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T-NEIGHBORHOOD GROUPS

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We generalize min-neighborhood groups to arbitrary T-neighborhood groups, where T is a continuous triangular norm. In particular, we point out that our results accommodate the previous theory on min-neighborhood groups due to T. M. G. Ahsanullah. We show that every T-neighborhood group is T-uniformizable, therefore, T-completely regular. We also present several results of T-neighborhood groups in conjunction with TI-groups due to J. N. Mordeson.

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1. Introduction. Menger in [19] introduces an important class of T-uniformities (T being a continuous t-norm) that is generated by a probabilistic metric [21]. Motivated by the Menger's T-uniformities, Höhle [13] brought into light in his celebrated article the idea of probabilistic metrization of fuzzy uniformities. While developing his theory, he showed that a fuzzy T-uniformity is probabilistic metrizable if and only if it is Hausdorff-separated and has a countable base. He also pointed out that when $T = \min$ is considered, his fuzzy T-uniformity reduces to min-fuzzy uniformity of R. Lowen—a fuzzy uniformity widely used over the years. One of the interesting features of this min-fuzzy uniformity [16] is that it gives rise to a fuzzy neighborhood space [17]; an interesting and very well-behaved class of fuzzy topological spaces [15], used by many authors in a wide variety of ways. Among the prominent classes of so called fuzzy neighborhood spaces are, for instance, Katsaras linear fuzzy neighborhood spaces [14], fuzzy metric neighborhood spaces [18], fuzzy neighborhood groups, rings, modules, algebras, and commutative division rings [1, 2, 3, 4, 5].

Very recently, following the famous articles of Menger [19], Höhle [13], Frank [9], Hashem and Morsi [10, 11, 12] introduced a class of fuzzy topological spaces [15] as they put it: fuzzy T-neighborhood spaces herein called T-neighborhood spaces—a natural generalization of min-fuzzy neighborhood spaces of Lowen [17]. Our main target here in this article is to generalize the notion of min-fuzzy neighborhood groups introduced in [2]. We show that every T-neighborhood group is a T-uniform space, and therefore, a T-complete regular space in the sense of Hashem and Morsi [12]. We also generalize the two important characterization theorems, which give necessary and sufficient condition for a T-neighborhood system and a group structure to be compatible, and a prefilter to be a T-neighborhood prefilter.

As an application, we present some results on T-neighborhood groups in conjunction with Mordeson's TI-groups [7], which we believe will open the opportunities to look into further work on fuzzy algebraic structures in connection with the T-neighborhood groups.

2. Preliminaries. Let T be a continuous two-place function (known as continuous triangular norm or t-norm) mapping from the closed unit square to the closed unit interval satisfying certain conditions. In other words, $T : I\times I \to I$, $(\alpha,\beta)\mapsto \alpha T\beta$, satisfying the following conditions:

(Ta) $0T0 = 0$, $\alpha T1 = \alpha$ for all $\alpha\in I$;
(Tb) $\alpha T\beta = \beta T\alpha$ for all $(\alpha,\beta)\in I\times I$;
(Tc) if $\alpha\le\beta$ and $\gamma\le\delta$, then $\alpha T\gamma\le\beta T\delta$ for all $\alpha,\beta,\gamma,\delta\in I$;
(Td) $(\alpha T\beta)T\gamma = \alpha T(\beta T\gamma)$ for all $\alpha,\beta,\gamma\in I$.

Definition 2.1 [6, 16]**.** A nonempty subset $\mathcal{B}\subset I^G$ is called a prefilterbase if and only if the following conditions are true:

(PB1) $0\notin\mathcal{B}$;
(PB2) for all $\nu_1,\nu_2\in\mathcal{B}$, there exists $\nu\in\mathcal{B}$ such that $\nu\le\nu_1\wedge\nu_2$.

If $\mathcal{B}$ is a prefilterbase in I^G, then by its saturation we understand the following collection:

$$\mathcal{B}^{\sim} = \{\nu : G\longrightarrow I;\ \forall\epsilon>0\ \exists\nu_\epsilon\in\mathcal{B}\ni\nu_\epsilon-\epsilon\le\nu\}. \tag{2.1}$$

Definition 2.2 [10, 11, 12]**.** A T-neighborhood space is an I-topological space [15] $(G,^-)$ whose closure operator "$-$" is induced by some indexed family $\Omega = (\Omega(x))_{x\in G}$ of prefilterbases in I^G defined by

$$\bar{\xi}(x) = \inf_{\nu\in\Omega(x)}\sup_{z\in G}\xi(z)T\nu(z)\quad \forall\xi\in I^G,\ x\in G. \tag{2.2}$$

Theorem 2.3 [10, 11, 12]**.** *A family $\Omega = (\Omega(x))_{x\in G}$ of prefilterbases in I^G is a T-neighborhood base in G if and only if it satisfies the following two properties for all $x\in G$:*

(TB1) *for all $\nu\in\Omega(x)$, $\nu(x) = 1$;*
(TB2) *for all $\nu\in\Omega(x)$, there exists a family $(\nu_{y\epsilon}\in\Omega(y))_{(y,\epsilon)\in G\times I_0}$ which satisfies for all $(y,\epsilon)\in G\times I_0$,*

$$\sup_{z\in G}\left[\nu_{x,\epsilon}(z)T\nu_{z,\epsilon}(y)\right]\le\nu(y)+\epsilon. \tag{2.3}$$

The family Ω is said to be a T-neighborhood basis for $(G,^-)$, and every $\nu\in\Omega(x)$ is called T-neighborhood at x. This I-topology is denoted by $t^T(\Omega)$. However, from now on we will be calling the triple $(G,^-,t(\Omega))$ the T-neighborhood space with Ω a T-neighborhood base in G.

Theorem 2.4 [10, 11, 12]**.** *Let $(G_1,^-,t(\Omega_1))$ and $(G_2,^-,t(\Omega_2))$ be T-neighborhood spaces with T-neighborhood bases Ω_1 and Ω_2, respectively. Then a function $f : G_1\to G_2$ is continuous at $x\in G_1$*

$$\begin{aligned}
&\Longleftrightarrow \forall\mu\in\Omega_2(f(x)),\ f^{-1}(\mu)\in\Omega_1(x)^{\sim},\\
&\Longleftrightarrow \forall\mu_2\in\Omega_2(f(x))\quad\forall\epsilon>0\ \exists\mu_1\in\Omega_1(x)\ni\mu_1-\epsilon\le f^{-1}(\mu_2),\\
&\Longleftrightarrow \overline{[f^{-1}(\sigma)]}(x)\le[f^{-1}(\bar{\sigma})](x)\quad\forall\sigma\in I^{G_2}.
\end{aligned}\tag{2.4}$$

If $\Lambda,\Gamma \in I^{G\times G}$ and $\nu \in I^G$, then T-section of Λ over ν is given by

$$\Lambda\langle \nu\rangle_T(x) = \sup_{y\in G} \nu(y) T \Lambda(y,x) \quad \forall x \in G. \tag{2.5}$$

The T-composition of Λ and Γ is defined as

$$\Lambda \circ_T \Gamma(x,y) = \sup_{z\in G}\left[\Gamma(x,z) T \Lambda(z,y)\right] \quad \forall (x,y) \in G\times G. \tag{2.6}$$

Γ is called symmetric if $\Gamma^s = \Gamma$, that is, $\Gamma(y,x) = \Gamma(x,y)$, for all $(x,y) \in G\times G$.

Definition 2.5 [10, 11, 12]. A subset $\mathcal{B} \subset I^{G\times G}$ is called a T-uniform base on a set G if and only if the following properties are fulfilled:

(TUB1) $\mathcal{B}$ is a prefilterbase;

(TUB2) for all $x \in G$, for all $\nu \in \mathcal{B}, \nu(x,x) = 1$;

(TUB3) for all $\beta \in \mathcal{B}$, for all $\epsilon > 0$, there exists $\beta_\epsilon \in \mathcal{B}$ such that $\beta_\epsilon \circ_T \beta_\epsilon - \epsilon \le \beta$;

(TUB4) for all $\beta \in \mathcal{B}$, for all $\epsilon > 0$, there exists $\beta_\epsilon \in \mathcal{B}$ such that $\beta_\epsilon - \epsilon \le \beta$.

The collection $\mathcal{B}$ of fuzzy subsets of $G\times G$ is called T-quasi-uniform base on a set G if and only if it fulfills the preceding conditions (TUB1), (TUB2), and (TUB3), while $\mathcal{U}$ is called T-quasi-uniformity if and only if $\tilde{\mathcal{B}} = \mathcal{U}$. A T-uniformity $\mathcal{U}$ is a saturated T-uniform base $\mathcal{B}$.

Theorem 2.6 [10, 11, 12]. *If $\mathcal{B}$ is a T-quasi-uniform base on a set G, then for all $x \in G$, the family*

$$\Sigma(x) = \{\beta\langle 1_x\rangle \mid \beta \in \mathcal{B}^\sim\} = \{\beta\langle 1_x\rangle \mid \beta \in \mathcal{B}\}^\sim \tag{2.7}$$

is a T-neighborhood system on G.

Proposition 2.7 [10, 11, 12]. *Let $(G,\mathcal{U})$ be a T-quasi-uniform space. Then the closure of the T-neighborhood space $(G,t(\mathcal{U}))$ is given by*

$$\bar{\mu} = \inf_{\sigma\in\mathcal{U}} \sigma\langle\mu\rangle_T \quad \forall \mu \in I^G. \tag{2.8}$$

Theorem 2.8 [10, 11, 12]. *If (G,τ) is a topological space and $\mathcal{V}_\tau = (\mathcal{V}_\tau(x))_{x\in G}$ is its associated neighborhood system in G, then $(G,^-,t(\Omega_\tau))$, a generated topological space, generated by τ, is a T-neighborhood space with a T-neighborhood basis $\Omega = (\Omega(x))_{x\in G}$, where for all $x \in G$,*

$$\begin{gathered}\Omega_1 := \Omega_1(x) = \{1_M : G \longrightarrow I;\ M \in \mathcal{V}_\tau(x)\} \subset I^G;\\ \Omega_2 := \Omega_2(x) = \{1_M : G \longrightarrow I;\ x \in M \in \tau\} \subset I^G;\\ \Omega_3 := \Omega_3(x) = \{\nu : G \longrightarrow I;\ \nu \text{ is l.s.c. in } x \text{ and } \nu(x) = 1\} \subset I^G.\end{gathered} \tag{2.9}$$

Just for the sake of convenience, we provide the proof of the following proposition.

Proposition 2.9. *A function $f : (G,\mathcal{V}_\tau) \to (G',\mathcal{V}'_{\tau'})$ between two topological spaces is continuous at a point $x \in G$ if and only if $f : (G,t(\Omega_\tau)) \to (G',t(\Omega'_{\tau'}))$ is continuous at $x \in G$ between two generated T-neighborhood spaces.*

Proof. Let $f:(G,\mathcal{V}_\tau)\to(G,\mathcal{V}_{\tau'})$ be continuous at $x\in G$ and $\mu'\in\Omega'_{\tau'}(f(x))$; in view of Theorem 2.4, we show that $f^{-1}(\mu')\in\widetilde{\Omega_\tau(x)}$.

Choose $M'\in\mathcal{V}'_\tau(f(x))$ such that $\mu'=1_{M'}$. This implies that there exists an $M\in\mathcal{V}_\tau(x)$ such that $f(M)\subset M'$, and hence for all $\epsilon>0$,

$$1_M(x)-\epsilon=1-\epsilon\le 1_{f^{-1}(M')}(x)=f^{-1}(\mu'). \tag{2.10}$$

With $\mu=1_M$, one obtains $\mu-\epsilon\le f^{-1}(\mu')$ implying that $f^{-1}(\mu')\in\widetilde{\Omega_\tau(x)}$.

Conversely, we show that the function $f:(G,\mathcal{V}_\tau)\to(G',\mathcal{V}'_{\tau'})$ is continuous at $x\in G$. If $U\in\mathcal{V}'_{\tau'}(f(x))$, then $1_U\in\Omega'_{\tau'}(f(x))$ implies $f^{-1}(1_U)\in\widetilde{\Omega_\tau(x)}$ by continuity of f between the generated spaces.

Thus, for all $\epsilon>0$, there is a $\mu=\mu_\epsilon\in\Omega_\tau(x)$ such that

$$\mu-\epsilon\le f^{-1}(1_U). \tag{2.11}$$

This implies that for all $\epsilon>0$, there exists a $V_\epsilon\in\mathcal{V}_\tau(x)$ such that $1_{V_\epsilon}=\mu=\mu_\epsilon$ and

$$1_{V_\epsilon}-\epsilon\le 1_{f^{-1}(U)}. \tag{2.12}$$

Now $V_\epsilon\in\mathcal{V}_\tau(x)$ implies $x\in V_\epsilon$ if and only if $1_{V_\epsilon}(x)=1$. Therefore,

$$\begin{aligned}0<1-\epsilon=1_{V_\epsilon}(x)-\epsilon\le 1_{f^{-1}(U)}(x)&\Longrightarrow 1_{f^{-1}(U)}(x)>0\Longrightarrow 1_{f^{-1}(U)}(x)\\ &=1\Longleftrightarrow x\in f^{-1}(U).\end{aligned} \tag{2.13}$$

This means that $V_\epsilon\subseteq f^{-1}(U)$ implies $f^{-1}(U)\in\mathcal{V}_\tau(x)$. That is, f is continuous at $x\in G$. □

Theorem 2.10 [11]. *Let $(G,^-)$ be an I-topological space. Then $(G,^-)$ is a T-neighborhood space if and only if $\overline{\alpha T\mu}=\alpha T\bar{\mu}$ for all $\mu\in I^G$ and for all $\alpha\in I$.*

Definition 2.11 [12]. An I-topological space (X,τ) is called T-completely regular if τ is the initial I-topology for the family of all continuous functions from (X,τ) to $(\mathcal{D}^+,t^T(\mathcal{F}_{\mathcal{H}}))$.

Here, $\mathcal{D}^+$ stands for the collection of all distance distribution functions from $\Re^+$ to the unit interval I, and the pair $(\mathcal{D}^+,t^T(\mathcal{F}_{\mathcal{H}}))$ is the T-neighborhood space induced by the well-known Höhle's probabilistic T-metric $\mathcal{F}_{\mathcal{H}}$. For details, we refer to [12, 13].

Theorem 2.12 [12]. *T-complete regularity is equivalent to T-uniformizability.*

3. Some results on T-neighborhood spaces

Theorem 3.1. *Let $(G_1,^-,t(\Omega_1))$ and $(G_2,^-,t(\Omega_2))$ be two T-neighborhood spaces with bases $\Omega_1=(\Omega_1(x))_{x\in G_1}$ and $\Omega_2=(\Omega_2(x))_{x\in G_2}$ in G_1 and G_2, respectively. Then their T-product $(G_1\times G_2,-^{\otimes_T},t(\Omega_1)\otimes_T t(\Omega_2))$ is the T-neighborhood space with base $\Omega=\Omega_1\otimes_T\Omega_2$ defined by*

$$\Omega(x,y)=\{\nu_1\otimes_T\nu_2\mid \nu_1\in\Omega_1(x),\ \nu_2\in\Omega_2(y)\}, \tag{3.1}$$

where $\nu_1 \otimes_T \nu_2$ is given as

$$\nu_1 \otimes_T \nu_2(x,y) = \nu_1(x)T\nu_2(y) \quad \forall (x,y) \in G_1 \times G_2. \tag{3.2}$$

Moreover,

$$\overline{\nu_1 \otimes_T \nu_2} = \overline{\nu_1} \otimes_T \overline{\nu_2} \quad \forall \nu_1 \in I^{G_1},\ \nu_2 \in I^{G_2}. \tag{3.3}$$

Conversely, if

$$\overline{\nu_1 \otimes_T \nu_2} = \overline{\nu_1} \otimes_T \overline{\nu_2} \quad \forall \nu_1 \in I^{G_1},\ \nu_2 \in I^{G_2}, \tag{3.4}$$

then both the I-topological spaces $(G_1, {}^-)$ and $(G_2, {}^-)$ are T-neighborhood spaces.

Proof. First we show that for all $(x,y) \in G_1 \times G_2$, $\Omega(x,y)$ is a prefilterbase.

(PB1) Obviously, $\Omega \neq \emptyset$ and $0 \notin \Omega$.

(PB2) Let $\xi_1, \xi_2 \in \Omega(x,y)$, then there are $\nu_1, \nu_2 \in \Omega_1(x)$ and $\mu_1, \mu_2 \in \Omega_2(y)$ such that $\xi_1 = \nu_1 \otimes_T \mu_1$ and $\xi_2 = \nu_2 \otimes_T \mu_2$.

Now, $\xi_1 \wedge \xi_2 = (\nu_1 \otimes_T \mu_1) \wedge (\nu_2 \otimes_T \mu_2)$. For any $(x,y) \in G_1 \times G_2$,

$$\begin{aligned}
\xi_1(x,y) \wedge \xi_2(x,y) &= (\nu_1 \otimes_T \mu_1)(x,y) \wedge (\nu_2 \otimes_T \mu_2)(x,y) \\
&= (\nu_1(x)T\mu_1(y)) \wedge (\nu_2(x)T\mu_2(y)) \\
&\geq (\nu_1(x) \wedge \nu_2(x))T(\nu_1(y) \wedge \mu_2(y)) \\
&= (\nu_1 \wedge \nu_2)(x)T(\mu_1 \wedge \mu_2)(y).
\end{aligned} \tag{3.5}$$

Therefore, $\xi_1 \wedge \xi_2(x,y) \geq \nu(x) \wedge \mu(y)$ for some $\nu \in \Omega_1(x)$ and $\mu \in \Omega_2(y)$, since both $\Omega_1(x)$ and $\Omega_2(y)$ are prefilterbases in G_1 and G_2, respectively.

This implies that $\xi_1 \wedge \xi_2(x,y) \geq \nu \otimes_T \mu(x,y) = \xi(x,y)$ and $\xi \in \Omega(x,y)$, and hence $\xi \leq \xi_1 \wedge \xi_2$, proving that $\Omega(x,y)$ is a prefilterbase in $G_1 \times G_2$.

Now we prove the conditions of Theorem 2.3.

(TB1) If $x \in G$ and $\xi \in \Omega(x,x)$, then for some $\nu \in \Omega_1(x)$ and $\mu \in \Omega_2(x)$, we have $\xi(x,x) = \nu \otimes_T \mu(x,x)$, $\nu(x)T\mu(x) = 1T1 = 1$.

(TB2) Let $(x,y) \in G_1 \times G_2$, $\xi \in \Omega(x,y)$, and $\epsilon \in I_0$. Then there exists $\nu \in \Omega_1(x)$ and $\mu \in \Omega_2(y)$ such that $\xi = \nu \otimes_T \mu$.

Consequently, there is a family $(\nu_{y_1\epsilon} \in \Omega_1(y_1))_{(y_1,\epsilon)\in G_1 \times I_0}$, a T-kernel for ν which satisfies for all $(y_1,\epsilon) \in G_1 \times I_0$,

$$\sup_{z_1 \in G_1} [\nu_{x,\epsilon}(z_1)T\nu_{z_1,\epsilon}(y_1)] \leq \nu(y_1) + \epsilon. \tag{3.6}$$

Also, there is a family $(\mu_{y_2\epsilon} \in \Omega_2(y_2))_{(y_2,\epsilon)\in G_2 \times I_0}$, a T-kernel for μ which satisfies for all $(y_2,\epsilon) \in G_2 \times I_0$,

$$\sup_{z_2 \in G_2} [\mu_{y,\epsilon}(z_2)T\mu_{z_2,\epsilon}(y_2)] \leq \mu(y_2) + \epsilon. \tag{3.7}$$

Now for all $(y_1,y_2)\in G_1\times G_2$,

$$(\nu\otimes_T\mu)(y_1,y_2)+\delta=[\nu(y_1)T\mu(y_2)]+\delta\geq(\nu(y_1)+\epsilon)T(\mu(y_2)+\epsilon) \tag{3.8}$$

with $\epsilon=\epsilon_{T,\delta}>0$.

This yields that

$$\begin{aligned}
&(\nu\otimes_T\mu)(y_1,y_2)+\delta\\
&\quad\geq\sup_{z_1\in G_1}[\nu_{x,\epsilon}(z_1)T\nu_{z_1,\epsilon}(y_1)]T\sup_{z_2\in G_2}[\mu_{y,\epsilon}(z_2)T\mu_{z_2,\epsilon}(y_2)]\\
&\quad=\sup_{(z_1,z_2)\in G_1\times G_2}[(\nu_{x,\epsilon}(z_1)T\mu_{y,\epsilon}(z_2))T(\nu_{z_1,\epsilon}(y_1)T\mu_{z_2,\epsilon}(y_2))]\\
&\quad=\sup_{(z_1,z_2)\in G_1\times G_2}[((\nu_{x,\epsilon}\otimes_T\mu_{y,\epsilon})(z_1,z_2))T((\nu_{z_1,\epsilon}\otimes_T\mu_{z_2,\epsilon})(y_1,y_2))]\\
&\quad\Longrightarrow\sup_{(z_1,z_2)\in G_1\times G_2}[\nu_{x,\epsilon}\otimes_T\mu_{y,\epsilon}(z_1,z_2)T\nu_{z_1,\epsilon}\otimes_T\mu_{z_2,\epsilon}(y_1,y_2)]\\
&\quad\leq\nu\otimes_T\mu(y_1,y_2)+\delta.
\end{aligned} \tag{3.9}$$

In order to prove the final part, we proceed as follows. Let $\nu_1\in I^{G_1}$, $\nu_2\in I^{G_2}$, and $(x,y)\in G_1\times G_2$.

Then in view of Definition 2.2, we have

$$\begin{aligned}
\overline{\nu_1\otimes_T\nu_2}(x,y)&=\inf_{\xi_1\in\Omega_1(x)}\inf_{\xi_2\in\Omega_2(y)}\sup_{z_1\in G_1}\sup_{z_2\in G_2}(\nu_1\otimes_T\nu_2)(z_1,z_2)T(\xi_1\otimes_T\xi_2)(z_1,z_2)\\
&=\inf_{\xi_1\in\Omega_1(x)}\inf_{\xi_2\in\Omega_2(y)}\sup_{z_1\in G_1}\sup_{z_2\in G_2}\{\nu_1(z_1)T\nu_2(z_2)\}T\{\xi_1(z_1)T\xi_2(z_2)\}\\
&=\inf_{\xi_1\in\Omega_1(x)}\inf_{\xi_2\in\Omega_2(y)}\sup_{z_1\in G_1}\sup_{z_2\in G_2}\nu_1(z_1)T\xi_1(z_1)T\nu_2(z_2)T\xi_2(z_2)\\
&=\inf_{\xi_1\in\Omega_1(x)}\sup_{z_1\in G_1}\nu_1(z_1)T\xi_1(z_2)T\inf_{\xi_2\in\Omega_2(y)}\sup_{z_2\in G_2}\nu_2(z_2)T\xi_2(z_2)\\
&=\overline{\nu_1}(x)T\overline{\nu_2}(y)=\overline{\nu_1}\otimes_T\overline{\nu_2}(x,y).
\end{aligned} \tag{3.10}$$

To prove the converse part, we proceed as follows. Since

$$\overline{\nu_1\otimes_T\nu_2}=\overline{\nu_1}\otimes_T\overline{\nu_2}\quad\forall\nu_1\in I^{G_1},\ \nu_2\in I^{G_2}, \tag{3.11}$$

in view of Theorem 2.10, we have

$$\begin{aligned}
\overline{(\alpha T\nu_1\otimes_T\nu_2)}(x)&=(\alpha T\overline{\nu_1\otimes_T\nu_2})(x)\\
&=\alpha(x)T(\overline{\nu_1}\otimes_T\overline{\nu_2})(x)\\
&=\alpha(x)T(\overline{\nu_1}(x)T\overline{\nu_2}(x)).
\end{aligned} \tag{3.12}$$

Since this holds for all x and for all ν_1 and ν_2, with $\nu_2=1$, we have

$$\begin{aligned}
\overline{(\alpha T\nu_1\otimes_T\nu_2)}(x)&=(\alpha T\overline{\nu_1\otimes_T\nu_2})(x)=\alpha(x)T(\overline{\nu_1}(x)T1)\\
&=\alpha(x)T\overline{\nu_1}(x)=(\alpha T\overline{\nu_1})(x)=\overline{(\alpha T\nu_1)}(x),
\end{aligned} \tag{3.13}$$

so $(G_1,^-)$ is a T-neighborhood space. Similarly, with $\nu_1 = 1$, we see that $(G_2,^-)$ is a T-neighborhood space. This completes the proof. $\square$

Proposition 3.2. *Let $(G_1,^-,t(\Omega_1))$ and $(G_2,^-,t(\Omega_2))$ be T-neighborhood spaces. Then the projections*

$$\begin{aligned}
\mathrm{pr}_1 &: (G_1\times G_2,-^{\otimes_T},t(\Omega_1)\otimes_T t(\Omega_2)) \longrightarrow (G_1,^-,t(\Omega_1)), \qquad (x_1,x_2)\longmapsto x_1,\\
\mathrm{pr}_2 &: (G_1\times G_2,-^{\otimes_T},t(\Omega_1)\otimes_T t(\Omega_2)) \longrightarrow (G_2,^-,t(\Omega_2)), \qquad (x_1,x_2)\longmapsto x_2,
\end{aligned} \tag{3.14}$$

are continuous.

Proof. Let $\nu\in\Omega_1(x_1)$ and $\epsilon>0$. Then

$$\begin{gathered}
\mathrm{pr}_1^{-1}(\nu_1)(x_1,x_2)=\nu_1(\mathrm{pr}_1(x_1,x_2))=\nu_1(x_1)T1\ge\nu_1(x_1)T\nu_2(x_2)\ge\nu_1\otimes_T\nu_2(x_1,x_2)-\epsilon\\
\Longrightarrow \nu_1\otimes_T\nu_2-\epsilon\le\mathrm{pr}_1^{-1}(\nu_1)\Longrightarrow\mathrm{pr}_1^{-1}(\nu_1)\in\Omega(x_1,x_2)^{\sim}.
\end{gathered} \tag{3.15}$$

This implies that $\mathrm{pr}_1 : (G_1\times G_2,-^{\otimes_T},t(\Omega_1)\otimes_T t(\Omega_2)) \to (G_1,^-,t(\Omega_1))$, $(x_1,x_2)\mapsto x_1$, is continuous, and similarly, one can prove that $\mathrm{pr}_2 : (G_1\times G_2,-^{\otimes_T},t(\Omega_1)\otimes_T t(\Omega_2)) \to (G_2,^-,t(\Omega_2))$, $(x_1,x_2)\mapsto x_2$, is continuous. $\square$

Definition 3.3. A T-neighborhood space $(G,^-,t(\Omega))$ is said to be a TN-regular space if and only if for all $z\in G$, for all $\mu\in\Omega(z)$, and for all $\epsilon>0$, there exists a $\nu\in\Omega(z)$ closed such that

$$\epsilon+\mu(z)\ge\inf_{\rho\in\Omega(z)}\sup_{t\in G}\nu(t)T\rho(t)\,(=\bar{\nu}(z)). \tag{3.16}$$

Theorem 3.4. *Every T-quasi-uniform space (G,Ψ) is TN-regular.*

Proof. Suppose that $z\in G$, $\psi\in\Psi$, and $\epsilon>0$, and choose $\psi_\epsilon\in\Psi$ such that

$$\psi_\epsilon\circ_T\psi_\epsilon\le\psi+\epsilon. \tag{3.17}$$

If $t\in G$, then by using Proposition 2.7,

$$\begin{aligned}
\overline{\psi_\epsilon\langle z\rangle}^{T}(t) &= \inf_{\psi'_\epsilon\in\Psi}\sup_{y\in G}\psi_\epsilon\langle z\rangle(y)T\psi'_\epsilon(y,t)\le\sup_{y\in G}\psi_\epsilon(z,y)T\psi_\epsilon(y,t)\\
&=\psi_\epsilon\circ_T\psi_\epsilon(z,t)\le\psi(z,t)+\epsilon\\
&=\psi\langle z\rangle(t)+\epsilon.
\end{aligned} \tag{3.18}$$

Hence the result follows. $\square$

4. T-neighborhood groups.

In what follows, we consider $(G,\cdot)$ as a multiplicative group with e as the identity element. If $\mu : G\to I$, then $\mu^{-1}(x)$ is defined as $\mu^{-1}(x)=\mu(-x)$, and μ is said to be symmetric if and only if $\mu=\mu^{-1}$.

Definition 4.1. Let $(G,\cdot)$ be a group and $(G,^{-},t(\Omega))$ a T-neighborhood space with T-neighborhood base Ω on G. Then the quadruple $(G,\cdot,^{-},t(\Omega))$ is called a T- neighborhood group if and only if the following properties are satisfied:

(TG1) the mapping $m:(G\times G,-^{\otimes_T},t(\Omega)\otimes_T t(\Omega))\to(G,^{-},t(\Omega))$, $(x,y)\mapsto xy$, is continuous;

(TG2) the inversion mapping $r:(G,^{-},t(\Omega))\to(G,^{-},t(\Omega))$, $x\mapsto x^{-1}$, is continuous.

A group structure and a T-neighborhood system is said to be compatible if and only if (TG1) and (TG2) are fulfilled.

Remarks 4.2. A T-neighborhood group may not be a fuzzy topological group in the sense of Foster [8] since we have used T-neighborhood topology, which differ from the product fuzzy topology.

Proposition 4.3. *Let $(G,\cdot)$ be a group and $(G,^{-},t(\Omega))$ a T-neighborhood space with a T-neighborhood base Ω. Then the quadruple $(G,\cdot,^{-},t(\Omega))$ is a T-neighborhood group if and only if the mapping*

$$h:(G\times G,-^{\otimes_T},t(\Omega)\otimes_T t(\Omega))\longrightarrow(G,^{-},t(\Omega)),\quad (x,y)\longmapsto xy^{-1}, \tag{4.1}$$

is continuous.

Proof. Observe that the conditions (TG1) and (TG2) are equivalent to the following single condition:

(TG3) the mapping $h:(G\times G,-^{\otimes_T},t(\Omega)\otimes_T t(\Omega))\to(G,^{-},t(\Omega))$, $(x,y)\mapsto xy^{-1}$, is continuous.

In fact, if we let $f(x,y)=(x,y^{-1})$, then by (TG2), f is continuous and hence in conjunction with (TG1), one obtains the continuity of h. On the other hand, (TG3)⇒(TG2) for $x\to ex^{-1}=x^{-1}$ is then continuous; while (TG1) follows from (TG3) and (TG2), because $(x,y)\to x(y^{-1})^{-1}=xy$ is then continuous. □

Proposition 4.4. *Let $(G,\cdot)$ be a group and $(G,^{-},t(\Omega))$ a T-neighborhood space with Ω a T-neighborhood base in G. Then*

(a) *the mapping $m:(G\times G,\cdot,-^{\otimes_T},t(\Omega)\otimes_T t(\Omega))\to(G,\cdot,^{-},t(\Omega))$, $(x,y)\mapsto xy$, is continuous at $(e,e)\in G\times G$ if and only if for all $\mu\in\Omega(e)$, for all $\epsilon>0$, there exists $\nu\in\Omega(e)$ such that*

$$\nu\odot_T\nu\le\mu+\epsilon; \tag{4.2}$$

(b) *the inversion mapping $r:(G,\cdot,^{-},t(\Omega))\to(G,\cdot,^{-},t(\Omega))$, $x\mapsto x^{-1}$, is continuous at $e\in G$ if and only if for all $\mu\in\Omega(e)$, for all $\epsilon>0$, there exists $\nu\in\Omega(e)$ such that*

$$\nu\le\mu^{-1}+\epsilon. \tag{4.3}$$

Proof. (a) In view of Theorem 2.4, continuity at $(e,e)\in G\times G$ is equivalent to

$$\begin{aligned}&\forall\mu\in\Omega(e)\Longrightarrow m^{-1}(\mu)\in\left(\Omega(e)\otimes_T\Omega(e)\right)^{\sim}\\&\qquad\Longleftrightarrow\forall\mu\in\Omega(e),\quad\forall\epsilon>0,\ \exists\nu=\nu_\epsilon\in\Omega(e)\ni\nu\otimes_T\nu\le m^{-1}(\mu)+\epsilon.\end{aligned} \tag{4.4}$$

But $m(\nu\otimes_T\nu)(z)=\sup_{(x,y)\in m^{-1}(z)}\nu(x)T\nu(y)=\sup_{xy=z}\nu(x)T\nu(y)=\nu\odot_T\nu(z)$. Thus, in this case, continuity at $(e,e)\in G\times G$ is in fact equivalent to

$$\nu\odot_T\nu\leq\mu+\epsilon. \tag{4.5}$$

(b) This follows almost in the same way as in (a). □

Corollary 4.5. *If $(G,\cdot,^-,t(\Omega))$ is a T-neighborhood group, then the mapping (4.1) is continuous at $(e,e)\in G\times G$ if and only if for all $\mu\in\Omega(e)$ and for all $\epsilon>0$, there exists a $\nu\in\Omega(e)$ such that*

$$\nu\odot_T\nu^{-1}\leq\mu+\epsilon. \tag{4.6}$$

Proof. This follows at once from the composition of (a) and (b) in Proposition 4.4. □

Proposition 4.6. *Let $(G,^-,t(\Omega))$ be a T-neighborhood space and $A\subset G$. Then $(A,^-,t(\Omega_{|A}))$ is a T-neighborhood space, a subspace of the T-neighborhood space $(G,^-,t(\Omega))$.*

Proof. The proof follows by easy verification. □

Theorem 4.7. *The triple $(G,\cdot,\tau)$ is a topological group if and only if the quadruple $(G,\cdot,^-,t(\Omega_\tau))$, where Ω is the generated T-neighborhood basis, is a T-neighborhood group.*

Proof. With the help of Proposition 2.9, it follows that the mapping

$$h:(G\times G,\mathcal{V}_\tau\times\mathcal{V}_\tau)\longrightarrow(G,\mathcal{V}_\tau),\quad(x,y)\longmapsto xy^{-1}, \tag{4.7}$$

is continuous if and only if

$$h:(G\times G,t(\Omega_\tau)\otimes_T t(\Omega_\tau))\longrightarrow(G,t(\Omega_\tau)),\quad(x,y)\longmapsto xy^{-1}, \tag{4.8}$$

is continuous, where Ω is the basis for the generated T-neighborhood spaces. □

Lemma 4.8. *Let $(G,\cdot,^-,t(\Omega))$ be a T-neighborhood group and $a\in G$. Then*

(1) *the left translation $\mathcal{L}_a:(G,\cdot,^-,t(\Omega))\to(G,\cdot,^-,t(\Omega))$, $x\mapsto ax$, and the right translation $\mathcal{R}_a:(G,\cdot,^-,t(\Omega))\to(G,\cdot,^-,t(\Omega))$, $x\mapsto ax$, are homeomorphisms;*
(2) *the inner automorphism $\mathcal{I}_a:(G,\cdot,^-,t(\Omega))\to(G,\cdot,^-,t(\Omega))$, $z\mapsto aza^{-1}$, is an isomorphism;*
(3) *$\nu\in\Omega(e)^\sim$ if and only if $\mathcal{L}_a(\nu)\in\Omega(a)^\sim$ if and only if $\mathcal{R}_a(\nu)\in\Omega(a)^\sim$. In other words, if Ω is saturated, then $\nu\in\Omega(e)$ if and only if $1_{\{a\}}\odot_T\nu=a\odot_T\nu\in\Omega(a)$ if and only if $\nu\odot_T a\in\Omega(a)$;*
(4) *$\nu\in\Omega(a)^\sim$ if and only if $\mathcal{L}_{-a}(\nu)\in\Omega(e)^\sim$ if and only if $\mathcal{R}_{-a}(\nu)\in\Omega(e)^\sim$. In other words, if Ω is saturated, then $\nu\in\Omega(a)$ if and only if $1_{\{a^{-1}\}}\odot_T\nu=a^{-1}\odot\nu\in\Omega(e)$ if and only if $\nu\odot_T a^{-1}\in\Omega(e)$;*
(5) *if $\nu\in\Omega(e)$, then $\nu^{-1}\in\Omega(e)$;*
(6) *$\nu\odot_T\nu^{-1}$ is symmetric.*

Proof. (1) follows at once from the definitions while (2) follows from the fact that $\mathcal{I}_a = \mathcal{L}_a \circ \mathcal{R}_{-a} = \mathcal{R}_{-a} \circ \mathcal{L}_a$ for all $a \in G$.

(3) Let $\nu \in \Omega(e)^{\sim} \subset \Omega(e)$, that is, $\nu \in \Omega(a^{-1}a) = \Omega(\mathcal{L}_a^{-1}(a))$. Since $\mathcal{L}_a^{-1}$ is continuous, then in view of Theorem 2.4, $\mathcal{L}_a(\nu) = (\mathcal{L}_a^{-1})^{-1}(\nu) \in \Omega(a)^{\sim}$ implies $\mathcal{L}_a(\nu) \in \Omega(a)^{\sim}$. Conversely, let $\mathcal{L}_a(\nu) \in \Omega(a)^{\sim} \subset \Omega(a)$ implies $\mathcal{L}_a(\nu) \in \Omega(a)$ implies $\mathcal{L}_a(\nu) \in \Omega(a) = \Omega(ae) = \Omega(\mathcal{L}_a(e))$, and since $\mathcal{L}_a : G \to G$ is continuous injection again by Theorem 2.4, $\nu = \mathcal{L}_a^{-1}(\mathcal{L}_a(\nu)) \in \Omega(e)^{\sim}$. For the calculations of the other part, see [7, Theorem 5.1.1].

(4) follows from (3) while (5) follows from the fact that the inversion mapping $r : G \to G$, $x \mapsto x^{-1}$ is a homeomorphism.

(6) We have $\nu \odot_T \nu^{-1} = (\nu \odot_T \nu^{-1})^{-1}$. If $x \in G$, then

$$\begin{aligned}
\left(\nu \odot_T \nu^{-1}\right)^{-1}(x) = \left(\nu \odot_T \nu^{-1}\right)(x^{-1}) &= \sup_{ab=x^{-1}} \nu(a)T\nu(b^{-1}) \\
&= \sup_{st^{-1}=x^{-1}} \nu(s)T\nu(t) = \sup_{ts^{-1}=x} \nu(t)T\nu(s) \\
&= \sup_{ts^{-1}=x} \nu(t)T\nu\left(\left(s^{-1}\right)^{-1}\right) \\
&= \sup_{ts^{-1}=x} \nu(t)T\nu^{-1}(s^{-1}) \\
&= \nu \odot_T \nu^{-1}(x).
\end{aligned} \tag{4.9}$$

This completes the proof. □

Definition 4.9. A T-neighborhood space $(G, ^-, t(\Omega))$ is called homogeneous space if and only if for all $(a,b) \in G \times G$, there exists a homeomorphism $f : (G, ^-, t(\Omega)) \to (G, ^-, t(\Omega))$ such that $f(a) = b$.

Theorem 4.10. *Every T-neighborhood group is a homogeneous space.*

Proof. This follows from the fact that for all $a, b \in G \times G$, the function

$$\mathcal{R}_{a^{-1}b} : G \longrightarrow G, \quad x \longmapsto xa^{-1}b, \tag{4.10}$$

is a homeomorphism. □

Lemma 4.11. *Let $(G,\cdot)$ be a group, and let, for all $\mu \in I^G$, $\mu_L : G \times G \to I$, $(x,y) \mapsto \mu_L(x,y) = \mu(x^{-1}y)$ (resp., $\mu_R : G \times G \to I$, $(x,y) \mapsto \mu_R(x,y) = \mu(yx^{-1})$) be the vicinities L-associated (resp., R-associated) with μ.*

Then for all $\mu, \theta, \nu \in I^G$, $(x,y) \in G \times G$, and triangular norm $T : I \times I \to I$, the following hold:

1. *$\mu_L\langle\theta\rangle_T = \theta \odot_T \mu$ (resp., $\mu_R\langle\theta\rangle_T = \mu \odot_T \theta$);*
2. *$\mu_L T \nu_L = (\mu T \nu)_L$ (resp., $\mu_R T \nu_R = (\mu T \nu)_R$);*
3. *$(\mu_L^s) = (\mu_L)^s$;*
4. *$\mu_L \odot_T \nu_L = (\nu \odot_T \mu)_L$ (resp., $\mu_R \odot_T \nu_R = (\nu \odot_T \mu)_R$).*

Proof. (1) For all $(x, \theta, \mu) \in G \times I^G \times I^G$,

$$\begin{aligned}
\mu_L\langle\theta\rangle_T(x) &= \sup_{y \in G}\left[\theta(y)T\mu_L(y,x)\right] = \sup_{y \in G}\left[\theta(y)T\mu(y^{-1}x)\right] \\
&= \theta \odot_T \mu(x) \quad (\text{by [7, Theorem 5.1.1]}).
\end{aligned} \tag{4.11}$$

(2) and (3) are obvious.

(4) For all $(x,y) \in G\times G$,

$$\begin{aligned} \mu_L \odot_T \nu_L(x,y) &= \sup_{z\in G}[\nu_L(x,z)T\mu_L(z,y)] = \sup_{z\in G}[\nu(x^{-1}z)T\mu(z^{-1}y)] \\ &= \sup_{st=x^{-1}zz^{-1}y=x^{-1}y}[\nu(s)T\mu(t)] = (\nu\odot_T\mu)(x^{-1}y) \\ &= (\nu\odot_T\mu)_L(x,y). \end{aligned} \tag{4.12}$$

□

Theorem 4.12. *Every T-neighborhood group is a T-uniform space.*

Proof. If $(G,\cdot,^-,t(\Omega))$ is a T-neighborhood group, then $(G,^-,t(\Omega))$ is a T-neighborhood space with the T-neighborhood basis Ω.

We consider the following collection:

$$\Omega = \{\mu_L \mid \mu\in\Omega(e)\} \subset I^{G\times G}. \tag{4.13}$$

We claim that Ω is a T-uniform basis.

(TUB1) Clearly Ω is a prefilterbasis.

(TUB2) If $\psi\in\Omega$, then there exists a $\mu\in\Omega(e)$ such that $\psi=\mu_L$, and for all $x\in G$,

$$\psi(x,x) = \mu_L(x,x) = \mu(e) = 1. \tag{4.14}$$

(TUB3) If $\psi\in\Omega$, then there exists a $\mu\in\Omega(e)$ such that $\psi=\mu_L$.

Thus, by virtue of Proposition 4.4(a), for all $\epsilon>0$, we can find $\nu^\epsilon\in\Omega(e)$ such that

$$\nu^\epsilon\odot_T\nu^\epsilon-\epsilon\le\mu. \tag{4.15}$$

If we let $\nu_L^\epsilon=\psi_\epsilon$, then one obtains

$$\begin{aligned} \psi_\epsilon\odot_T\psi_\epsilon-\epsilon = \nu_L^\epsilon\odot_T\nu_L^\epsilon-\epsilon = (\nu^\epsilon\odot_T\nu^\epsilon)_L-\epsilon\le\mu_L \\ \Longrightarrow \psi_\epsilon\odot_T\psi_\epsilon-\epsilon\le\psi. \end{aligned} \tag{4.16}$$

(TUB4) If $\psi\in\Omega$, then there is a $\mu\in\Omega(e)$ such that $\psi=\mu_L$. Consequently, by Proposition 4.4(b), for all $\epsilon>0$, there exists a $\nu^\epsilon\in\Omega(e)$ such that

$$\nu^\epsilon-\epsilon\le\mu^{-1}. \tag{4.17}$$

Therefore, $\nu_L^\epsilon-\epsilon\le(\mu^{-1})_L=(\mu_L)^{-1}$ implies $\psi_\epsilon-\epsilon\le\psi_s$.

This shows in accordance with Definition 2.5 that Ω is a T-uniform basis, which in turn gives rise to a left T-uniformity $\mathcal{U}_L=\Omega^\sim$.

In fact, we have for all $x\in G$,

$$\mathcal{U}_L(x) = \{\mu_L\langle 1_x\rangle \mid \mu\in\Omega(e)\}^\sim = \{\mathcal{L}_x(\mu)\mid\mu\in\Omega(e)^\sim\} = \Omega(x)^\sim, \tag{4.18}$$

which is a T-neighborhood system on G and that $(G,t(\Omega)=t(\mathcal{U}_L))$ is a T-uniform space. Similarly, one can obtain right T-uniformity. □

Theorem 4.13. *Every T-neighborhood group is T-completely regular.*

Proof. This follows from the preceding theorem in conjunction with Theorem 2.12 because every T-neighborhood group is T-uniformizable and every T-uniformizable space is T-completely regular. □

Theorem 4.14. *Let $(G,\cdot)$ be a group, $(G,^-,t(\Omega))$ a T-neighborhood space with T-neighborhood base Ω in G. Then the quadruple $(G,\cdot,^-,t(\Omega))$ is a T-neighborhood group if and only if the following are true:*

(1) *for all $a\in G,\Omega(a)^\sim=\{\mathcal{L}_a(\mu)\mid\mu\in\Omega(e)\}^\sim$ (resp., for all $a\in G,\Omega(a)^\sim=\{\mathcal{R}_a(\mu)\mid \mu\in\Omega(e)\}^\sim$);*

(2) *for all $\mu\in\Omega(e)$, for all $\epsilon>0$, there exists a $\nu\in\Omega(e)$ such that*

$$\nu\odot_T\nu\le\mu+\epsilon, \tag{4.19}$$

that is, the mapping $m:(x,y)\mapsto xy$ is continuous at $(e,e)\in G\times G$;

(3) *for all $\mu\in\Omega(e)$, for all $\epsilon>0$, there exists a $\nu\in\Omega(e)$ such that*

$$\nu\le\mu^{-1}+\epsilon, \tag{4.20}$$

that is, the mapping $r:x\mapsto x^{-1}$ is continuous at $e\in G$;

(4) *for all $\mu\in\Omega(e)$, for all $\epsilon>0$, and for all $a\in G$ such that*

$$a\odot_T\nu\odot_T a^{-1}\le\mu+\epsilon, \tag{4.21}$$

that is, the mapping $\mathcal{I}_a:x\mapsto axa^{-1}$ is continuous at $e\in G$.

Proof. Let $(G,\cdot,^-,t(\Omega))$ be a T-neighborhood group. Then the conditions (1), (2), (3), and (4) are clearly true.

To prove the converse part, we remark that from Corollary 4.5, it follows that the mapping $h:G\times G\to G$; $(x,y)\mapsto xy^{-1}$ is continuous at (e,e), and since the translations $\mathcal{L}_a$ and $\mathcal{R}_a$ are continuous at a and e, respectively, the continuity of m follows from the following chain:

$$G\times G\xrightarrow{\mathcal{L}_{a^{-1}}\times\mathcal{L}_{b^{-1}}}G\times G\xrightarrow{m}G\xrightarrow{\mathcal{I}_b}G\xrightarrow{\mathcal{L}_{ab^{-1}}}G, \tag{4.22}$$

where $(a,b)\to(e,e)\to e\to e\to ab^{-1}$. □

Theorem 4.15. *Let $(G,\cdot)$ be a group and $\mathcal{F}$ a collection of nonempty subsets of I^G, that is, $\varnothing\ne\mathcal{F}\subset I^G$ such that*

(1) *$\mathcal{F}$ is a prefilterbasis and $\mu(e)=1$ for all $\mu\in\mathcal{F}$;*
(2) *for all $\mu\in\mathcal{F}$, for all $\epsilon>0$, there exists a $\nu\in\mathcal{F}$ such that $\nu-\epsilon\le\mu^{-1}$;*
(3) *for all $\mu\in\mathcal{F}$, for all $\epsilon>0$, there exists a $\nu\in\mathcal{F}$ such that $\nu\odot_T\nu-\epsilon\le\mu$;*
(4) *for all $\mu\in\mathcal{F}$, for all $a\in G$, for all $\epsilon>0$, there exists $\nu\in\mathcal{F}$ such that $a\odot_T\nu\odot_T a^{-1}-\epsilon\le\mu$.*

Then there exists a unique T-neighborhood system compatible with the group structure of G such that $\mathcal{F}$ is a T-neighborhood basis at $e\in G$.

Proof. For all $\mu\in\mathcal{F}$, let $\mu_L:G\times G\to I$ be the vicinities L-associated with μ. Evidently, $\mu_L(a,a)=\mu(a^{-1}a)=\mu(e)=1$.

We let

$$\mathcal{B} = \{\mu_L \mid \mu \in \mathcal{F}\} \subset I^{G\times G}. \tag{4.23}$$

We show that $\mathcal{B}$ is a T-quasi-uniform basis for a T-quasi-uniformity. We verify Definition 2.5 upto (TUB3).

(TUB1) $\mathcal{B}$ is a prefilterbasis; for $0 \notin \mathcal{B}$ which is clearly true, since $\mathcal{F}$ is a prefilterbasis. Next, let $\lambda, \xi \in \mathcal{B}$, then $\lambda = \mu_L$ for some $\mu \in \mathcal{F}$ and $\xi = \eta_L$ for some $\eta \in \mathcal{F}$. Since $\mathcal{F}$ is a prefilterbasis, there exists a $\theta \in \mathcal{F}$ such that $\theta \le \mu \wedge \eta$ and $\theta_L \le \mu_L \wedge \eta_L = \lambda \wedge \xi$, proving that $\mathcal{B}$ is indeed a prefilterbasis.

(TUB2) For all $x \in G$, and $\psi \in \mathcal{B}$, we have $\psi = \mu_L$ for some $\mu \in \mathcal{F}$ and $\psi(x,x) = \mu_L(x,x) = \mu(e) = 1$ by (1).

(TUB3) Let $\psi \in \mathcal{B}$. Then there exists a $\mu \in \mathcal{F}$ such that $\psi = \mu_L$.

Now by (3), for all $\epsilon > 0$, we can find a $\nu \in \mathcal{F}$ such that

$$\nu \odot_T \nu - \epsilon \le \mu. \tag{4.24}$$

But then by virtue of Lemma 4.11(4), we get $\nu_L \odot_T \nu_L - \epsilon \le \mu_L$. So, if we put $\psi_\epsilon = \nu_L$, then

$$\psi_\epsilon \odot_T \psi_\epsilon - \epsilon \le \psi. \tag{4.25}$$

This completes the proof that $\mathcal{B}$ is a T-quasi-uniform basis which in turn gives rise to a T-quasi-uniformity and hence a T-quasi-uniform space. Then in view of the Theorem 2.6, since every T-quasi-uniform space is a T-neighborhood space, in this case, we have the T-neighborhood system as given by the family

$$\begin{aligned}\{\mu_L\langle 1_x\rangle_T \mid \mu_L \in \mathcal{B}^{\sim}\} &= \{\mu_L\langle 1_x\rangle_T \mid \mu_L \in \mathcal{B}\}^{\sim} \\ &= \{1_x \odot_T \mu \mid \mu \in \mathcal{F}\}^{\sim} \\ &= \{1_x \odot \mu \mid \mu \in \mathcal{F}\}^{\sim}.\end{aligned} \tag{4.26}$$

Thus one obtains the T-neighborhood system with the following family: $\Omega(x) = \{1_x \odot_T \mu \mid \mu \in \mathcal{F}\}$, a basis for the system in question. □

Theorem 4.16. *Let $(G,\cdot,^-,t(\Omega))$ be a T-neighborhood group. Then for all $\mu : G \to I$,*

$$\bar{\mu} = \inf\{\mu \odot_T \nu \mid \nu \in \Omega(e)^{\sim}\} = \inf\{\mu \odot_T \nu \mid \nu \in \Omega(e)\}^{\sim}. \tag{4.27}$$

Proof. Observe that every T-neighborhood group is a T-quasi-uniform space. Therefore, by virtue of Theorem 2.6, we can write, in particular, that

$$\bar{\mu} = \inf\{\nu_L\langle\mu\rangle_T \mid \nu \in \Omega(e)^{\sim}\}. \tag{4.28}$$

Then by using Lemma 4.11(1), we have the following:

$$\bar{\mu} = \inf\{\mu \odot_T \nu \mid \nu \in \Omega(e)^{\sim}\} = \inf\{\mu \odot_T \nu \mid \nu \in \Omega(e)\}^{\sim}. \tag{4.29}$$

□

Corollary 4.17. *In a T-neighborhood group $(G,\cdot,^{-},t(\Omega))$, the following property holds:*

$$\bar{\mu} = \inf\{\nu \odot_T \mu \mid \nu \in \Omega(e)^{\sim}\} = \inf\{\nu \odot_T \mu \mid \mu \in \Omega(e)\}^{\sim}. \tag{4.30}$$

Proof. This follows at once from the preceding results. □

Theorem 4.18. *If $(G,\cdot,^{-},t(\Omega))$ is a T-neighborhood group, then $(G,^{-},t(\Omega))$ is T-regular.*

Proof. Let $\mu \in \Omega(e)$ and $\epsilon > 0$. Since the map $(x,y) \mapsto xy^{-1}$ is continuous at $(e,e) \in G \times G$, in view of Corollary 4.5, we can find a $\nu \in \Omega(e)$ such that

$$\nu \odot_T \nu^{-1} \le \mu + \epsilon. \tag{4.31}$$

Then using Theorem 4.16, we obtain

$$\bar{\mu}(x) = \inf_{\omega \in \Omega(e)} \nu \odot_T \omega^{-1} \le \nu \odot_T \nu^{-1} \le \mu(x) + \epsilon, \tag{4.32}$$

which ends the proof. □

Proposition 4.19. *If $(G,\cdot,^{-},t(\Omega))$ is a T-neighborhood group, then for all $\mu,\nu \in I^G$, we have the following:*

(i) $\bar{\mu} \odot_T \bar{\nu} \le \overline{\mu \odot_T \nu}$;

(ii) $\overline{\mu^{-1}} = \bar{\mu}^{-1}$;

(iii) $\overline{x \odot_T \mu \odot_T y} = x \odot_T \bar{\mu} \odot_T y$ *for all $x,y \in G$.*

Proof. (i) If $z \in G$, then we have

$$\begin{aligned} \bar{\mu} \odot_T \bar{\nu}(z) &= \sup_{xy=z} \bar{\mu}(x) T \bar{\nu} y = \sup_{(x,y) \in m^{-1}(z)} [\bar{\mu} \otimes_T \bar{\nu}](x,y) \\ &= m[\bar{\mu} \otimes_T \bar{\nu}](z) = m[\overline{\mu \otimes_T \nu}](z) \\ &\le \overline{m[\mu \otimes_T \nu]}(z) = \overline{\mu \odot_T \nu}(z). \end{aligned} \tag{4.33}$$

(ii) and (iii) follow immediately. □

Lemma 4.20. *If $(G,\cdot)$ and $(G',\cdot)$ are groups and $f : G \to G'$ is a group homomorphism, then*

$$f(x \odot_T a^{-1} \odot_T \mu) = f(x) \odot_T f(a)^{-1} \odot_T f(\mu). \tag{4.34}$$

Proof. This follows the same way as in [2, Lemma 2.15]; see also [7]. □

Theorem 4.21. *Let $(G,\cdot,^{-},t(\Omega))$ and $(H,\cdot,^{-},t(\Xi))$ be T-neighborhood groups with bases Ω and Ξ in G and H, respectively. If $f : G \to H$ is a group homomorphism, then f is continuous if and only if it is continuous at one point.*

Proof. Let $f : G \to H$ be continuous at the point $a \in G$. We need to show that f is continuous at each $x \in G$. Let $\xi \in \Xi(f(x))$ and $\epsilon > 0$. Then we have $f(x)^{-1} \odot_T \xi \in \Xi(e)$ and hence $f(a) \odot_T f(x)^{-1} \odot_T \xi \in \Xi(f(a))$. Then by Theorem 2.4, the continuity at one point $a \in G$ yields that $f^{-1}(f(a) \odot_T f(x)^{-1} \odot_T \xi) \in \Omega(a)^{\sim}$, which in turn implies that there exists a $\sigma = \sigma_\epsilon \in \Omega(a)$ such that

$$\sigma - \epsilon \leq f^{-1}(f(a) \odot_T f(x)^{-1} \odot_T \xi). \tag{4.35}$$

Now we have $\mu := x \odot_T a^{-1} \odot_T \sigma \in \Omega(x)$.

Thus, one obtains

$$\begin{aligned} \mu(z) - \epsilon &= x \odot_T a^{-1} \odot_T \sigma(z) - \epsilon = \sigma(ax^{-1}z) - \epsilon \\ &\leq f(a) \odot_T f(x)^{-1} \odot_T \xi(f(ax^{-1}z)) \\ &= f(ax^{-1}) \odot_T \xi(f(ax^{-1})f(z)) \\ &= \xi(f(z)) = f^{-1}(\xi)(z), \end{aligned} \tag{4.36}$$

that is, $\mu - \epsilon \leq f^{-1}(\xi)$, which implies that $f^{-1}(\xi) \in \Omega(x)^{\sim}$. □

Now we present some results on T-neighborhood groups in conjunction with Mordeson's TI-group.

5. Application of T-neighborhood groups in TI-groups

Definition 5.1 [7, 20]. An I-subset μ of G is called a TI-subgroup of G if it fulfills the following conditions:

(G1) $\mu(e) = 1$;
(G2) $\mu(x^{-1}) \geq \mu(x)$, for all $x \in G$;
(G3) $\mu(xy) \geq \mu(x)T\mu(y)$, for all $x, y \in G$.

We denote the set of all TI-subgroups of G by $TI(G)$ and that of the set of all normal TI-subgroups by NTI-subgroups, while by NI-subgroup we mean normal I-subgroups, the one introduced by Rosenfeld [20] in which case $T = \min$ is used.

Proposition 5.2. *Let $(G, \cdot, ^-, t(\Omega))$ be a T-neighborhood group and $\mu \in TI(G)$. Then $\bar{\mu}^{t(\Omega)} \in TI(G)$.*

Proof. In view of [7, Theorem 5.1.4], it suffices to prove that

$$\bar{\mu}^{t(\Omega)} \odot_T (\bar{\nu}^{t(\Omega)})^{-1} \leq \bar{\mu}^{t(\Omega)}. \tag{5.1}$$

Since $\mu \in TI(G)$, we have $\mu \odot_T \mu^{-1} \leq \mu$. Then by an easy calculation, one obtains

$$\mu \odot_T \mu^{-1} = h(\mu \otimes_T \mu^{-1}), \tag{5.2}$$

which in conjunction with Theorem 3.1, yields the following:

$$(\bar{\mu}^{t(\Omega)}) \odot_T (\bar{\mu}^{t(\Omega)})^{-1} = h\Big[(\bar{\mu}^{t(\Omega)}) \otimes_T (\bar{\mu}^{t(\Omega)})^{-1}\Big] \leq \bar{\mu}^{t(\Omega)}, \tag{5.3}$$

which proves that $\bar{\mu}^{t(\Omega)} \in TI(G)$. □

Proposition 5.3. *If $(G,\cdot,^{-},t(\Omega))$ is a T-neighborhood group and $\mu\in NTI(G)$, then $\bar{\mu}^{t(\Omega)}\in NTI(G)$.*

Proof. Since $\mu\in NI(G)$, we have $I_x(\mu)=x\odot_T\mu\odot_T x^{-1}=\mu$, where $I_x:G\to G$, $z\mapsto xzx^{-1}$ is an inner automorphism. But then using Proposition 4.19(iii), we obtain

$$x\odot_T\bar{\mu}\odot_T x^{-1}=\overline{x\odot_T\mu\odot x^{-1}}=\bar{\mu}. \tag{5.4}$$

Hence the result follows from [7, Theorem 5.2.1(N5)]. □

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References

[1] T. M. G. Ahsanullah, *Some results on fuzzy neighborhood spaces*, Pure Math. Manuscript **4** (1985), 97–106.

[2] ______, *On fuzzy neighborhood groups*, J. Math. Anal. Appl. **130** (1988), no. 1, 237–251.

[3] ______, *On fuzzy neighborhood modules and algebra*, Fuzzy Sets and Systems **35** (1990), no. 2, 219–229.

[4] ______, *Minimal locally bounded fuzzy neighborhood rings and commutative division rings*, Fuzzy Sets and Systems **87** (1997), no. 1, 87–97.

[5] T. M. G. Ahsanullah and S. Ganguly, *On fuzzy neighborhood rings*, Fuzzy Sets and Systems **34** (1990), no. 2, 255–262.

[6] M. H. Burton and J. Gutiérrez García, *Extensions of uniform space notions*, Mathematics of Fuzzy Sets. Logic, Topology, and Measure Theory (U. Höhle and S. E. Rodabaugh, eds.), Handb. Fuzzy Sets Ser., vol. 3, Kluwer Academic Publishers, Massachusetts, 1999, pp. 581–606.

[7] S.-C. Cheng, J. N. Mordeson, and Yu. Yandong, *Elements of L-Algebra*, Lecture Notes in Fuzzy Mathematics and Computer Science, Creighton University, Nebraska, 1994.

[8] D. H. Foster, *Fuzzy topological groups*, J. Math. Anal. Appl. **67** (1979), no. 2, 549–564.

[9] M. J. Frank, *Probabilistic topological spaces*, J. Math. Anal. Appl. **34** (1971), 67–81.

[10] K. A. Hashem and N. N. Morsi, *Fuzzy T-neighbourhood spaces. I. T-proximities*, Fuzzy Sets and Systems **127** (2002), no. 3, 247–264.

[11] ______, *Fuzzy T-neighbourhood spaces. II. T-neighbourhood systems*, Fuzzy Sets and Systems **127** (2002), no. 3, 265–280.

[12] ______, *Fuzzy T-neighbourhood spaces. III. T-separation axioms*, Fuzzy Sets and Systems **133** (2003), no. 3, 333–361.

[13] U. Höhle, *Probabilistic uniformization of fuzzy topologies*, Fuzzy Sets and Systems **1** (1978), no. 4, 311–332.

[14] A. K. Katsaras, *Linear fuzzy neighborhood spaces*, Fuzzy Sets and Systems **16** (1985), no. 1, 25–40.

[15] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl. **56** (1976), no. 3, 621–633.

[16] ______, *Fuzzy uniform spaces*, J. Math. Anal. Appl. **82** (1981), no. 2, 370–385.

[17] ______, *Fuzzy neighborhood spaces*, Fuzzy Sets and Systems **7** (1982), no. 2, 165–189.

[18] A. S. Mashhour and N. N. Morsi, *Fuzzy metric neighbourhood spaces*, Fuzzy Sets and Systems **45** (1992), no. 3, 367–388.

[19] K. Menger, *Statistical metrics*, Proc. Natl. Acad. Sci. USA **28** (1942), 535–537.

[20] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.

[21] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing, New York, 1983.

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