ON BIRATIONAL MONOMIAL TRANSFORMATIONS OF PLANE

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We study birational monomial transformations of the form $\varphi(x : y : z) = (\varepsilon_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} : \varepsilon_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} : x^{\alpha_3} y^{\beta_3} z^{\gamma_3})$, where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. These transformations form a group. We describe this group in terms of generators and relations and, for every such transformation φ , we prove a formula, which represents the transformation φ as a product of generators of the group. To prove this formula, we use birationally equivalent polynomials $Ax + By + C$ and $Ax^p + By^q + Cx^r y^s$. If φ is the transformation which carries one polynomial onto another, then the integral powers of generators in the product, which represents the transformation φ , can be calculated by the expansion of p/q in the continued fraction.

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1. Introduction. Birational monomial transformations of the projective plane have already found a lot of applications. For example, such transformations are actively used for construction of real algebraic curves and surfaces (see, e.g., [1, 4, 5, 6, 8, 9]). We think that formula (3.6) will be helpful for description of construction of algebraic objects.

In Section 2 we give a little exposition on projective polynomials in three variables. In Section 3 we describe the birational monomial group in terms of its generators and relations and give the statement of a theorem of decomposition of birational monomial transformations. In Section 4 we give the proof of the theorem.

2. Preliminaries. A nonzero homogeneous polynomial of degree n in three variables, x, y, z , is the expression

$$f(x, y, z) = \sum_{\omega_1 + \omega_2 + \omega_3 = n} f_{\omega} x^{\omega_1} y^{\omega_2} z^{\omega_3}, \quad \omega = (\omega_1, \omega_2, \omega_3). \quad (2.1)$$

The convex hull of the set $\{(\omega_1, \omega_2) \in \mathbb{R}^2 \mid f_{\omega} \neq 0\}$ is called the *Newton polygon* of the polynomial $f(x, y, z)$ and is denoted as $N(f)$. The plane with coordinates (ω_1, ω_2) is called the *plane of Newton's polygons*.

Every polynomial $f(x, y, 1)$ can be represented in the form $f(x, y, 1) = x^i y^j \hat{f}(x, y, 1)$, where i and j are nonnegative integers, and the polynomial $\hat{f}(x, y, 1)$ has no factors x and y . If φ is a transformation, then clearly $(\hat{f} \circ \varphi)^{\sim} = (\hat{f} \circ \varphi)^{\sim}$. It is also clear that the Newton polygon $N(\hat{f}^{\sim})$ can be obtained from the Newton polygon $N(\hat{f})$ by translation in the plane of Newton's polygons by the vector $(-i, -j)$.

3. The birational monomial group. Let $(x : y : z)$ be homogeneous point coordinates in the projective plane $\mathbb{K}P^2$ over a field $\mathbb{K}$ and let (x, y) be affine coordinates in the affine chart $\mathbb{K}^2 = \mathbb{K}P^2 \setminus \{z = 0\}$. A projective transformation φ is defined by the

formula $\varphi(x : y : z) = (\varphi_1(x, y, z) : \varphi_2(x, y, z) : \varphi_3(x, y, z))$, where $\varphi_1, \varphi_2, \varphi_3$ are homogeneous polynomials of the same degree, assumed to have no common factors. For the transformation $\varphi(x : y : z)$, we define its natural restriction $\varphi(x, y)$ to the affine chart $\mathbb{K}^2 = \mathbb{K}P^2 \setminus \{z = 0\}$ by the formula $\varphi(x, y) = (\varphi_1(x, y, 1)/\varphi_3(x, y, 1), \varphi_2(x, y, 1)/\varphi_3(x, y, 1))$.

Let $\text{id} : \mathbb{K}P^2 \rightarrow \mathbb{K}P^2$ be the identity map. If φ is a birational transformation, then we denote as usual

$$\varphi^0 = \text{id}, \quad \underbrace{\varphi \circ \dots \circ \varphi}_{n \text{ times}} = \varphi^n, \quad \underbrace{\varphi^{-1} \circ \dots \circ \varphi^{-1}}_{n \text{ times}} = \varphi^{-n}. \quad (3.1)$$

Let $r_1, r_2, r_3 : \mathbb{K}P^2 \rightarrow \mathbb{K}P^2$ be maps defined by formulas $r_1(x : y : z) = ((-x) : y : z)$, $r_2(x : y : z) = (x : (-y) : z)$, and $r_3(x : y : z) = (x : y : (-z))$. The set of maps $R = \{\text{id}, r_1, r_2, r_1 \circ r_2\}$ with the operation of composition of the maps, with generators r_1 and r_2 , and with relations

$$r_1^2 = r_2^2 = \text{id}, \quad r_1 \circ r_2 = r_2 \circ r_1, \quad (3.2)$$

is a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note that $r_3 = r_1 \circ r_2$.

Let $s_1, s_2, s_3 : \mathbb{K}P^2 \rightarrow \mathbb{K}P^2$ be maps defined by formulas $s_1(x : y : z) = (x : z : y)$, $s_2(x : y : z) = (z : y : x)$, and $s_3(x : y : z) = (y : x : z)$. The set of maps $S = \{\text{id}, s_1 \circ s_2, s_2 \circ s_1, s_1, s_2, s_1 \circ s_2 \circ s_1\}$ with the operation of composition of the maps, with generators s_1 and s_2 , and with relations

$$s_1^2 = s_2^2 = \text{id}, \quad s_1 \circ s_2 \circ s_1 = s_2 \circ s_1 \circ s_2, \quad (3.3)$$

is a group isomorphic to the symmetric group S_3 . Note that $s_3 = s_1 \circ s_2 \circ s_1$.

Let hy be the birational transformation defined by the formula $hy(x : y : z) = (x^2 : yz : xz)$, whose inverse transformation is $hy^{-1}(x : y : z) = (xz : xy : z^2)$. Due to Newton, the transformation hy is called a *hyperbolism*. The set $H = \{\dots, hy^{-2}, hy^{-1}, \text{id}, hy, hy^2, \dots\}$ of integral powers of hy is a free group isomorphic to $\mathbb{Z}$.

Let $G = R * S * H$ be the free product of groups R, S , and H . This means that the set of generators of G is the union of the generators of R, S , and H , and the set of relations of G is the union of the relations of R, S , and H .

DEFINITION 3.1. The factor group $G/\mathcal{R}$ with generators r_1, r_2, s_1, s_2, hy , where $\mathcal{R}$ is the system of relations

$$\mathcal{R} : \begin{cases} r_1 \circ s_1 = s_1 \circ r_1, \\ r_2 \circ s_1 = s_1 \circ r_2 \circ r_1, \\ r_1 \circ s_2 = s_2 \circ r_1 \circ r_2, \\ r_2 \circ s_2 = s_2 \circ r_2, \\ r_1 \circ hy = hy \circ r_1 \circ r_2, \\ r_2 \circ hy = hy \circ r_2, \\ s_1 \circ hy = hy \circ s_1 \circ s_2 \circ hy \circ s_1, \\ hy \circ s_2 \circ hy = s_2, \end{cases} \quad (3.4)$$

is called the *group of birational monomial transformations* of $\mathbb{K}P^2$ and denoted by $T(\mathbb{K}P^2)$.

The group of birational monomial transformations $T(\mathbb{K}P^2)$ is a subgroup of the Cremona group $\text{Cr}(\mathbb{K}P^2)$ (see [2, 3]).

Below in this paper, the word “transformation” without an adjective always means a “birational monomial transformation.”

Every transformation $\varphi \in T(\mathbb{K}P^2)$ can be represented as a composition $\varphi_1 \circ \dots \circ \varphi_s$, where each of $\varphi_1, \dots, \varphi_s$ is a positive integral power of one of the generators of the group $T(\mathbb{K}P^2)$, because $r_1^{-1} = r_1$, $r_2^{-1} = r_2$, $s_1^{-1} = s_1$, $s_2^{-1} = s_2$, and $hy^{-n} = s_2 \circ hy^n \circ s_2$. Every transformation φ can be represented in the form

$$\varphi(x : y : z) = (\varepsilon_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} : \varepsilon_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} : x^{\alpha_3} y^{\beta_3} z^{\gamma_3}), \quad (3.5)$$

where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$; α_i, β_i , and γ_i are nonnegative integers; and the monomials $x^{\alpha_1} y^{\beta_1} z^{\gamma_1}$, $x^{\alpha_2} y^{\beta_2} z^{\gamma_2}$, $x^{\alpha_3} y^{\beta_3} z^{\gamma_3}$ have no common factors. We stress this convention, for example, $(hy \circ hy)(x : y : z) = (x^4 : xy^2z^2 : x^3z) = (x^3 : yz^2 : x^2z)$, and accept only the last form. It means that one or two of $\alpha_1, \alpha_2, \alpha_3$, one or two of $\beta_1, \beta_2, \beta_3$, one or two of $\gamma_1, \gamma_2, \gamma_3$ are equal to 0, and $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = \alpha_3 + \beta_3 + \gamma_3$. The integer $\alpha_1 + \beta_1 + \gamma_1$ is a degree of the transformation φ . For example, the degree of the transformations r_1, r_2, s_1, s_2 equals 1, and the degree of hy^n equals $|n| + 1$, where $n \in \mathbb{Z}$.

Denote the element $s_3 \circ hy \circ s_3 \in T(\mathbb{K}P^2)$ as hx . Its inverse is $hx^{-1} = s_2 \circ s_1 \circ hy \circ s_1 \circ s_2$. In homogeneous coordinates it is defined by formulae $hx(x : y : z) = (xz : y^2 : yz)$ and $hx^{-1}(x : y : z) = (xy : yz : z^2)$.

In the following theorem and below, the phrase “a polynomial $f(x, y, 1)$ subjected to the transformation φ is carried onto the polynomial $l(x, y, 1)$ ” means that $l(x, y, 1) = [(f \circ \varphi^{-1})(x, y, z)|_{z=1}]^\wedge$.

THEOREM 3.2. *Let p and q be mutually prime natural integers, $0 < q < p$. Let r and s be integers which satisfy the following conditions: (1) $0 < r < p$, $0 \leq s < q$, (2) $r/p + s/q < 1$, and (3) $r \equiv -q^{\phi(p)-1} \pmod{p}$ and $s \equiv -p^{\phi(q)-1} \pmod{q}$, where $\phi(m)$ is the Euler function. Then every polynomial $f(x, y, 1) = Ax^p + By^q + Cx^r y^s$, where at least two of A, B, C are not zero, subjected to the transformation*

$$\varphi = \left(hx^{(1-(-1)^{k+1})/2} \circ hy^{(1-(-1)^k)/2} \right)^{a_k} \circ \dots \circ hx^{a_4} \circ hy^{a_3} \circ hx^{a_2} \circ hy^{a_1} \quad (3.6)$$

is carried onto the polynomial $l(x, y, 1) = Ax + By + C$, where the integers $a_1, a_2, \dots, a_k$ are provided by expansion of p/q in the continued fraction with adjusted last denominator

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{k-1} + \frac{1}{a_k + 1}}}}; \quad (3.7)$$

in other words, $l = (f \circ \varphi^{-1})^\wedge$.

COROLLARY 3.3. (1) If the polynomial $f(x, y, 1)$ subjected to a transformation ψ is carried onto the polynomial $\varepsilon_1 Ax + \varepsilon_2 By + \varepsilon_3 C$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$, then $\psi = r_1^{(1/2)(1-\varepsilon_1)} \circ r_2^{(1/2)(1-\varepsilon_2)} \circ (r_1 \circ r_2)^{(1/2)(1-\varepsilon_3)} \circ \varphi$.

(2) If the polynomial $f(x, y, 1)$ subjected to a transformation ψ is carried onto the polynomial $A + Bx + Cy$, $Ay + B + Cx$, $Ax + B + Cy$, $A + By + Cx$, or $Ay + Bx + C$, then $\psi = s_1 \circ s_2 \circ \varphi$, $\psi = s_2 \circ s_1 \circ \varphi$, $\psi = s_1 \circ \varphi$, $\psi = s_2 \circ \varphi$, or $\psi = s_1 \circ s_2 \circ s_1 \circ \varphi$, respectively.

(3) If condition (2) of [Theorem 3.2](#) is changed to condition (2'), $r/p + s/q > 1$, and other conditions and notations are kept, and if the polynomial $f(x, y, 1)$ subjected to a transformation ψ is carried onto the polynomial $Ax + By + C$, then $\psi = \varphi \circ \text{tr} = \text{tr} \circ \varphi$, where $\text{tr} = s_1 \circ h y^{-1} \circ s_1 \circ s_2 \circ s_1 \circ h y$ is well-known standard (triangular) quadratic transformation $\text{tr}(x : y : z) = (yz : xz : xy)$.

REMARK 3.4. There is only one more possible case: when $p = q = 1$, which does not satisfy the theorem. In this case either $r = s = 0$, and the polynomial $Ax + By + C$ is carried onto itself by the identity transformation: $\varphi = \text{id}$, or $r = s = 1$, and the polynomial $Ax + By + Cxy$ is carried onto the polynomial $Ax + By + C$ by the transformation $\varphi = s_3 \circ \text{tr} = s_3 \circ s_1 \circ h y^{-1} \circ s_1 \circ s_2 \circ s_1 \circ h y$.

4. Proof of the theorem. A birational monomial transformation ψ maps a polynomial f onto a polynomial $(f \circ \psi^{-1})$. We find the connection between $N(f \circ \psi^{-1})$ and $N(f)$.

Every transformation ψ^{-1} written in the form (3.5) induces a generic linear mapping $A(\psi) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of the plane of Newton's polygons, which can be defined on any monomial. Namely, if g is a monomial, say $g(x, y, z) = x^{\omega_1} y^{\omega_2} z^{\omega_3}$, then

$$(g \circ \psi^{-1})(x, y, z) = x^{\alpha_1 \omega_1 + \alpha_2 \omega_2 + \alpha_3 \omega_3} y^{\beta_1 \omega_1 + \beta_2 \omega_2 + \beta_3 \omega_3} z^{\gamma_1 \omega_1 + \gamma_2 \omega_2 + \gamma_3 \omega_3}, \quad (4.1)$$

thus, the linear mapping $A(\psi)$ is defined by the matrix

$$A_\psi = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}. \quad (4.2)$$

And thus, $N(f \circ \psi^{-1}) = A_\psi(N(f))$ for every polynomial f .

Remark that the generators of the birational monomial group have matrices

$$A_{\text{id}} = A_{r_1} = A_{r_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.3)$$

$$A_{s_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{s_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_{h y} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The set of matrices $A(T(\mathbb{K}P^2)) = \{A_\varphi \mid \varphi \in T(\mathbb{K}P^2)\}$ is a subset in the linear group $GL(3, \mathbb{R})$ of 3×3 -matrices. The operation $\diamond$ defined by the formula $A_\varphi \diamond A_\psi = A_{\varphi \circ \psi}$ converts the set $A(T(\mathbb{K}P^2))$ into a group, which is natural ($\varphi \mapsto A_\varphi$) homomorphic image of $T(\mathbb{K}P^2)$.

It is clear that every birational monomial transformation φ induces one-to-one correspondence between monomials of the polynomials f and $(f \circ \varphi^{-1})^\wedge$. Thus, to represent a transformation φ as a composition of generators of birational monomial group, it is enough to study the action of the transformation φ on a polynomial f whose Newton's polygon $N(f)$ is a triangle with the area $1/2$.

We consider a polynomial $f(x, y, 1) = Ax^p + By^q + Cx^r y^s$, with $ABC \neq 0$, where p and q are mutually prime integers, $0 < q < p$, and r and s are integers, which satisfy the following conditions: (1) $0 < r < p$, $0 \leq s < q$, (2) $r/p + s/q < 1$, and (3) $r \equiv -q^{\phi(p)-1} \pmod{p}$ and $s \equiv -p^{\phi(q)-1} \pmod{q}$, where $\phi(m)$ is the Euler function. The Newton polygon $N(f)$ is the triangle with integer vertices $(p, 0)$, $(0, q)$, (r, s) which has no other integer points belonging to its interior and boundary but its three vertices. According to the Pick theorem [7], the area of such a triangle equals $1/2$. The genus of any curve $f(x, y, 1) = Ax^p + By^q + Cx^r y^s = 0$ with such properties is zero and thus, all such curves are birationally equivalent.

Note that $hy^{-a}(x : y : 1) = (x : x^a y : 1)$ and $hx^{-a}(x : y : 1) = (xy^a : y : 1)$. We evaluate $(f \circ \varphi^{-1})(x, y, 1)$ as follows.

The first step.

$$\begin{aligned} (f \circ hy^{-a_1})(x, y, 1) &= f(x, x^{a_1} y, 1) = x^{a_1 q} (Ax^{b_1} + By^q + Cx^{c+a_1 d-a_1 q} y^d) \\ &= x^{u_1} y^{v_1} (Ax^{b_1} + By^q + Cx^{c_1} y^{d_1}), \end{aligned} \quad (4.4)$$

where $u_1 = a_1 q$, $v_1 = 0$, $c_1 = c + a_1 d - a_1 q$, and $d_1 = d$.

The second step.

$$\begin{aligned} (f \circ hy^{-a_1} \circ hx^{-a_2})(x, y, 1) &= (f \circ hy^{-a_1})(xy^{a_2}, y, 1) \\ &= x^{u_1} y^{a_2 u_1 + v_1 + a_2 b_1} (Ax^{b_1} + By^{b_2} + Cx^{c_1} y^{a_2 c_1 + d_1 - a_2 b_1}) \\ &= x^{u_2} y^{v_2} (Ax^{b_1} + By^{b_2} + Cx^{c_2} y^{d_2}), \end{aligned} \quad (4.5)$$

where $u_2 = u_1$, $v_2 = a_2 u_1 + v_1 + a_2 b_1$, $c_2 = c_1$, and $d_2 = a_2 c_1 + d_1 - a_2 b_1$.

The third step.

$$\begin{aligned} (f \circ hy^{-a_1} \circ hx^{-a_2} \circ hy^{-a_3})(x, y, 1) &= (f \circ hy^{-a_1} \circ hx^{-a_2})(x, x^{a_3} y, 1) \\ &= x^{u_3} y^{v_3} (Ax^{b_3} + By^{b_2} + Cx^{c_3} y^{d_3}), \end{aligned} \quad (4.6)$$

where $u_3 = u_2 + a_2 v_2 + a_3 b_2$, $v_3 = v_2$, $c_3 = c_2 + a_3 d_2 - a_3 b_2$, and $d_3 = d_2$.

The fourth step.

$$\begin{aligned} (f \circ hy^{-a_1} \circ hx^{-a_2} \circ hy^{-a_3} \circ hx^{-a_4})(x, y, 1) &= (f \circ hy^{-a_1} \circ hx^{-a_2} \circ hy^{-a_3})(xy^{a_4}, y, 1) \\ &= x^{u_1} y^{a_2 u_1 + v_1 + a_2 b_1} (Ax^{b_1} + By^{b_2} + Cx^{c_1} y^{a_2 c_1 + d_1 - a_2 b_1}) \\ &= x^{u_2} y^{v_2} (Ax^{b_1} + By^{b_2} + Cx^{c_2} y^{d_2}), \end{aligned} \quad (4.7)$$

where $u_2 = u_1$, $v_2 = a_2 u_1 + v_1 + a_2 b_1$, $c_2 = c_1$, and $d_2 = a_2 c_1 + d_1 - a_2 b_1$. We then proceed until the $(k-1)$ th step.

The $(k-1)$ th step. We have two cases.

The first case: k is even.

$$\begin{aligned} & (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \dots \circ hx^{-a_{k-2}} \circ hy^{-a_{k-1}})(x, y, 1) \\ &= (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \dots \circ hx^{-a_{k-2}})(x, x^{a_{k-1}}y, 1) \\ &= x^{u_{k-1}}y^{v_{k-1}}(Ax + By^{b_{k-2}} + Cx^{c_{k-1}}y^{d_{k-1}}), \end{aligned} \quad (4.8)$$

where $u_{k-1} = u_{k-2} + a_{k-1}v_{k-2} + a_{k-1}b_{k-2}$, $v_{k-1} = v_{k-2}$, $c_{k-1} = c_{k-2} + a_{k-1}d_{k-2} - a_{k-1}b_{k-2}$, and $d_{k-1} = d_{k-2}$.

The second case: k is odd.

$$\begin{aligned} & (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \dots \circ hy^{-a_{k-2}} \circ hx^{-a_{k-1}})(x, y, 1) \\ &= (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \dots \circ hy^{-a_{k-2}})(xy^{a_{k-1}}, y, 1) \\ &= x^{u_{k-1}}y^{v_{k-1}}(Ax^{b_{k-2}} + By + Cx^{c_{k-1}}y^{d_{k-1}}), \end{aligned} \quad (4.9)$$

where $u_{k-1} = u_{k-2}$, $v_{k-1} = a_{k-1}u_{k-2} + v_{k-2} + a_{k-1}b_{k-2}$, $c_{k-1} = c_{k-2}$, and $d_{k-1} = a_{k-1}c_{k-2} + d_{k-2} - a_{k-1}b_{k-2}$.

The k th step. We have two cases.

The first case: k is even.

$$\begin{aligned} & (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \dots \circ hy^{-a_{k-1}} \circ hx^{-a_k})(x, y, 1) \\ &= (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \dots \circ hy^{-a_{k-1}})(xy^{a_k}, y, 1) \\ &= x^{u_k}y^{v_k}(Ax + By + Cx^{c_k}y^{d_k}), \end{aligned} \quad (4.10)$$

where $u_k = u_{k-1}$, $v_k = a_k u_{k-1} + v_{k-1} + a_k$, $c_k = c_{k-1}$, and $d_k = a_k c_{k-1} + d_{k-1} - a_k$.

The second case: k is odd.

$$\begin{aligned} & (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \dots \circ hx^{-a_{k-1}} \circ hy^{-a_k})(x, y, 1) \\ &= (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \dots \circ hx^{-a_{k-2}})(x, x^{a_k}y, 1) \\ &= x^{u_k}y^{v_k}(Ax + By + Cx^{c_k}y^{d_k}), \end{aligned} \quad (4.11)$$

where $u_k = u_{k-1} + a_k v_{k-1} + a_k$, $v_k = v_{k-1}$, $c_k = c_{k-1} + a_k d_{k-1} - a_k$, and $d_k = d_{k-1}$.

This calculation shows that the integers $a_1, a_2, a_3, \dots, a_k$ satisfy the Euclidean algorithm with adjusted last row

$$\begin{aligned} p &= a_1 q + b_1, \\ q &= a_2 b_1 + b_2, \\ b_1 &= a_3 b_2 + b_3, \\ &\vdots \\ b_{k-4} &= a_{k-2} b_{k-3} + b_{k-2}, \\ b_{k-3} &= a_{k-1} b_{k-2} + 1, \\ b_{k-2} &= a_k + 1, \end{aligned} \quad (4.12)$$

which provides the desired continued fraction.

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