ON THE CLASS OF SQUARE PETRIE MATRICES INDUCED BY CYCLIC PERMUTATIONS

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Let $n \geq 2$ be an integer and let $P = \{1, 2, \dots, n, n+1\}$. Let Z_p denote the finite field $\{0, 1, 2, \dots, p-1\}$, where $p \geq 2$ is a prime. Then every map σ on P determines a real $n \times n$ Petrie matrix A_σ which is known to contain information on the dynamical properties such as topological entropy and the Artin-Mazur zeta function of the linearization of σ . In this paper, we show that if σ is a *cyclic* permutation on P , then all such matrices A_σ are similar to one another over Z_2 (but not over Z_p for any prime $p \geq 3$) and their characteristic polynomials over Z_2 are all equal to $\sum_{k=0}^n x^k$. As a consequence, we obtain that if σ is a *cyclic* permutation on P , then the coefficients of the characteristic polynomial of A_σ are all odd integers and hence nonzero.

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1. Introduction. Throughout this paper, let $n \geq 2$ be a fixed integer and let $P = \{1, 2, \dots, n, n+1\}$. For every integer $1 \leq i \leq n$, let $J_i = [i, i+1]$. Let σ be a map from P into itself. The linearization of σ on P is defined as the continuous map f_σ from $[1, n+1]$ into itself such that $f_\sigma(k) = \sigma(k)$ for every integer $1 \leq k \leq n+1$ and f_σ is linear on J_i for every integer $1 \leq i \leq n$. Let $A_\sigma = (a_{ij})$ be the *real* $n \times n$ matrix defined by $a_{ij} = 1$ if $f_\sigma(J_i) \supset J_j$ and $a_{ij} = 0$ otherwise. The definition of A_σ may seem opaque. But if we take J_i 's as the vertices of a directed graph and draw an arrow from the vertex J_i to the vertex J_j if $f_\sigma(J_i) \supset J_j$, then A_σ will be the adjacency matrix [4, page 17] of the resulting directed graph. For example, the adjacency matrix of the cyclic permutation $\sigma : 1 \rightarrow 2 \rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 1$ is given as

$$A_\sigma = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (1.1)$$

In the theory of discrete dynamical systems on the interval, this adjacency matrix A_σ turns out to contain much information on the dynamical properties of the map f_σ . For example, for some special types (including cyclic permutations) of σ , if $x^n + \sum_{k=0}^{n-1} a_k x^k$ is the characteristic polynomial of A_σ , then it is shown in [6] that the Artin-Mazur zeta function $\zeta(z)$ [2] of f_σ is $\zeta(z) = 1/(1 + \sum_{k=1}^n a_{n-k} z^k)$. On the other hand, it follows from [1, Theorem 4.4.5, page 222] or [4, Proposition 19, page 204] that the topological entropy of f_σ equals $\max\{0, \log \lambda\}$, where λ is the maximal eigenvalue of A_σ . Since every cyclic graph defines a communication channel, as defined by Shannon, we can claim that the logarithm of the largest eigenvalue of A_σ gives its channel capacity. This motivates further investigation of such matrices A_σ .

Due to the continuity of f_σ , it is clear that such matrices A_σ have entries either zeros or ones such that the ones in each *row* occur consecutively. Actually, we have $a_{ij} = 1$ for all $a_i \leq j \leq b_i - 1$, where $a_i = \min\{f_\sigma(i), f_\sigma(i+1)\}$ and $b_i = \max\{f_\sigma(i), f_\sigma(i+1)\}$, and $a_{ij} = 0$ elsewhere. For our purpose, we define a Petrie matrix [5] to be a matrix whose entries are either zeros or ones such that the ones in each *row* occur consecutively. So, the matrix A_σ induced by a map σ on P is a square Petrie matrix whose determinant is easily seen (by induction) [7] to be either 0 or ± 1 . For any prime number $p \geq 2$, let $Z_p = \{0, 1, 2, \dots, p-1\}$ denote the usual finite field and let $W_{Z_p} = \{\sum_{i=1}^n r_i J_i \mid r_i \in Z_p, 1 \leq i \leq n\}$ be the n -dimensional vector space over Z_p with $\{J_i \mid 1 \leq i \leq n\}$ as a set of basis. Then the matrix $A_\sigma \pmod{2}$ defines a linear transformation ψ_σ on W_{Z_2} such that, for every integer $1 \leq i \leq n$, $\psi_\sigma(J_i) = \sum_{j=1}^n a_{ij} J_j$.

If both σ and ρ are just permutations on P , then it is easy to see that A_σ may not be similar to A_ρ over Z_2 . But if both σ and ρ are *cyclic* permutations on P , then we show, in this paper, that A_σ is similar to A_ρ over Z_2 (but A_σ may not be similar to A_ρ over Z_p for any prime $p \geq 3$) and their characteristic polynomials over Z_2 are all equal to $\sum_{k=0}^n x^k$. As a consequence, we obtain that if σ is a *cyclic* permutation, then the coefficients of the characteristic polynomial of A_σ are all odd integers and hence nonzero (not true in general if σ is not cyclic) with constant term ± 1 .

2. On the Petrie matrix A_σ over Z_2 with any map σ on P . In the following, we let $[x : y]$ denote the closed interval on the real line with x and y as endpoints. For integers $1 \leq k < j \leq n+1$, we let $[k, j]$ denote the element $\sum_{i=k}^{j-1} J_i$ of W_{Z_2} and call k and j the endpoints (this terminology will be used in the proof of Theorem 3.2 in Section 3) of the element $\sum_{i=k}^{j-1} J_i$. Part (2) of the following lemma is proved in [4, pages 22-23], which will be needed in Section 3. Here, we present a different proof (see also [3]).

LEMMA 2.1. *Let n, P, J_i 's, $\sigma, f_\sigma, W_{Z_2}, \psi_\sigma, A_\sigma$ be defined as in Section 1. Let ρ be a map from P into itself and let ψ_ρ and A_ρ be defined similarly. Then the following hold.*

(1) *Let $1 \leq k < j \leq n+1$ be any integers. Then for any element $[k, j] = \sum_{i=k}^{j-1} J_i$ in W_{Z_2} , $\psi_\sigma([k, j]) = [f_\sigma(k) : f_\sigma(j)]$.*

(2) *$\psi_\rho \circ \psi_\sigma = \psi_{\rho \circ \sigma}$ and $(A_\sigma)(A_\rho) \equiv A_{\rho \circ \sigma} \pmod{2}$. Consequently, if σ is a permutation on P , then ψ_σ is invertible with inverse $\psi_{\sigma^{-1}}$ and A_σ is nonsingular with determinant ± 1 .*

PROOF. It follows from the definition of ψ_σ in Section 1 that $\psi_\sigma(J_i) = [f_\sigma(i) : f_\sigma(i+1)]$ for every integer $1 \leq i \leq n$. Thus, we obtain that $\psi_\sigma([k, j]) = \psi_\sigma(\sum_{i=k}^{j-1} J_i) = \sum_{i=k}^{j-1} \psi_\sigma(J_i) = \sum_{i=k}^{j-1} [f_\sigma(i) : f_\sigma(i+1)] = [f_\sigma(k) : f_\sigma(j)]$ since $1+1=0$ in Z_2 . This proves part (1).

By part (1), $\psi_\sigma([k, j]) = [f_\sigma(k) : f_\sigma(j)]$. Similarly, $\psi_\rho([k, j]) = [f_\rho(k) : f_\rho(j)]$. So, $(\psi_\rho \circ \psi_\sigma)(J_i) = \psi_\rho([f_\sigma(i) : f_\sigma(i+1)]) = [f_\rho(f_\sigma(i)) : f_\rho(f_\sigma(i+1))] = [(\rho \circ \sigma)(i) : (\rho \circ \sigma)(i+1)] = \psi_{\rho \circ \sigma}(J_i)$ since, on the finite set P , $f_\sigma = \sigma$ and $f_\rho = \rho$. This shows that $\psi_\rho \circ \psi_\sigma = \psi_{\rho \circ \sigma}$ on W_{Z_2} . Thus, if σ is a permutation on P , then $\psi_{\sigma^{-1}} \circ \psi_\sigma = \psi_{\sigma^{-1} \circ \sigma}$ is the identity map on W_{Z_2} , and so ψ_σ is an invertible linear transformation on W_{Z_2} with inverse $\psi_{\sigma^{-1}}$. The rest of part (2) can be easily proved and is omitted. This proves part (2) and completes the proof of Lemma 2.1. $\square$

3. On the Petrie matrix A_σ with any cyclic permutation σ on P . We will need the following elementary result. We include its proof for completeness.

LEMMA 3.1. *Let $1 \leq j \leq n$ be any fixed integer and let b denote the greatest common divisor of j and $n+1$. Let $s = (n+1)/b$. For every integer $1 \leq k \leq s-1$, let $1 \leq m_k \leq n$ be the unique integer such that $kj \equiv m_k \pmod{n+1}$. Then the m_k 's are all distinct and $\{m_k \mid 1 \leq k \leq s-1\} = \{kb \mid 1 \leq k \leq s-1\}$.*

PROOF. Let $B = \{m_k \mid 1 \leq k \leq s-1\}$ and $C = \{kb \mid 1 \leq k \leq s-1\}$. For every integer $1 \leq k \leq s-1$, since j/b and $(n+1)/b$ are relatively prime, the congruence equation $(j/b)x \equiv k \pmod{(n+1)/b}$ has a solution in $1 \leq x \leq s-1 = (n+1)/b - 1$. Consequently, for every integer $1 \leq k \leq s-1$, the congruence equation $jx \equiv kb \pmod{n+1}$ has a solution in $1 \leq x \leq s-1$. Since $1 \leq kb \leq n$ for every $1 \leq k \leq s-1$, we obtain that $C \subset B$. Since both B and C contain exactly $s-1$ elements, we have $B = C$. That is, $\{m_k \mid 1 \leq k \leq s-1\} = \{kb \mid 1 \leq k \leq s-1\}$. This completes the proof. $\square$

THEOREM 3.2. *Let n, P, J_i 's, $\sigma, f_\sigma, W_{Z_2}, \psi_\sigma, A_\sigma$ be defined as in Section 1. Assume that σ is also a cyclic permutation on P . Then the following hold.*

- (1) *For every integer $1 \leq i \leq n$, $\sum_{k=0}^n \psi_\sigma^k(J_i) = \mathbf{0}$. Consequently, $\sum_{k=0}^n \psi_\sigma^k(w) = \mathbf{0}$ for all $w \in W_{Z_2}$.*
- (2) *Let $1 \leq i \leq n-1$ and $1 \leq j \leq n$ be two fixed integers such that $1 \leq i < f_\sigma^j(i) \leq n$ and let $J = [i, f_\sigma^j(i)] = \sum_{k=i}^{f_\sigma^j(i)-1} J_k$. Assume that j and $n+1$ are relatively prime. Then the set $\{\psi_\sigma^k(J) \mid 0 \leq k \leq n-1\}$ is a basis for W_{Z_2} .*
- (3) *For any cyclic permutations σ and ρ on P , ψ_σ and ψ_ρ are similar on W_{Z_2} . Consequently, the Petrie matrices over Z_2 of all cyclic permutations on P are similar to one another and have the same characteristic polynomial $\sum_{k=0}^n x^k$.*
- (4) *The coefficients of the characteristic polynomial of A_σ are all odd integers (and hence nonzero) with constant term ± 1 .*

REMARK 3.3. Part (3) of the above theorem does not hold if the Petrie matrices of cyclic permutations are over the finite field Z_p for any prime $p \geq 3$. For example, if $P = \{1, 2, 3, 4, 5\}$, σ denotes the cyclic permutation $1 \rightarrow 2 \rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 1$, and ρ denotes the cyclic permutation $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$, then A_σ and A_ρ are not similar over Z_p for any prime $p \geq 3$ because the characteristic polynomials of A_σ and A_ρ are $x^4 - x^3 - 3x^2 - 3x - 1$ and $x^4 - x^3 - x^2 - x - 1$, respectively, which are distinct over Z_p for any prime $p \geq 3$.

PROOF. For any fixed integer $1 \leq i \leq n$, let $1 \leq j \leq n$ be the unique integer such that $f_\sigma^j(i) = i+1$, and so $J_i = [i, i+1] = [i, f_\sigma^j(i)]$. Let b denote the greatest common divisor of j and $n+1$ and let $s = (n+1)/b$. For every integer $1 \leq k \leq s-1$, let $1 \leq m_k \leq n$ be the unique integer such that $kj \equiv m_k \pmod{n+1}$. Then, by Lemma 3.1, we obtain that $\{m_k \mid 1 \leq k \leq s-1\} = \{kb \mid 1 \leq k \leq s-1\}$. Let $m_0 = 0$. Then $\{m_k \mid 0 \leq k \leq s-1\} = \{kb \mid 0 \leq k \leq s-1\}$. Hence, the set $\{0, 1, 2, 3, \dots, n-1, n\}$ is the disjoint union of the sets $\{m_k + m \mid 0 \leq k \leq s-1, 0 \leq m \leq b-1\}$. Therefore, $\sum_{k=0}^{s-1} \psi_\sigma^{m_k}(J_i) = \sum_{k=0}^{s-1} \psi_\sigma^{kj}(J_i)$ (since $kj \equiv m_k \pmod{n+1}$) $= [i : f_\sigma^j(i)] + [f_\sigma^j(i) : f_\sigma^{2j}(i)] + [f_\sigma^{2j}(i) : f_\sigma^{3j}(i)] + \dots + [f_\sigma^{(s-2)j}(i) : f_\sigma^{(s-1)j}(i)] + [f_\sigma^{(s-1)j}(i) : i] = \mathbf{0}$. So, $\sum_{\ell=0}^n \psi_\sigma^\ell(J_i) = \sum_{m=0}^{b-1} \psi_\sigma^m(\sum_{k=0}^{s-1} \psi_\sigma^{m_k}(J_i)) = \mathbf{0}$. This proves part (1).

For the proof of part (2), we first show that if E is a nonempty subset of $\{1, 2, 3, \dots, n-1, n\}$ such that $J + \sum_{k \in E} \psi_\sigma^k(J) = \mathbf{0}$, then $E = \{1, 2, 3, \dots, n-1, n\}$. Indeed, for every integer $1 \leq k \leq n$, let $1 \leq m_k \leq n$ be the unique integer such that $kj \equiv m_k \pmod{n+1}$. Assume that $m_1 = j \notin E$. Then, for any $m \in E$, $m \neq 0, j$. Since $\psi_\sigma^m(J) = \psi_\sigma^m([i, f_\sigma^j(i)]) = [f_\sigma^m(i) : f_\sigma^{m+j}(i)]$, the endpoints of $\psi_\sigma^m(J)$ do not contain the point $f_\sigma^j(i)$. Thus, in the expression of $\psi_\sigma^m(J)$ as a sum of the basis elements J_k 's, it contains either both the basis elements $J_{f_\sigma^j(i)-1}$ and $J_{f_\sigma^j(i)}$ or none of them. But, since $J = [i, f_\sigma^j(i)] = J_i + J_{i+1} + \dots + J_{f_\sigma^j(i)-1}$ contains the element $J_{f_\sigma^j(i)-1}$, but not the element $J_{f_\sigma^j(i)}$, in its expression as a sum of the basis elements J_k 's, we obtain that in the expression of $J + \sum_{m \in E} \psi_\sigma^m(J)$ as a sum of the basis elements J_k 's, the coefficient of $J_{f_\sigma^j(i)-1}$ is different from that of $J_{f_\sigma^j(i)}$ by 1. This implies that $J + \sum_{m \in E} \psi_\sigma^m(J) \neq \mathbf{0}$, which is a contradiction. Therefore, $m_1 = j \in E$.

Thus,

$$\begin{aligned}
 \mathbf{0} &= J + \sum_{m \in E} \psi_\sigma^m(J) \\
 &= J + \psi_\sigma^j(J) + \sum_{m \in E \setminus \{m_1\}} \psi_\sigma^m(J) \\
 &= [i, f_\sigma^j(i)] + [f_\sigma^j(i) : f_\sigma^{2j}(i)] + \sum_{m \in E \setminus \{m_1\}} \psi_\sigma^m(J) \\
 &= [i : f_\sigma^{2j}(i)] + \sum_{m \in E \setminus \{m_1\}} \psi_\sigma^m(J).
 \end{aligned} \tag{3.1}$$

Proceeding in this manner finitely many times, we obtain that $\{m_1, m_2, \dots, m_{n-1}\} \subset E$ and

$$\begin{aligned}
 \mathbf{0} &= J + \sum_{m \in E} \psi_\sigma^m(J) \\
 &= [i : f_\sigma^{2j}(i)] + \sum_{m \in E \setminus \{m_1\}} \psi_\sigma^m(J) \\
 &= [i : f_\sigma^{3j}(i)] + \sum_{m \in E \setminus \{m_1, m_2\}} \psi_\sigma^m(J) \\
 &= \dots = [i : f_\sigma^{nj}(i)] \\
 &\quad + \sum_{m \in E \setminus \{m_1, m_2, \dots, m_{n-1}\}} \psi_\sigma^m(J).
 \end{aligned} \tag{3.2}$$

In particular, $\mathbf{0} = [i : f_\sigma^{nj}(i)] + \sum_{m \in E \setminus \{m_1, m_2, \dots, m_{n-1}\}} \psi_\sigma^m(J)$. If $m \in E$ and $m \neq m_n$, then, as above, since $m \neq 0$ and $m \neq m_n \equiv nj \pmod{n+1}$, the endpoints of $\psi_\sigma^m(J)$ do not contain the point $f_\sigma^{nj}(i)$. Hence, in the expression of $\psi_\sigma^m(J)$ as a sum of the basis elements J_k 's, it contains either both the basis elements $J_{f_\sigma^{nj}(i)-1}$ and $J_{f_\sigma^{nj}(i)}$ or none of them. But, since $[i, f_\sigma^{nj}(i)] = J_i + J_{i+1} + \dots + J_{f_\sigma^{nj}(i)-1}$ contains the element $J_{f_\sigma^{nj}(i)-1}$, not the element $J_{f_\sigma^{nj}(i)}$, in its expression as a sum of the basis elements J_k 's, we obtain that in the expression of $[i : f_\sigma^{nj}(i)] + \sum_{m \in E \setminus \{m_1, m_2, \dots, m_{n-1}\}} \psi_\sigma^m(J)$ as a sum of the basis elements J_k 's, the coefficient of $J_{f_\sigma^{nj}(i)-1}$ is different from that

of $J_{f_\sigma^{nj}(i)}$ by 1. This implies that $[i : f_\sigma^{nj}(i)] + \sum_{m \in E \setminus \{m_1, m_2, \dots, m_{n-1}\}} \psi_\sigma^m(J) \neq \mathbf{0}$, which is a contradiction. Thus, $m_n = nj \in E$. Since, by assumption, j and $n+1$ are relatively prime, we see that, by Lemma 3.1, $\{m_1, m_2, \dots, m_n\} = \{1, 2, \dots, n-1, n\}$. Since $\{m_1, m_2, \dots, m_n\} \subset E \subset \{1, 2, \dots, n-1, n\}$, we obtain that $E = \{1, 2, \dots, n-1, n\}$. This proves our assertion.

Now, assume that $\sum_{k=0}^{n-1} \alpha(k) \psi_\sigma^k(J) = \mathbf{0}$, where $\alpha(k) = 0$ or 1 in $\mathbb{Z}_2$, $0 \leq k \leq n-1$. If $\alpha(0) = 0$ and $\alpha(\ell) \neq 0$ for some integer $1 \leq \ell < n-1$, let ℓ be the smallest such integer; then, since ψ_σ is invertible by Lemma 2.1(2), we obtain that $J + \sum_{k=1}^{n-\ell-1} \alpha(k) \psi_\sigma^k(J) = \mathbf{0}$. So, without loss of generality, we may assume that $\alpha(0) \neq 0$. That is, we assume that $J + \sum_{k=1}^{n-1} \alpha(k) \psi_\sigma^k(J) = \mathbf{0}$. Let $E = \{k \mid 1 \leq k \leq n-1, \alpha(k) \neq 0\}$. Then, we have $J + \sum_{k \in E} \psi_\sigma^k(J) = \mathbf{0}$. But then it follows from what we have just proved above that $E = \{1, 2, \dots, n-1, n\}$. This contradicts the assumption that $E \subset \{1, 2, \dots, n-1\}$. So, the set $\{\psi_\sigma^k(J) \mid 0 \leq k \leq n-1\}$ is linearly independent and hence, by [8], is a basis for $W_{\mathbb{Z}_2}$. This proves part (2).

Let θ denote the cyclic permutation $1 \rightarrow 2 \rightarrow 3 \rightarrow \dots \rightarrow i \rightarrow i+1 \rightarrow \dots \rightarrow n \rightarrow n+1 \rightarrow 1$ on P and let σ be any cyclic permutation on P . Choose any fixed integer $1 \leq j \leq n$ such that j and $n+1$ are relatively prime and let $J = [1, f_\sigma^j(1)]$. Then, by part (2), the set $\{\psi_\sigma^k(J) \mid 0 \leq k \leq n-1\}$ is a basis for $W_{\mathbb{Z}_2}$. Let ϕ be the linear transformation on $W_{\mathbb{Z}_2}$ defined by $\phi(J_k) = \psi_\sigma^{k-1}(J)$, $1 \leq k \leq n$. Then ϕ is an isomorphism on $W_{\mathbb{Z}_2}$. Furthermore, $(\phi \circ \psi_\theta)(J_n) = \phi(\sum_{k=1}^n J_k) = \sum_{k=1}^n \phi(J_k) = \sum_{k=1}^n \psi_\sigma^{k-1}(J) = \psi_\sigma^n(J)$ (by part (1)) $= \psi_\sigma(\psi_\sigma^{n-1}(J)) = \psi_\sigma(\phi(J_n)) = (\psi_\sigma \circ \phi)(J_n)$ and, for every integer $1 \leq k \leq n-1$, $(\phi \circ \psi_\theta)(J_k) = \phi(\psi_\theta(J_k)) = \phi(J_{k+1}) = \psi_\sigma^k(J) = \psi_\sigma(\psi_\sigma^{k-1}(J)) = \psi_\sigma(\phi(J_k)) = (\psi_\sigma \circ \phi)(J_k)$. Thus, ψ_σ is similar to ψ_θ through ϕ . Since the property of similarity is obviously transitive, we obtain that if ρ is any cyclic permutation on P , then ψ_σ and ψ_ρ are similar on $W_{\mathbb{Z}_2}$. Consequently, by [8], the Petrie matrices (over $\mathbb{Z}_2$) of all cyclic permutations on P are similar to one another and so have the same characteristic polynomial $\sum_{k=0}^n x^k$ since $\sum_{k=0}^n x^k$ is easily verified to be the characteristic polynomial of the Petrie matrix A_θ over $\mathbb{Z}_2$. This proves part (3).

Finally, let σ be a cyclic permutation on P . Since A_σ is a real $n \times n$ matrix with entries either zeros or ones, the coefficients of the characteristic polynomial of A_σ are all integers. By taking every entry in A_σ modulo 2 and applying part (3) and the fact that the determinants of Petrie matrices are either 0 or ± 1 , we obtain that the characteristic polynomial of $A_\sigma \pmod{2}$ is equal to $\sum_{k=0}^n x^k$. Consequently, the coefficients of the characteristic polynomial of A_σ are all odd integers with constant term ± 1 . This proves part (4) and completes the proof of Theorem 3.2. $\square$

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