CLASSES OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS

SAEID SHAMS, S. R. KULKARNI, and JAY M. JAHANGIRI

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Some classes of uniformly starlike and convex functions are introduced. The geometrical properties of these classes and their behavior under certain integral operators are investigated.

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1. Introduction. Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disk $U = \{z : |z| < 1\}$. A function f in A is said to be starlike of order β , $0 \leq \beta < 1$, written as $f \in S^*(\beta)$, if $\operatorname{Re}[(zf'(z))/(f(z))] > \beta$. A function $f \in A$ is said to be convex of order β , or $f \in K(\beta)$, if and only if $zf' \in S^*(\beta)$.

Let $\operatorname{SD}(\alpha, \beta)$ be the family of functions f in A satisfying the inequality

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta, \quad z \in U, \quad \alpha \geq 0, \quad 0 \leq \beta < 1. \quad (1.1)$$

We note that for $\alpha > 1$, if $f \in \operatorname{SD}(\alpha, \beta)$, then $zf'(z)/f(z)$ lies in the region $G \equiv G(\alpha, \beta) \equiv \{w : \operatorname{Re} w > \alpha|w-1| + \beta\}$, that is, part of the complex plane which contains $w = 1$ and is bounded by the ellipse $(u - (\alpha^2 - \beta)/(\alpha^2 - 1))^2 + (\alpha^2/(\alpha^2 - 1))v^2 = \alpha^2(1 - \beta)^2/(\alpha^2 - 1)^2$ with vertices at the points $((\alpha + \beta)/(\alpha + 1), 0)$, $((\alpha - \beta)/(\alpha - 1), 0)$, $((\alpha^2 - \beta)/(\alpha^2 - 1), (\beta - 1)/\sqrt{\alpha^2 - 1})$, and $((\alpha^2 - \beta)/(\alpha^2 - 1), (1 - \beta)/\sqrt{\alpha^2 - 1})$. Since $\beta < (\alpha + \beta)/(\alpha + 1) < 1 < (\alpha - \beta)/(\alpha - 1)$, we have $G \subset \{w : \operatorname{Re} w > \beta\}$ and so $\operatorname{SD}(\alpha, \beta) \subset S^*(\beta)$. For $\alpha = 1$ if $f \in \operatorname{SD}(\alpha, \beta)$, then $zf'(z)/f(z)$ belongs to the region which contains $w = 2$ and is bounded by parabola $u = (v^2 + 1 - \beta^2)/2(1 - \beta)$.

Using the relation between convex and starlike functions, we define $\operatorname{KD}(\alpha, \beta)$ as the class of functions $f \in A$ if and only if $zf' \in \operatorname{SD}(\alpha, \beta)$. For $\alpha = 1$ and $\beta = 0$, we obtain the class $\operatorname{KD}(1, 0)$ of uniformly convex functions, first defined by Goodman [1]. Rønning [3] investigated the class $\operatorname{KD}(1, \beta)$ of uniformly convex functions of order β . For the class $\operatorname{KD}(\alpha, 0)$ of α -uniformly convex function, see [2]. In this note, we study the coefficient bounds and Hadamard product or convolution properties of the classes $\operatorname{SD}(\alpha, \beta)$ and $\operatorname{KD}(\alpha, \beta)$. Using these results, we further show that the classes $\operatorname{SD}(\alpha, \beta)$ and $\operatorname{KD}(\alpha, \beta)$ are closed under certain integral operators.

2. Main results. First we give a sufficient coefficient bound for functions in $\operatorname{SD}(\alpha, \beta)$.

THEOREM 2.1. *If $\sum_{n=2}^{\infty} [n(1 + \alpha) - (\alpha + \beta)]|a_n| < 1 - \beta$, then $f \in \operatorname{SD}(\alpha, \beta)$.*

PROOF. By definition, it is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - (1 + \beta) - \alpha \right| \left| \frac{zf'(z)}{f(z)} - 1 \right| < \left| \frac{zf'(z)}{f(z)} + (1 - \beta) - \alpha \right| \left| \frac{zf'(z)}{f(z)} - 1 \right|. \quad (2.1)$$

For the right-hand side and left-hand side of (2.1) we may, respectively, write

$$\begin{aligned} R &= \left| \frac{zf'(z)}{f(z)} + (1 - \beta) - \alpha \right| \left| \frac{zf'(z)}{f(z)} - 1 \right| \\ &= \frac{1}{|f(z)|} |zf'(z) + (1 - \beta)f(z) - \alpha e^{i\theta} |zf'(z) - f(z)| | \\ &\geq \frac{1}{|f(z)|} \left[(2 - \beta)|z| - \sum_{n=2}^{\infty} (n + 1 - \beta) |a_n| |z|^n - \alpha \sum_{n=2}^{\infty} (n - 1) |a_n| |z|^n \right] \\ &> \frac{|z|}{|f(z)|} \left[2 - \beta - \sum_{n=2}^{\infty} (n + 1 - \beta + n\alpha - \alpha) |a_n| \right], \end{aligned} \quad (2.2)$$

and similarly

$$L = \left| \frac{zf'(z)}{f(z)} - (1 + \beta) - \alpha \right| \left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{|z|}{|f(z)|} \left[\beta + \sum_{n=2}^{\infty} (n - 1 - \beta + n\alpha - \alpha) |a_n| \right]. \quad (2.3)$$

Now, the required condition (2.1) is satisfied, since

$$R - L > \frac{|z|}{|f(z)|} \left[2(1 - \beta) - 2 \sum_{n=2}^{\infty} [n(1 + \alpha) - (\alpha + \beta)] |a_n| \right] > 0. \quad (2.4)$$

The following two theorems follow from the above Theorem 2.1 in conjunction with a convolution result of Ruscheweyh and Sheil-Small [5] and the already discussed relation between the classes $SD(\alpha, \beta)$ and $KD(\alpha, \beta)$. $\square$

THEOREM 2.2. If $\sum_{n=2}^{\infty} n[n(1 + \alpha) - (\alpha + \beta)] |a_n| < 1 - \beta$, then $f \in KD(\alpha, \beta)$.

THEOREM 2.3. The classes $SD(\alpha, \beta)$ and $KD(\alpha, \beta)$ are closed under Hadamard product or convolution with convex functions in U .

From Theorem 2.3 and the fact that

$$F(z) = \frac{1 + \lambda}{z^\lambda} \int_0^z t^{\lambda-1} f(t) dt = f(z) * \sum_{n=1}^{\infty} \frac{1 + \lambda}{n + \lambda} z^n, \quad \operatorname{Re} \lambda \geq 0, \quad (2.5)$$

we obtain the following corollary upon noting that $\sum_{n=1}^{\infty} ((1 + \lambda)/(n + \lambda)) z^n$ is convex in U .

COROLLARY 2.4. If f is in $SD(\alpha, \beta)$ or $KD(\alpha, \beta)$, so is $F(z)$ given by (2.5).

Similarly, the following corollary is obtained for

$$G(z) = \int_0^z \frac{f(t) - f(\mu t)}{t(1 - \mu)} dt = f(z) * \left(z + \sum_{n=2}^{\infty} \frac{1 - \mu^n}{n(1 - \mu)} z^n \right), \quad |\mu| \leq 1, \mu \neq 1. \quad (2.6)$$

COROLLARY 2.5. *If f is in $SD(\alpha, \beta)$ or $KD(\alpha, \beta)$, so is $G(z)$ given by (2.6).*

We observed that if $\alpha > 1$ and if $f \in SD(\alpha, \beta)$, then $(zf'(z)/f(z))_{z \in U} \subset E$, where E is the region bounded by the ellipse $(u - (\alpha^2 - \beta)/(\alpha^2 - 1))^2 + (\alpha^2/(\alpha^2 - 1))v^2 = \alpha^2(1 - \beta)^2/(\alpha^2 - 1)^2$ with the parametric form

$$w(t) = \frac{\alpha^2 - \beta}{\alpha^2 - 1} + \frac{\alpha(1 - \beta)}{\alpha^2 - 1} \cos t + \frac{i(1 - \beta)}{\sqrt{\alpha^2 - 1}} \sin t, \quad 0 \leq t < 2\pi. \quad (2.7)$$

Thus for $\alpha > 1$ and z in the punctured unit disk $U - \{0\}$, we have $f \in SD(\alpha, \beta)$ if and only if $zf'(z)/f(z) \neq w(t)$ or $zf'(z) - w(t)f(z) \neq 0$. By Ruscheweyh derivatives (see [4]), we obtain $f \in SD(\alpha, \beta)$, if and only if $f(z) * [z/(1 - z)^2 - w(t)z/(1 - z)] \neq 0$, $z \in U - \{0\}$. Consequently, $f \in SD(\alpha, \beta)$, $\alpha > 1$, if and only if $f(z) * h(z)/z \neq 0$, $z \in U$ where h is given by the normalized function

$$h(z) = \frac{1}{1 - w(t)} \left[\frac{z}{(1 - z)^2} - w(t) \frac{z}{1 - z} \right] \quad (2.8)$$

and w is given by (2.7). Conversely, if $f(z) * h(z)/z \neq 0$, then $zf'(z)/f(z) \neq w(t)$, $0 \leq t < 2\pi$. Hence $(zf'(z)/f(z))_{z \in U}$ lie completely inside E or its complement E^c . Since $(zf'(z)/f(z))_{z=0} = 1 \in E$, $(zf'(z)/f(z))_{z \in U} \subset E$, which implies that $f \in SD(\alpha, \beta)$. This proves the following theorem.

THEOREM 2.6. *The function f belongs to $SD(\alpha, \beta)$, $\alpha > 1$, if and only if $f(z) * h(z)/z \neq 0$, $z \in U$ where $h(z)$ is given by (2.8).*

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Saeid Shams: Department of Mathematics, Fergusson College, Pune - 411004, India

S. R. Kulkarni: Department of Mathematics, Fergusson College, Pune - 411004, India

E-mail address: kulkarni_ferg@yahoo.com

Jay M. Jahangiri: Department of Mathematics, Kent State University, Burton, OH 44021-9500, USA

E-mail address: jay@geauga.kent.edu