THE RADIUS OF STARLIKENESS FOR CONVEX FUNCTIONS OF COMPLEX ORDER

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Received 4 November 2003

We will give the relation between the class of Janowski starlike functions of complex order and the class of Janowski convex functions of complex order. As a corollary of this relation, we obtain the radius of starlikeness for the class of Janowski convex functions of complex order.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let F be the class of analytic functions in $D = \{z \mid |z| < 1\}$, and let S denote those functions in F that are univalent and normalized by $f(0) = 0$, $f'(0) = 1$. Furthermore, let Ω be the family of functions $\omega(z)$ regular in D and satisfying $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in D$.

For arbitrary fixed numbers $-1 \leq B < A \leq 1$, denoted by $P(A, B)$, the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \quad (1.1)$$

which is regular in D on the condition such that

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (1.2)$$

for some functions $\omega(z) \in \Omega$ and every $z \in D$. This class was introduced by Janowski [7].

Moreover, let $S^*(A, B, b)$ be denoted by the family of functions $f(z) \in S$ such that $f(z)$ is in $S^*(A, B, b)$ if and only if $f(z)/z \neq 0$,

$$1 + \frac{1}{b} \left(z \cdot \frac{f'(z)}{f(z)} - 1 \right) = p(z), \quad (b \neq 0, \text{ complex}) \quad (1.3)$$

for some functions $p(z) \in P(A, B)$ and all z in D .

Finally, let $C(A, B, b)$ denote the family of functions which are regular:

$$1 + \frac{1}{b} \cdot z \cdot \frac{f''(z)}{f'(z)} = p(z), \quad (b \neq 0, \text{ complex}) \quad (1.4)$$

for some functions $p(z) \in P(A, B)$ and every z in D .

We note that $P(-1, 1)$ is the class of Caratheodory functions, and therefore the class $C(A, B, b)$ contains the following classes. $b = 1$, $C(1, -1, 1) = C$ is the well-known class of convex functions [2], and $C(1, -1, b) = C(b)$ is the class of convex functions of complex order [7, 8]. $C(1, -1, 1 - \beta)$, $(0 \leq \beta < 1)$ is the class of convex functions of order β [9]. For $A = 1$, $B = -1$, $b = e^{-i\lambda} \cdot \cos \lambda$, $|\lambda| < \pi/2$ is the class of functions for which $zf'(z)$ is λ -spirallike [3, 6, 11, 12, 13, 14]. For $A = 1$, $B = -1$, $b = (1 - \beta)e^{-i\lambda} \cdot \cos \lambda$, $0 \leq \beta < 1$, $|\lambda| < \pi/2$ is the class of functions for which $zf'(z)$ is λ -spirallike of order β [3, 6, 11, 12, 13, 14].

2. Representation theorem for the class $S^*(A, B, b)$. The following lemma, well known as Jack's lemma, is required in our investigation.

LEMMA 2.1 [4, 5]. *Let $w(z)$ be a nonconstant and analytic function in the unit disc D with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0 , then $z_0 w'(z_0) = kw(z_0)$ and $k \geq 1$.*

LEMMA 2.2. $e^{-i\alpha} f(e^{i\alpha} z)$, $\alpha \in [0, 2\pi)$ is in $C(A, B, b)$ whenever $f(z)$ is in $C(A, B, b)$.

PROOF. If $f(z) \in C(A, B, b)$, then

$$g(z) = e^{-i\alpha} f(e^{i\alpha} z) \Rightarrow 1 + \frac{1}{b} z \frac{g'(z)}{g(z)} = 1 + \frac{1}{b} (e^{i\alpha} z) \frac{f'(e^{i\alpha} z)}{f(e^{i\alpha} z)}. \quad (2.1)$$

□

We note that similarly the class $S^*(A, B, b)$ is invariant under the rotation so that $e^{-i\alpha} f(e^{i\alpha} z)$, $\alpha \in [0, 2\pi)$ is in $S^*(A, B, b)$ whenever $f(z)$ is in $S^*(A, B, b)$.

LEMMA 2.3. *If $g(z) \in S^*(A, B, b)$, then*

$$g(z) = \begin{cases} z(1 + Bw(z))^{b(A-B)/B}, & B \neq 0, k = 1, \\ ze^{bAw(z)}, & B = 0, k = 1, \end{cases} \quad (2.2)$$

for some $w(z) \in \Omega$ and for all z in D , and conversely.

PROOF. The proof of this lemma is completed in four steps, and we have used Nicola Tuneski's technique for the special case of $k = 1$ [15].

FIRST STEP. If $B \neq 0$ and

$$g(z) = z(1 + Bw(z))^{b(A-B)/B}, \quad (2.3)$$

then by taking logarithmic derivative of (2.3) followed by a brief computation using Jack's lemma and the definition of subordination, we obtain

$$1 + \frac{1}{b} \left(z \frac{g'(z)}{g(z)} - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad \text{for } k = 1, \quad (2.4)$$

and so from the definition of $S^*(A, B, b)$ it follows that $g(z) \in S^*(A, B, b)$. (See [10].)

SECOND STEP. If $B = 0$, then we have $g(z) = ze^{bAw(z)}$. Similarly, we get

$$1 + \frac{1}{b} \left(z \frac{g'(z)}{g(z)} - 1 \right) = 1 + Aw(z), \quad \text{for } k = 1. \quad (2.5)$$

The equality shows that $g(z) \in S^*(A, B, b)$.

THIRD STEP. Conversely, if $g(z) \in S^*(A, B, b)$ and $B \neq 0$, then we have

$$1 + \frac{1}{b} \left(z \frac{g'(z)}{g(z)} - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}. \quad (2.6)$$

Equation (2.6) can be written in the form

$$\frac{g'(z)}{g(z)} = \frac{b(A-B)(w(z)/z)}{1 + Bw(z)} + \frac{1}{z}. \quad (2.7)$$

If we use Jack's lemma in (2.7) for $k = 1$, we obtain

$$\frac{g'(z)}{g(z)} = \frac{b(A-B)w'(z)}{1 + Bw(z)} + \frac{1}{z}. \quad (2.8)$$

Integrating both sides of equality (2.8), we get (2.3).

FOURTH STEP. Again, conversely, if $g(z) \in S^*(A, B, b)$ and $B = 0$, then in the same way we obtain $g(z) = ze^{bAw(z)}$ which completes the proof. $\square$

LEMMA 2.4. Let $f(z)$ be regular and analytic in D , and normalized so that $f(0) = 0$, $f'(0) = 1$. A necessary and sufficient condition for $f(z) \in C(A, B, b)$ is that for each member $g(z) = z + b_1z + b_2z^2 + \dots$ of $S^*(A, B, b)$ the following equation holds:

$$g(z, \zeta) = z \left(\frac{f(z) - f(\zeta)}{z - \zeta} \right)^2, \quad \zeta, z \in D, \zeta \neq z, \zeta = nz, |n| \leq 1. \quad (2.9)$$

PROOF. If $f(z) \in C(A, B, b)$, then this function is analytic, regular, and continuous in the unit disc. Therefore, equality (2.9) can be written in the form

$$g(z) = z(f'(z))^2. \quad (2.10)$$

If we take the logarithmic derivative of equality (2.10) followed by simple calculations, we get

$$1 + \frac{1}{2b} \left(z \frac{g'(z)}{g(z)} - 1 \right) = 1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}. \quad (2.11)$$

On the other hand, b is a complex number and $b \neq 0$. Therefore, $b_1 = 2b$ is a complex number and $2b \neq 0$, thus (2.11) can be written in the form

$$1 + \frac{1}{b_1} \left(z \frac{g'(z)}{g(z)} - 1 \right) = 1 + \frac{1}{b} z \frac{f''(z)}{f'(z)}. \quad (2.12)$$

Considering equality (2.12), the definition of $C(A, B, b)$, and the definition of $S^*(A, B, b)$, we obtain $g(z) \in S^*(A, B, 2b)$.

Conversely, if $g(z) \in S^*(A, B, b)$, and $g(z) = z((f(z) - f(\zeta))/(z - \zeta))$ holds, then from Lemma 2.3 we get

$$g(z) = z \left(\frac{f(z) - f(\zeta)}{z - \zeta} \right)^2 = \begin{cases} z(1 + Bw(z))^{b(A-B)/B}, & B \neq 0, \\ ze^{bAw(z)}, & B = 0. \end{cases} \quad (2.13)$$

If we take the logarithmic derivative with respect to z of (2.13) followed by simple calculations, we get

$$\begin{aligned} 1 + \frac{1}{b} \left(z \frac{g'(z)}{g(z)} - 1 \right) &= \frac{1}{b} \left[\frac{2zf'(z)}{f(z) - f(\zeta)} - \frac{z + \zeta}{z - \zeta} \right] + 1 - \frac{1}{b} \\ &= \frac{1 + Aw(z)}{1 + Bw(z)}, \quad B \neq 0, \\ 1 + \frac{1}{b} \left(z \frac{g'(z)}{g(z)} - 1 \right) &= \frac{1}{b} \left[\frac{2zf'(z)}{f(z) - f(\zeta)} - \frac{z + \zeta}{z - \zeta} \right] + 1 - \frac{1}{b} \\ &= 1 + Aw(z), \quad B = 0. \end{aligned} \quad (2.14)$$

Furthermore, if we write $F(z, \zeta) = (1/b)[2zf'(z)/(f(z) - f(\zeta)) - (z + \zeta)/(z - \zeta)] + 1 - 1/b$, then we have

$$\lim_{\zeta \rightarrow z} F(z, \zeta) = 1 + \frac{1}{b} z \frac{f''(z)}{f'(z)}. \quad (2.15)$$

Considering relations (2.14) and (2.15) together, we obtain $f(z) \in C(A, B, b)$. $\square$

COROLLARY 2.5. *If $f(z) \in C(A, B, b)$, then*

$$2 \left[1 + \frac{1}{b} \left(z \frac{f'(z)}{f(z)} - 1 \right) \right] - 1 = p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \quad (2.16)$$

PROOF. If we take $\zeta = 0$ in $F(z, \zeta)$, we obtain the desired result of this corollary. $\square$

3. The radius of starlikeness for the class $C(A, B, b)$

LEMMA 3.1. *If $f(z) \in C(A, B, b)$, then*

$$\left| z \frac{f'(z)}{f(z)} - \frac{2 - [B^2 - b(2B^2 - AB)r^2]}{2(1 - B^2r^2)} \right| \leq \frac{|b|(A - B)r}{2(1 - B^2r^2)}. \quad (3.1)$$

PROOF. If $p(z) \in P(A, B)$, then

$$\left| p(z) - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leq \frac{(A - B)r}{1 - B^2r^2}. \quad (3.2)$$

The inequality (3.2) was proved by Janowski [7]. Considering Corollary 2.5 and inequality (3.1), then we get

$$\left| 2 \left[1 + \frac{1}{b} \left(z \frac{f'(z)}{f(z)} - 1 \right) - 1 \right] - \frac{1 - AB r^2}{1 - B^2 r^2} \right| \leq \frac{(A - B)r}{1 - B^2 r^2}. \quad (3.3)$$

After brief calculations from (3.3), we obtain (3.1). $\square$

THEOREM 3.2. *The radius of starlikeness for the class $C(A, B, b)$ is*

$$r_s = \frac{4}{|b|(A - B) + \sqrt{|b|^2(A - B)^2 + 8[2B^2 + (AB - B^2)\operatorname{Re} b]}}. \quad (3.4)$$

This radius is sharp, because the extremal function is

$$f_*(z) = \begin{cases} \int_0^z (1 + B\zeta)^{b(A-B)/B} d\zeta, & B \neq 0, \\ \int_0^z e^{Ab\zeta} d\zeta, & B = 0. \end{cases} \quad (3.5)$$

PROOF. After the brief calculations from inequality (3.1), we get

$$\operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) \geq \frac{2 - |b|(A - B)r - [2B^2 + (AB - B^2)\operatorname{Re} b]r^2}{1 - B^2 r^2}. \quad (3.6)$$

Hence for $r < r_s$ the right-hand side of inequality (3.6) is positive. This implies that (3.4) holds.

Also note that inequality (3.6) becomes an equality for the function $f_*(z)$. It follows that (3.4) holds. $\square$

COROLLARY 3.3. *If $A = 1$, $B = -1$, $b = 1$, then $r_s = 1$. This is the radius of starlikeness of convex functions which is well known (see [1, Volume II, page 88]).*

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