the operator norm. The following statements are equivalent:

(a) $\mathcal{M}$ is uniformly completely continuous,

(b) the family $\{m_T : T \in \mathcal{M}\}$ is uniformly countably additive,
(c) $\mathcal{M}^* = \{T^* : T \in \mathcal{M}\}$ is collectively weakly compact, that is, the set $\bigcup_{T \in \mathcal{M}} T^*(B_{Y^*})$ is relatively weakly compact in $\mathcal{C}(\Omega)^*$.

**Proof.** (a)$\Rightarrow$(b). By [6, Proposition I.1.17], the family $\{m_T : T \in \mathcal{M}\}$ is uniformly countably additive if and only if $\mathcal{N} = \{y^* \circ m_T : T \in \mathcal{M}, y^* \in B_{Y^*}\}$ is. According to [6, Lemma VI.1.13], we have to prove that

$$\lim_{n \to \infty} y^* \circ m_T(O_n) = 0 \text{ uniformly in } \mathcal{N},$$

for all disjoint sequences $(O_n)$ of open subsets of $\Omega$. By contradiction, suppose there exists such a sequence $(O_n)$ for which $\lim_{n \to \infty} y^* \circ m_T(O_n) = 0$ but not uniformly in $\mathcal{N}$. Then, there exists $\varepsilon > 0$ and sequences $(y^*_n) \subset B_{Y^*}$, $(T_n) \in \mathcal{M}$, and $(O_{kn}) \subset (O_n)$ such that

$$|y^*_n \circ m_{T_n}(O_{kn})| > \varepsilon, \quad \forall n \in \mathbb{N}. \quad (2.16)$$

Now, using the regularity of each $m_{T_n}$, we can find a sequence of compact sets $(H_n)$ with $H_n \subset O_{kn}$ and

$$\|m_{T_n}\|(O_{kn} \setminus H_n) < \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{N}, \quad (2.17)$$

($\|m\|$ denotes the semivariation of $m$, that is, $\|m\|(E) = \sup\{|y^* \circ m|(E) : y^* \in B_{Y^*}\}$). By Urysohn’s lemma, for every $n \in \mathbb{N}$ there exists a continuous function $\varphi_n : \Omega \to [0,1]$ such that $\varphi_n(H_n) = 1$ and $\varphi_n(\Omega \setminus O_{kn}) = 0$. Obviously, the series $\sum_{n=1}^{\infty} \varphi_n$ is unconditionally convergent in $\mathcal{C}(\Omega)$. Since $\mathcal{M}$ is uniformly completely continuous, there exists $n_0 \in \mathbb{N}$ such that

$$\|T \varphi_n\| < \frac{\varepsilon}{2}, \quad \forall n \geq n_0, \forall T \in \mathcal{M}. \quad (2.18)$$

Then, we have

$$\|m_{T_n}(O_{kn})\| \leq \|m_{T_n}(O_{kn}) - T_n \varphi_n\| + \|T_n \varphi_n\|$$

$$= \left| \int_{O_{kn}} \chi_{O_{kn}} \, dm_{T_n} - \int_{O_{kn}} \varphi_n \, dm_{T_n} \right| + \|T_n \varphi_n\|$$

$$= \left| \int_{O_{kn}} (1 - \varphi_n) \, dm_{T_n} \right| + \|T_n \varphi_n\|$$

$$= \left| \int_{O_{kn} \setminus H_n} (1 - \varphi_n) \, dm_{T_n} \right| + \|T_n \varphi_n\|$$

$$\leq \|m_{T_n}\|(O_{kn} \setminus H_n) + \|T_n \varphi_n\|$$

$$< \varepsilon,$$

for all $n \geq n_0$. This is in contradiction with $(2.16)$.

(b)$\Rightarrow$(a). By [6, Theorem I.2.4], there exists a scalar countably additive measure $\mu : \Sigma \to [0,\infty)$ such that $\{m_T : T \in \mathcal{M}\}$ is uniformly $\mu$-continuous. Then, if $(\varphi_n)$ is a sequence
that tends to zero weakly in \( \ell(\Omega) \), it is obvious that zero is the pointwise limit of the sequence \((\varphi_n(t))\). Now, using Egorov’s theorem and proceeding along similar lines as the proof of (b)\(\Rightarrow\)(a) in \textbf{Theorem 2.4}, the proof concludes.

(b)\(\Leftrightarrow\)(c). The set \( \bigcup_{T \in \mathcal{H}} T^*(B_{Y^*}) = \{y^* \circ m_T : T \in \mathcal{M}, \ y^* \in B_{Y^*}\} \subset \ell(\Omega)^* \) is relatively weakly compact if and only if it is bounded and uniformly countably additive [4, \textit{Theorem VII.13}]. A call to [6, \textit{Proposition I.1.17}] makes clear that \( \bigcup_{T \in \mathcal{H}} T^*(B_{Y^*}) \) is uniformly countably additive if and only if condition (b) is satisfied.

\textbf{Corollary 2.6.} If \( \mathcal{M} \subset \Pi_1(\ell(\Omega), Y) \) is uniformly 1-summing, then \( \mathcal{M} \) is uniformly completely continuous.

The converse of the last result is not true in general.

\textbf{Proposition 2.7.} Suppose that the cardinal of \( \Omega \) is infinite. The following statements are equivalent:

(a) each subset of \( \Pi_1(\ell(\Omega), Y) \) uniformly completely continuous is uniformly 1-summing,

(b) \( Y \) is finite-dimensional.

\textbf{Proof.} (a)\(\Rightarrow\)(b). By contradiction, suppose there is an unconditionally summable serie \( \sum_k y_k \) in \( Y \) such that \( \sum_k \|y_k\| = \infty \). Let \( (\omega_k) \) be a sequence in \( \Omega \) with \( \omega_k \neq \omega_l \) when \( k \neq l \). For each \( m \in \mathbb{N} \) consider the operator \( T_m : \ell(\Omega) \to Y \) defined by

\[ T_m\varphi = \sum_{k=1}^{m} \varphi(\omega_k)y_k. \quad (2.20) \]

It is not difficult to show that \( \mathcal{M} = (T_m) \) is uniformly completely continuous. Nevertheless,

\[ \pi_1(T_m) = \sum_{k=1}^{m} \|y_k\| \to \infty, \quad (2.21) \]

so \( \mathcal{M} \) cannot be uniformly 1-summing because it is not \( \pi_1 \)-bounded.

(b)\(\Rightarrow\)(a). This follows easily in view of conditions (b) in \textit{Theorems 2.4 and 2.5}. \(\square\)

We have showed that the converse of \textbf{Corollary 2.6} is not true in general. However, a direct argument using \textit{Theorems 2.4} and 2.5 leads up to conclude that every uniformly completely continuous set \( \mathcal{M} \subset \Pi_1(\ell(\Omega), Y) \) verifying the following condition is uniformly 1-summing:

(i) given \( T \in \mathcal{M} \) and a finite subset \( \{(\varphi_1, y_1^*), \ldots, (\varphi_m, y_m^*)\} \) of \( \ell(\Omega) \times B_{Y^*} \), there exist \( S \in \mathcal{M} \) and \( z^* \in B_{Y^*} \) such that \( |\langle y_n^*, T\varphi_n \rangle| \leq |\langle z^*, S\varphi_n \rangle|, \ n = 1, \ldots, m. \)

\textbf{3. Final notes and examples.} The Grothendieck-Pietsch domination theorem states that an operator \( T : X \to Y \) is \( p \)-summing if and only if there exists a positive Radon measure \( \mu \) defined on the (weak\(^*\)) compact space \( B_{X^*} \) such that

\[ \|Tx\|^p \leq \int_{B_{X^*}} |\langle x^*, x \rangle|^p \ d\mu(x^*), \quad (3.1) \]
for all \(x \in X\) [5, Theorem 2.12]. Since the appearance of this theorem, there is a great interest in finding out the structure of uniformly \(p\)-dominated sets. A subset \(\mathcal{M}\) of \(\Pi_p(X,Y)\) is uniformly \(p\)-dominated if there exists a positive Radon measure \(\mu\) such that the inequality (3.1) holds for all \(x \in X\) and all \(T \in \mathcal{M}\). In [3, 8, 9], the reader can find some of the most recent steps given on this subject. Now we are going to show that these sets are uniformly \(p\)-summing.

**Proposition 3.1.** If \(\mathcal{M} \subset \Pi_p(X,Y)\) is a uniformly \(p\)-dominated set, then \(\mathcal{M}^{**} = \{T^{**}: T \in \mathcal{M}\}\) is uniformly \(p\)-summing.

**Proof.** Let \(\mu\) be a measure for which \(\mathcal{M}\) is uniformly \(p\)-dominated. In a similar way as in the Pietsch factorization theorem [5, Theorem 2.13], we can obtain, for all \(T \in \mathcal{M}\), operators \(U_T: L_p(\mu) \to \ell_\infty(B_Y^{**})\), \(\|U_T\| \leq \mu(B_X^{**})^{1/p}\), and an operator \(V: X \to L_\infty(\mu)\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow V & & \downarrow i_Y \\
L_\infty(\mu) & \xrightarrow{i_p} & L_p(\mu) \\
\end{array}
\]

Here, \(i_p\) is the canonical injection from \(L_\infty(\mu)\) into \(L_p(\mu)\) and \(i_Y\) is the isometry from \(Y\) into \(\ell_\infty(B_Y^{**})\) defined by \(i_Y(y) = ((y^*, y))_{y^* \in B_Y^*}\). Notice that \(i_p^{**}\) can be viewed as \(i_p\) composed with the canonical projection \(P: L_\infty(\mu)^{**} \to L_\infty(\mu)\) which is simply the adjoint of the usual embedding \(L_1(\mu) \to L_\infty(\mu)\). By weak compactness, we may and do consider \(T^{**}\) as a map from \(X^{**}\) into \(Y\) for which

\[
i_Y \circ T^{**} = U_T \circ i_p \circ P \circ V^{**}.
\]

Given \(\varepsilon > 0\) and \((x_n^{**}) \in \ell_p^w(X^{**})\), we can choose \(n_0 \in \mathbb{N}\) so that

\[
\sum_{n \geq n_0} \|i_p \circ P \circ V^{**} (x_n^{**})\|^p < \frac{\varepsilon}{\mu(B_X^{**})},
\]

because \(i_p \circ P \circ V^{**}\) is \(p\)-summing. Then, we have

\[
\sum_{n \geq n_0} \|T^{**}x_n^{**}\|^p = \sum_{n \geq n_0} \|i_Y \circ T^{**} (x_n^{**})\|^p = \sum_{n \geq n_0} \|U_T \circ i_p \circ P \circ V^{**} (x_n^{**})\|^p
\]

\[
\leq \mu(B_X^{**}) \sum_{n \geq n_0} \|i_p \circ P \circ V^{**} (x_n^{**})\|^p < \varepsilon,
\]

for all \(T \in \mathcal{M}\). So, \(\mathcal{M}^{**}\) is uniformly \(p\)-summing. \(\square\)
It is easy to show that the study of uniformly \( p \)-summing sets can be reduced to the behavior of its sequences. Indeed, a bounded set \( \mathcal{M} \) in \( \Pi_p(X,Y) \) is uniformly \( p \)-summing if and only if every sequence \( (T_n) \) in \( \mathcal{M} \) admits a uniformly \( p \)-summing subsequence. Thus, it seems to be interesting to make clear the relationship between uniformly \( p \)-summing sets and relatively weakly compact sets. For \( p = 1 \), we have the following result.

**Proposition 3.2.** Every relatively weakly compact set in \( \Pi_1(X,Y) \) is uniformly 1-summing.

**Proof.** Let \( \mathcal{M} \) be a relatively weakly compact set in \( \Pi_1(X,Y) \). Given \( \hat{x} = (x_n) \in \ell^1_w(X) \), consider the (weak-weak) continuous operator \( U_{\hat{x}} : \Pi_1(X,Y) \to \ell^1_a(Y) \) defined by \( U_{\hat{x}}(T) = (Tx_n) \). Then, \( U_{\hat{x}}(\mathcal{M}) \) is relatively weakly compact in \( \ell^1_a(Y) \); according to [2, Theorem 2], we can conclude that \( \mathcal{M} \) is uniformly 1-summing. \( \square \)

Proposition 3.2 does not remain true if \( p = 2 \). For example, for each \( \beta = (\beta_n) \in \ell^2 \), consider the operator \( T_\beta : c_0 \to \ell^2 \) defined by \( T(\alpha_n) = (\alpha_n \cdot \beta_n) \) and put \( \mathcal{M} = \{ T_\beta : \beta \in \ell_2 \} \subset \Pi_2(c_0,\ell^2) \) [5, Theorem 3.5]. If we consider \( \ell^2 \) as a subspace of \( \Pi_2(c_0,\ell^2) \), the set \( \mathcal{M} = \ell_2 \) is relatively weakly compact. Nevertheless, no matter how we choose \( k \in \mathbb{N} \),

\[
\sum_{n \geq k} \| T_{\hat{x}_k} e_n \|^2 = 1, \tag{3.6}
\]

so \( \mathcal{M} \) cannot be uniformly 2-summing.

Now we show that there are uniformly \( p \)-summing sets failing to be relatively weakly compact.

**Proposition 3.3.** If every uniformly \( p \)-summing set is relatively weakly compact in \( \Pi_p(X,Y) \), then \( Y \) is reflexive.

**Proof.** Fixing \( x_0^* \in X^* \) with \( \| x_0^* \| = 1 \), the isometry \( y \in Y \to x_0^* \otimes y \in x_0^* \otimes Y \) allows us to see \( Y \) as a subspace of \( \Pi_p(X,Y) \). If \( \varepsilon > 0 \) and \( (x_n) \in \ell^p_w(X) \), there exists \( n_0 \in \mathbb{N} \) so that

\[
\sum_{n \geq n_0} | \langle x_0^*, x_n \rangle |^p < \varepsilon; \tag{3.7}
\]

hence, for every \( y \in B_Y \),

\[
\sum_{n \geq n_0} \| (x_0^* \otimes y)(x_n) \|^p = \sum_{n \geq n_0} | \langle x_0^*, x_n \rangle |^p \| y \|^p < \varepsilon. \tag{3.8}
\]

This yields that \( B_Y \) is uniformly \( p \)-summing and, by hypothesis, weakly compact. \( \square \)

The converse of Proposition 3.3 is not always true. By contradiction, suppose every uniformly 1-summing set in \( \Pi_1(\ell_1,\ell_2) \) is relatively weakly compact. Because \( \ell_1 \) does not contain any copy of \( c_0 \), every bounded set in \( \Pi_1(\ell_1,\ell_2) \) is relatively weakly compact. Then, we conclude that \( \Pi_1(\ell_1,\ell_2) \) is reflexive, which is not possible since \( \ell_1^* \), viewed as a subspace of \( \Pi_1(\ell_1,\ell_2) \), is not.

However, if \( p = 1 \) and \( X = \ell^1(\Omega) \), the reflexivity of \( Y \) is a sufficient condition for a uniformly 1-summing set to be relatively weakly compact. Indeed, if \( r b v c a(\Sigma, Y) \) denotes
the set of all regular, countably additive, $Y$-valued measures $m$ on $\Sigma$ with bounded variation, recall that relatively weakly compact sets $\mathcal{M}$ in $rbvca(\Sigma, Y)$ are those verifying the following conditions: (i) $\mathcal{M}$ is bounded; (ii) the family of nonnegative measures $\{m| : m \in \mathcal{M}\}$ is uniformly countably additive; and (iii) for each $E \in \Sigma$, the set $\{m(E) : m \in \mathcal{M}\}$ is relatively weakly compact in $Y$ [6, Theorem IV.2.5]. Having in mind the identification between $\Pi_1(\ell^1(\Omega), Y)$ and $rbvca(\Sigma, Y)$, and making use of the characterization of uniformly 1-summing sets obtained in Theorem 2.4, we conclude the next characterization.

**Corollary 3.4.** The following statements are equivalent:

(a) $Y$ is reflexive,

(b) every set $\mathcal{M}$ in $\Pi_1(\ell^1(\Omega), Y)$ is uniformly 1-summing if and only if $\mathcal{M}$ is relatively weakly compact.

It is well known that a linear operator $T$ is 1-summing if and only if $T^{**}$ is. So, it is natural to ask if a set $\mathcal{M}$ is uniformly 1-summing whenever $\mathcal{M}^{**} = \{T^{**} : T \in \mathcal{M}\}$ is. Unfortunately, we are going to show that this is not true in general. It suffices to take $X$ as the separable $L_\infty$-space of Bourgain and Delbaen [1]. This space has the Radon-Nikodym property, so it does not contain any copy of $c_0$. Nevertheless, $X^*$ is isomorphic to $\ell_1$ and, therefore, $X^{**}$ contains a copy of $c_0$. Let $(e_n)$ be the canonical basis of $\ell_1$ and $J : \ell_1 \to X^*$ an isomorphism. Put $T_n = Je_n \in \Pi_1(X, \mathbb{R})$; the set $\mathcal{M} = \{T_n : n \in \mathbb{N}\}$ is uniformly 1-summing since it is bounded and $X$ does not contain any copy of $c_0$. Notice that the elements of $\mathcal{M}^{**}$ are the linear forms $x^{**} \in X^{**} \to (x^{**}, Je_n) \in \mathbb{R}$, for all $n \in \mathbb{N}$. If $(e^*_n)$ is the canonical basis of $c_0$, then $((J^*)^{-1}(e^*_n)) \in \ell_1^1(X^{**})$; hence, no matter how we choose $k \in \mathbb{N}$, it turns out that

$$\sum_{n=k} \left| T_n^{**} ((J^*)^{-1}(e^*_n)) \right| = \sum_{n=k} \left| \langle (J^*)^{-1}(e^*_n), Je_k \rangle \right| = \sum_{n=k} |\langle e^*_n, e_k \rangle| = 1,$$

and $\mathcal{M}^{**}$ cannot be uniformly 1-summing.

Nevertheless, if $\mathcal{M}$ is a set of operators defined on $c_0$, then it is true that $\mathcal{M}$ is uniformly 1-summing if and only if $\mathcal{M}^{**}$ is too. To see this, notice that for a 1-summing operator $T : (\alpha_n) \in c_0 \to \sum_{n=1}^\infty \alpha_n x_n \in X$, the second adjoint $T^{**} : \ell_\infty \to X$ is defined by $T^{**}(\beta_n) = \sum_{n=1}^\infty \beta_n x_n$, for all $(\beta_n) \in \ell_\infty$.

When $\mathcal{M}$ is a set of operators defined on a $\ell^1(\Omega)$-space, we do not know whether $\mathcal{M}^{**}$ inherits the property or not. Anyway, we are going to prove the following weaker result.

We inject isometrically $B(\Sigma)$ into $\ell^1(\Omega)^{**}$ in the natural way.

**Proposition 3.5.** If $\mathcal{M} \subset \Pi_1(\ell^1(\Omega), X)$ is uniformly 1-summing, then $\tilde{\mathcal{M}} = \{\tilde{T} : B(\Sigma) \to X : T \in \mathcal{M}\}$ is uniformly 1-summing too.

**Proof.** The argument is similar to the one used in the proof of (b)$\Rightarrow$(a) in Theorem 2.4.

Finally, we give an example to show that Corollary 2.6 is not true if $\ell^1(\Omega)$ is replaced by a general Banach space $X$. It suffices to take $X = \ell_2$ and $\mathcal{M} = \{e^*_n : n \in \mathbb{N}\}$, where $(e^*_n)$ is the unit basis of $\ell_2^\perp \cong \ell_2$. The set $\mathcal{M}$ is bounded in $\Pi_1(\ell_2, \mathbb{R})$ and, therefore, uniformly 1-summing but it is not uniformly completely continuous.
References


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