INFINITE MATRICES, WAVELET COEFFICIENTS AND FRAMES

N. A. SHEIKH and M. MURSALEEN

Received 2 December 2003

We study the action of $A$ on $f \in L^2(\mathbb{R})$ and on its wavelet coefficients, where $A = (a_{lmjk})_{lmjk}$ is a double infinite matrix. We find the frame condition for $A$-transform of $f \in L^2(\mathbb{R})$ whose wavelet series expansion is known.

2000 Mathematics Subject Classification: 42C15, 41A17, 42C40.

1. Introduction. The notation of frame goes back to Duffin and Schaeffer [7] in the early 1950s to deal with the problems in nonharmonic Fourier series. There has been renewed interest in the subject related to its role in wavelet theory. For a glance of the recent development and work on frames and related topics, see [3, 4, 5, 6, 9]. In this note, we will use the regular double infinite matrices (see [9, 10]) to obtain the frame conditions and wavelet coefficients.

2. Notations and known results. $\mathbb{N}$ is the set of positive integers, $\mathbb{Z}$ is the set of integers, $\mathbb{R}$ is the set of real numbers. The space $L^2(\mathbb{R})$ of measurable function $f$ is defined on the real line $\mathbb{R}$, that satisfies

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty. \quad (2.1)$$

The inner product of two square integrable functions $f, g \in L^2(\mathbb{R})$ is defined as

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx,$$

$$\|f\|^2 = \langle f, f \rangle^{1/2}. \quad (2.2)$$

Every function $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{j,k \in \mathbb{Z}} C_{j,k} \psi_{j,k}(x). \quad (2.3)$$

This series representation of $f$ is called wavelet series. Analogous to the notation of Fourier coefficients, the wavelet coefficients $C_{j,k}$ are given by

$$C_{j,k} = \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) \, dx = \langle f, \psi_{j,k} \rangle,$$

$$\psi_{j,k} = 2^{j/2} \psi(2^j x - k). \quad (2.4)$$
Now, if we define an integral transform
\[(W\psi f)(b,a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x) \psi\left(\frac{x-b}{a}\right) dx, \quad f \in L^2(\mathbb{R}), \quad (2.5)\]
then the wavelet coefficients become
\[C_{j,k} = (W\psi f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right), \quad (2.6)\]

A sequence \(\{x_n\}\) in a Hilbert space \(H\) is a frame if there exist constants \(c_1\) and \(c_2\), \(0 < c_1 \leq c_2 < \infty\), such that
\[c_1 \|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, x_n \rangle|^2 \leq c_2 \|f\|^2, \quad (2.7)\]
for all \(f \in H\). The supremum of all such numbers \(c_1\) and infimum of all such numbers \(c_2\) are called the frame bounds of the frame. The frame is called tight frame when \(c_1 = c_2\) and is called normalized tight frame when \(c_1 = c_2 = 1\). Any orthonormal basis in a Hilbert space \(H\) is a normalized tight frame. The connection between frames and numerically stable reconstruction from discretized wavelet was pointed out by Grossmann et al. [8]. In 1985, they defined that a wavelet function \(\psi \in L^2(\mathbb{R})\), constitutes a frame with frame bounds \(c_1\) and \(c_2\), if any \(f \in L^2(\mathbb{R})\) such that
\[c_1 \|f\|^2 \leq \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 \leq c_2 \|f\|^2. \quad (2.8)\]
Again, it is said to be tight if \(c_1 = c_2\) and is said to be exact if it ceases to be frame by removing any of its elements. There are many examples proposed by Daubechies et al. [6]. For further details, one can refer to [1, 5, 6]. Chui and Shi [2] proved that \(\{\psi_{j,k}\}\) is a frame for \(L^2(\mathbb{R})\) with bounds \(c_1\) and \(c_2\), if for some \(a > 1\) and \(b > 0\), the Fourier transform \(\hat{\psi}\) satisfies
\[c_1 \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} |\hat{\psi}(a^j w)|^2 \leq c_2 \text{ a.e.}, \quad (2.9)\]
for some constants \(c_1\) and \(c_2\). By integrating each term in
\[\frac{c_1}{|w|} \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} \frac{|\hat{\psi}(a^j w)|^2}{|w|} \leq \frac{c_2}{|w|}, \quad (2.10)\]
over \(1 \leq |w| \leq a\), we have
\[2c_1 \log a \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} \int_{1 \leq |w| \leq a} \frac{|\hat{\psi}(a^j w)|^2}{|w|} dw \leq 2c_2 \log a, \quad (2.11)\]
which immediately yields
\[c_1 \leq \frac{1}{2b \log a} \int_{-\infty}^{\infty} \frac{|\hat{\psi}(a^j w)|^2}{|w|} dw \leq c_2. \quad (2.12)\]
The above condition known as compactibility condition was also observed by Daubechies [4] by using techniques from trace class operators. The above constants were given by frame bounds, see [2].

Let \( A = (a_{mnjk}) \) be a double infinite matrix of real numbers. Then, \( A \)-transform of a double sequence \( x = (x_{jk}) \) is

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{mnjk} x_{jk},
\]

which is called \( A \)-means or \( A \)-transform of the sequence \( x = (x_{ij}) \). This definition is due to Móricz and Rhoades [9].

A double matrix \( A = (a_{mnjk}) \) is said to be regular (see [10]) if the following conditions hold:

(i) \( \lim_{m,n \to \infty} \sum_{j,k=0}^{\infty} a_{mnjk} = 1 \),
(ii) \( \lim_{m,n \to \infty} \sum_{j=0}^{\infty} |a_{mnjk}| = 0, \) (\( k = 0, 1, 2, \ldots \)),
(iii) \( \lim_{m,n \to \infty} \sum_{k=0}^{\infty} |a_{mnjk}| = 0, \) (\( j = 0, 1, 2, \ldots \)),
(iv) \( \| A \| = \sup_{m,n > 0} \sum_{j,k=0}^{\infty} |a_{m,n}| < \infty \).

Either of conditions (ii) and (iii) implies that

\[
\lim_{m,n \to \infty} a_{mnjk} = 0.
\]

In this note, we establish the frame condition by using \( A \)-transform of nonnegative regular matrix, also we find action of the matrix \( A \) on wavelet coefficients.

3. Main results. In this section, we prove the following theorems.

**Theorem 3.1.** Let \( A = (a_{iljk}) \) be a double nonnegative regular matrix. If

\[
f(x) = \sum_{j,k \in \mathbb{Z}} C_{j,k} \psi_{j,k}(x)
\]

is a wavelet expansion of \( f \in L^2(\mathbb{R}) \) with wavelet coefficients

\[
C_{j,k} = \int_{-\infty}^{\infty} f(x) \overline{\psi_{j,k}(x)} dx = \langle f, \psi_{j,k} \rangle,
\]

then the frame condition for \( A \)-transform of \( f \in L^2(\mathbb{R}) \) is

\[
c_1 \| f \|^2 \leq \sum_{i,l \in \mathbb{Z}} | \langle Af, \psi_{i,l} \rangle |^2 \leq c_2 \| f \|^2,
\]

where \( Af \) is the \( A \)-transform of \( f \) and \( 0 < c_1 \leq c_2 < \infty \).

**Theorem 3.2.** If \( C_{j,k} \) are the wavelet coefficients of \( f \in L^2(\mathbb{R}) \), that is, \( C_{j,k} = \langle f, \psi_{j,k} \rangle \), then the \( d_{l,m} \) are the wavelet coefficients of \( Af \), where \( \{d_{l,m}\} \) is defined as the \( A \)-transform of \( \{C_{j,k}\} \) by

\[
d_{l,m} = \sum_{j,k=-\infty}^{0} a_{lmjk} C_{jk}.
\]
**Theorem 3.3.** Let $A = (a_{lm})$ be a double nonnegative matrix whose elements are $(\langle \psi_{j,k}, \psi_{l,m} \rangle)$. Then, $\{\psi_{j,k}\}$ constitutes a frame of $L^2(\mathbb{R})$ if and only if $\{\psi_{l,m}\}$ constitutes a frame of $L^2(\mathbb{R})$, where $C_{j,k} = \langle f, \psi_{j,k} \rangle$ and $d_{l,m} = \langle f, \psi_{l,m} \rangle$.

**Proof of Theorem 3.1.** We can write

$$f(x) = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}. \quad (3.5)$$

If we take $A$-transform of $f$, we get

$$Af(x) = \sum_{i,l \in \mathbb{Z}} \langle Af, \psi_{i,l} \rangle \psi_{i,l}, \quad (3.6)$$

and therefore

$$\sum_{i,l \in \mathbb{Z}} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \leq \sum_{i,l \in \mathbb{Z}} \int_{-\infty}^{\infty} \left| Af(x) \right|^2 \left| \psi_{i,l}(x) \right|^2 dx \leq \|A\|^2 \|f\|_2^2 \sum_{i,l \in \mathbb{Z}} \|\psi_{i,l}\|_2^2. \quad (3.7)$$

Since $A$ is regular matrix and $\|\psi_{i,l}\|_2 = 1$, therefore

$$\sum_{i,l \in \mathbb{Z}} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \leq c_2 \|f\|_2^2, \quad (3.8)$$

where $c_2$ is positive constant.

Now, for any arbitrarily $f \in L^2(\mathbb{R})$, define

$$\tilde{f} = \left[ \sum_{i,l \in \mathbb{Z}} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \right]^{-1/2} f. \quad (3.9)$$

Clearly,

$$\langle A\tilde{f}, \psi_{i,l} \rangle = \left[ \sum_{i,l \in \mathbb{Z}} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \right]^{-1/2} \langle Af, \psi_{i,l} \rangle, \quad (3.10)$$

then

$$\sum_{i,l \in \mathbb{Z}} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \leq 1. \quad (3.11)$$
Hence, if there exists $\alpha$ a positive constant, then

$$
\| A\hat{f} \|_2^2 \leq \alpha,
$$

(3.12)

$$
\left[ \sum_{i,l \in \mathbb{Z}} |\langle A f, \psi_{i,l} \rangle|^2 \right]^{-1} \| A f \|_2^2 \leq \alpha.
$$

(3.13)

where $\alpha$ is another positive constant. Therefore,

$$
c_1 \| f \|_2^2 \leq \sum_{i,l \in \mathbb{Z}} |\langle A f, \psi_{i,l} \rangle|^2,
$$

(3.14)

where $c_1 = \alpha > 0$.

Combining (3.8) and (3.14), we have

$$
c_1 \| f \|_2^2 \leq \sum_{i,l \in \mathbb{Z}} |\langle A f, \psi_{i,l} \rangle|^2 \leq c_2 \| f \|_2^2.
$$

(3.15)

This completes the proof. \qed

**Proof of Theorem 3.2.** We can write

$$
\langle A f, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} A f(x) \overline{\psi_{l,m}(x)} dx
$$

$$
= \int_{-\infty}^{\infty} \sum_{j,k=-\infty}^{\infty} a_{l,m} c_{j,k} \psi_{j,k}(x) \overline{\psi_{l,m}(x)} dx.
$$

(3.16)

Now,

$$
\sum_{l,m=-\infty}^{\infty} \langle A f, \psi_{l,m} \rangle \psi_{l,m} = \sum_{l,m=-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j,k=-\infty}^{\infty} a_{l,m} c_{j,k} \psi_{j,k}(x) \psi_{l,m}(x) dx
$$

$$
= \sum_{l,m=-\infty}^{\infty} d_{l,m} \psi_{l,m} \int_{-\infty}^{\infty} |\psi_{l,m}(x)|_2^2
$$

(3.17)

$$
= \sum_{l,m=-\infty}^{\infty} d_{l,m} \psi_{l,m}.
$$

Therefore,

$$
\sum_{l,m=-\infty}^{\infty} d_{l,m} \psi_{l,m} = \sum_{l,m=-\infty}^{\infty} \langle A f, \psi_{l,m} \rangle \psi_{l,m}.
$$

(3.18)

This implies that $d_{l,m}$ are wavelet coefficients of $A f$. 
Thus,

\[ d_{l,m} = \langle f, \psi_{l,m} \rangle. \]  

(3.19)

This completes the proof. \( \square \)

**Proof of Theorem 3.3.** We observe that

\[
\alpha_{lmjk}C_{j,k} = \langle \psi_{j,k}, \psi_{l,m} \rangle \langle f, \psi_{j,k} \rangle \\
= \int_{-\infty}^{\infty} \psi_{j,k}(x) \overline{\psi_{l,m}(x)} dx \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx \\
= \int_{-\infty}^{\infty} f(x) \overline{\psi_{l,m}(x)} dx \\
= \langle f, \psi_{l,m} \rangle,
\]

that is, \( \alpha_{lmjk}C_{j,k} = d_{l,m} \).

Now,

\[
\sum_{l,m} |d_{l,m}|^2 = \sum_{l,m} |\alpha_{lmjk}C_{j,k}|^2 = \sum_{l,m} |\langle f, \psi_{l,m} \rangle|^2 = \frac{1}{(2\pi)^2} \sum_{l,m} |\langle \hat{f}, \psi_{l,m} \rangle|^2,
\]

(3.20)

by Parseval’s formula for trigonometric Fourier series.

Now

\[
\frac{1}{(2\pi)^2} \sum_{l,m} \left| \int_0^{2\pi} \sum_{p=\infty}^{\infty} \hat{f}(w + 2\pi p) \overline{\psi(w + 2\pi p)} e^{ilmw} dw \right|^2
\]

(3.21)

\[ = p = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{p=\infty}^{\infty} \hat{f}(w + 2\pi p) \overline{\psi(w + 2\pi p)} d\omega \right|^2,
\]

by Parseval’s formula for trigonometric Fourier series.

Now

\[
\left| \sum_{p=\infty}^{\infty} \hat{f}(w + 2\pi p) \overline{\psi(w + 2\pi p)} \right|^2 = \left( \sum_{p=\infty}^{\infty} \hat{f}(w + 2\pi p) \overline{\psi(w + 2\Pi p)} \right) \times \left( \sum_{q=\infty}^{\infty} \hat{f}(w + 2\pi q) \overline{\psi(w + 2\pi q)} \right).
\]

(3.22)

Let \( f(w) = \sum_{p=\infty}^{\infty} \hat{f}(w + 2\pi p) \overline{\psi(w + 2\pi p)} \).
Therefore, 
\[
p = \frac{1}{2\pi} \left( \sum_{p = -\infty}^{\infty} \int_{0}^{2\pi} \left| \hat{f}(w + 2\pi p) \hat{\psi}(w + 2\pi p) \right|^2 dw \right)
\]
\[
= \frac{1}{2\pi} \left( \sum_{p = -\infty}^{\infty} \int_{0}^{2\pi} \hat{f}(w + 2\pi p) \hat{\psi}(w + 2\pi p) F(w) dw \right)
\]
\[
= \frac{1}{2\pi} \left( \sum_{q = -\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(w + 2\pi q) \hat{\psi}(w + 2\pi q) dw F(w) dw \right)
\]
\[
= \frac{1}{2\pi} \left( \sum_{q = -\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{f}(w)|^2 |\hat{\psi}(w)|^2 dw \right)
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw
\]
\[
= \| f \|_2^2,
\]
that is,
\[
\sum_{l,m} |d_{lm}|^2 = \| f \|_2^2, \quad f \in L^2(\mathbb{R}). \tag{3.24}
\]
Therefore, for a regular matrix \( A = (a_{lmjk}) \), we have
\[
c_1 \| f \|_2^2 \leq \sum_{l,m} |d_{lm}|^2 \leq c_2 \| f \|_2^2 \tag{3.25}
\]
if and only if
\[
c_1' \| f \|_2^2 \leq \sum_{j,k} |c_{jk}|^2 \leq c_2' \| f \|_2^2, \tag{3.26}
\]
where, \( 0 \leq c_1', c_2' < \infty \). This completes the proof. \( \square \)

**REFERENCES**


N. A. Sheikh: Department of Mathematics, National Institute of Technology, Srinagar, Kashmir 190006, Jammu and Kashmir, India

E-mail address: neyaznit@yahoo.co.in

M. Mursaleen: Department of Mathematics, Aligarh Muslim University, Aligarh 202002, Uttar Pradesh, India

Current address: Department of Mathematics, Faculty of Science, P.O. Box 80203, King Abdul Aziz University, Jeddah, Kingdom of Saudi Arabia

E-mail address: mursaleen@postmark.net