UNIT 1-STABLE RANGE FOR IDEALS

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We investigate necessary and sufficient conditions under which a ring satisfies unit 1-stable range for an ideal. As an application, we prove that R satisfies unit 1-stable range for I if and only if $\text{QM}_2(R)$ satisfies unit 1-stable range for $\text{QM}_2(I)$.

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Let R be a ring with identity 1. We say that R satisfies unit 1-stable range in case $ax + b = 1$ with $a, x, b \in R$ implying that $a + bu \in U(R)$. Many authors studied unit 1-stable range such as those of [1, 2, 3, 4, 5, 6]. Following the authors, a ring R satisfies unit 1-stable range for an ideal I provided that $ax + b = 1$ with $a \in I$, $x, b \in R$ implying that $x + ub \in U(R)$ for some unit $u \in U(R)$. Let $R = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $I = 0 \oplus \mathbb{Z}/3\mathbb{Z}$. Then R satisfies unit 1-stable range for I , while in fact R does not satisfy unit 1-stable range. Thus the concept of unit 1-stable range for an ideal is a nontrivial generalization of that of rings satisfying such stable range condition. In this note, we investigate necessary and sufficient conditions under which a ring R satisfies unit 1-stable range for an ideal. It is shown that R satisfies unit 1-stable range for I if and only if $\text{QM}_2(R)$ satisfies unit 1-stable range for $\text{QM}_2(I)$.

Throughout, all rings are associative with identity. $M_n(R)$ denotes the ring of $n \times n$ matrices over R , $\text{GL}_n(R)$ denotes the n -dimensional general linear group of R , $\text{TM}_n(R)$ denotes the ring of all $n \times n$ lower triangular matrices over R , and $\text{TM}_n(I)$ denotes the ideal of all $n \times n$ lower triangular matrices over I . Clearly, $\text{TM}_n(I)$ is an ideal of $\text{TM}_n(R)$. We begin with the following.

LEMMA 1. *Let I be an ideal of a ring R . Then the following are equivalent:*

- (1) *R satisfies unit 1-stable range for I ;*
- (2) *for any $x \in R$, $y \in I$, there exists $a \in R$ such that $1 + xa, y + a \in U(R)$.*

PROOF. (1) $\Rightarrow$ (2). Given any $x \in R$, $y \in I$, we have $yx + (1 - yx) = 1$; hence, there exists $u \in U(R)$ such that $x + u(1 - yx) \in U(R)$, so $1 + (u^{-1} - y)x \in U(R)$. Set $a = u^{-1} - y$. Then $1 + ax, y + a \in U(R)$. Clearly, $1 + ax \in U(R)$ if and only if $1 + xa \in U(R)$. Therefore, $1 + xa, y + a \in U(R)$.

(2) $\Rightarrow$ (1). Suppose that $ax + b = 1$ with $a \in I$, $x, b \in R$. Then we have $r \in R$ such that $a + r, 1 + (-x)r \in U(R)$. Set $u = a + r$. We get $1 - x(a - u) = 1 + (-x)r \in U(R)$; hence, $1 - (a - u)x \in U(R)$. This infers that $b + ux \in U(R)$, and so $x + u^{-1}b \in U(R)$, as asserted. $\square$

In [3, Theorem 2], the authors proved that if R satisfies unit 1-stable range for I , then so does $M_n(R)$ for $M_n(I)$. Now we give a simple proof of this fact.

THEOREM 2. *If R satisfies unit 1-stable range for I , then so does $M_n(R)$ for $M_n(I)$.*

PROOF. Assume that $\begin{pmatrix} a_1 & N_1 \\ M_1 & B_1 \end{pmatrix} \in M_n(I)$, $\begin{pmatrix} a_2 & N_2 \\ M_2 & B_2 \end{pmatrix} \in M_n(R)$ with $a_1 \in I$, $a_2 \in R$, $B_1 \in M_{n-1}(I)$, $B_2 \in M_{n-1}(R)$, $N_1 \in M_{1 \times (n-1)}(I)$, $N_2 \in M_{1 \times (n-1)}(R)$, $M_1 \in M_{(n-1) \times 1}(I)$, and $M_2 \in M_{(n-1) \times 1}(R)$. Using Lemma 1, we can choose $a \in R$ such that $a_1 + a = u_1 \in U(R)$, $1 + a_2a = v_1 \in U(R)$. Assume that $M_{n-1}(R)$ satisfies unit 1-stable range for $M_{n-1}(I)$. Clearly, $B_1 - M_1 u_1^{-1} N_1 \in M_{n-1}(I)$. So we have $B \in M_{n-1}(R)$ such that $(B_1 - M_1 u_1^{-1} N_1) + B = U_2 \in \text{GL}_{n-1}(R)$ and $I_{n-1} + (-M_2 v_1^{-1} N_2 + B_2)B = V_2 \in \text{GL}_{n-1}(R)$. Hence,

$$\begin{pmatrix} a_1 & N_1 \\ M_1 & B_1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} u_1 & N_1 \\ M_1 & B_1 + B \end{pmatrix}, \quad (1)$$

$$I_n + \begin{pmatrix} a_2 & N_2 \\ M_2 & B_2 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} v_1 & N_2 B \\ M_2 a & I_{n-1} + B_2 B \end{pmatrix}.$$

We check that

$$\begin{pmatrix} u_1 & N_1 \\ M_1 & B_1 + B \end{pmatrix}^{-1} = \begin{pmatrix} u_1^{-1} + u_1^{-1} N_1 U_2^{-1} M_1 u_1^{-1} & -u_1^{-1} N_1 U_2^{-1} \\ -U_2^{-1} M_1 u_1^{-1} & U_2^{-1} \end{pmatrix}, \quad (2)$$

$$\begin{pmatrix} v_1 & N_2 B \\ M_2 a & I_{n-1} + B_2 B \end{pmatrix}^{-1} = \begin{pmatrix} v_1^{-1} + v_1^{-1} N_2 B V_2^{-1} M_2 a v_1^{-1} & -v_1^{-1} N_2 B V_2^{-1} \\ -V_2^{-1} M_2 a v_1^{-1} & V_2^{-1} \end{pmatrix}.$$

By induction and Lemma 1, $M_n(R)$ satisfies unit 1-stable range for $M_n(I)$. □

COROLLARY 3. *If R satisfies unit 1-stable range, then so does $M_n(R)$ for all $n \in \mathbb{N}$.*

PROOF. By choosing $I = R$ in Theorem 2, we complete the proof. □

THEOREM 4. *Let I be an ideal of a ring R . Then the following are equivalent:*

- (1) R satisfies unit 1-stable range for I ;
- (2) $\text{TM}_n(R)$ satisfies unit 1-stable range for $\text{TM}_n(I)$.

PROOF. It suffices to show that the result holds for $n = 2$. Suppose that R satisfies unit 1-stable range for I . Assume that $\begin{pmatrix} a_1 & 0 \\ m_1 & b_1 \end{pmatrix} \in \text{TM}_2(I)$, $\begin{pmatrix} a_2 & 0 \\ m_2 & b_2 \end{pmatrix} \in \text{TM}_2(R)$ with $a_1, b_1, m_1 \in I$, $a_2, b_2, m_2 \in R$. Using Lemma 1, we can choose $a \in R$ such that $a_1 + a = u_1 \in U(R)$, $1 + a_2a = v_1 \in U(R)$ and we have $b \in R$ such that $b_1 + b = u_2 \in U(R)$ and $1 + b_2b = v_2 \in U(R)$. Hence,

$$\begin{pmatrix} a_1 & 0 \\ m_1 & b_1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} u_1 & 0 \\ m_1 & b_1 + b \end{pmatrix}, \quad (3)$$

$$I_2 + \begin{pmatrix} a_2 & 0 \\ m_2 & b_2 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} v_1 & 0 \\ m_2 a & I_2 + b_2 b \end{pmatrix}.$$

Analogously to [Theorem 2](#), we have

$$\begin{pmatrix} u_1 & 0 \\ m_1 & b_1 + b \end{pmatrix}, \begin{pmatrix} v_1 & 0 \\ m_2 a & I_2 + b_2 b \end{pmatrix} \in \text{GL}_2(\text{TM}_2(R)). \quad (4)$$

By [Lemma 1](#) again, $\text{TM}_2(I)$ is unit 1-stable.

We now establish the converse. Given any $x \in R$, $y \in I$, we have $\text{diag}(x, 1) \in \text{TM}_2(R)$ and $\text{diag}(y, 0) \in \text{TM}_2(I)$. So there exists $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \text{TM}_2(R)$ such that $\text{diag}(1, 1) + \text{diag}(x, 1) + \text{diag}(y, 0) + \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \text{GL}_2(\text{TM}_2(R))$. Therefore, $1 + xa, y + a \in U(R)$, as required. $\square$

Let I be an ideal of a unital complex C^* -algebra R . By [Theorem 4](#) and [\[3, Corollary 6\]](#), we prove that if every element of I is a sum of a unitary and a unit, then every square lower-triangular matrix over I is a sum of two invertible matrices. Let I be an ideal of a ring R . Define $\text{QM}_2(R) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + c = b + d, a, b, c, d \in R \}$ and $\text{QM}_2(I) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + c = b + d, a, b, c, d \in I \}$. It is easy to verify that $\text{QM}_2(I)$ is an ideal of $\text{QM}_2(R)$.

COROLLARY 5. *Let I be an ideal of a ring R . Then the following are equivalent:*

- (1) R satisfies unit 1-stable range for I ;
- (2) $\text{QM}_2(R)$ satisfies unit 1-stable range for $\text{QM}_2(I)$.

PROOF. (1) $\Rightarrow$ (2). We construct a map $\psi: \text{QM}_2(R) \rightarrow \text{TM}_2(R)$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+c & 0 \\ c & d-c \end{pmatrix}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{QM}_2(R)$. For any $\begin{pmatrix} x & 0 \\ z & y \end{pmatrix} \in \text{TM}_2(R)$, we have $\psi(\begin{pmatrix} x-z & x-y-z \\ z & y+z \end{pmatrix}) = \begin{pmatrix} x & 0 \\ z & y \end{pmatrix}$. Thus we prove that ψ is a ring isomorphism. Also $\psi|_{\text{QM}_2(I)}: \text{QM}_2(I) \cong \text{TM}_2(I)$. Thus we complete the proof by [Theorem 4](#). $\square$

As an immediate consequence of [Corollary 5](#), we prove that R satisfies unit 1-stable range if and only if so does $\text{QM}_2(R)$. This result gives a new kind of rings satisfying unit 1-stable range.

THEOREM 6. *Let I be an ideal of a ring R . Then the following are equivalent:*

- (1) R satisfies unit 1-stable range for I ;
- (2) there exists a complete set of idempotents $\{e_1, \dots, e_n\}$ such that all $e_i R e_i$ satisfy unit 1-stable range for $e_i I e_i$.

PROOF. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). One easily checks that

$$\begin{aligned} I &\cong \begin{pmatrix} e_1 I e_1 & \cdots & e_1 I e_n \\ \vdots & \ddots & \vdots \\ e_n I e_1 & \cdots & e_n I e_n \end{pmatrix}, \\ R &\cong \begin{pmatrix} e_1 R e_1 & \cdots & e_1 R e_n \\ \vdots & \ddots & \vdots \\ e_n R e_1 & \cdots & e_n R e_n \end{pmatrix}. \end{aligned} \quad (5)$$

By induction, it suffices to prove that the result holds for $n = 2$. Assume that $\begin{pmatrix} a_1 & n_1 \\ m_1 & b_1 \end{pmatrix} \in \begin{pmatrix} e_1 I e_1 & e_1 I e_2 \\ e_2 I e_1 & e_2 I e_2 \end{pmatrix}$, $\begin{pmatrix} a_2 & n_2 \\ m_2 & b_2 \end{pmatrix} \in \begin{pmatrix} e_1 R e_1 & e_1 R e_2 \\ e_2 R e_1 & e_2 R e_2 \end{pmatrix}$. According to Lemma 1, we can choose $a \in e_1 R e_1$ such that $a_1 + a = u_1 \in U(e_1 R e_1)$, $e_1 + a_2 a = v_1 \in U(e_1 R e_1)$. Also we have $b \in e_2 R e_2$ such that $(b_1 - m_1 u_1^{-1} n_1) + b = u_2 \in U(e_2 R e_2)$ and $e_2 + (-m_2 v_1^{-1} n_2 + b_2) b = v_2 \in U(e_2 R e_2)$. Hence,

$$\begin{pmatrix} a_1 & n_1 \\ m_1 & b_1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} u_1 & n_1 \\ m_1 & b_1 + b \end{pmatrix}, \quad (6)$$

$$\text{diag}(e_1, e_2) + \begin{pmatrix} a_2 & n_2 \\ m_2 & b_2 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} v_1 & n_2 b \\ m_2 a & e_2 + b_2 b \end{pmatrix}.$$

Similarly to Theorem 2, we show that

$$\begin{pmatrix} u_1 & n_1 \\ m_1 & b_1 + b \end{pmatrix}, \begin{pmatrix} v_1 & n_2 b \\ m_2 a & e_2 + b_2 b \end{pmatrix} \in U \begin{pmatrix} e_1 R e_1 & e_1 R e_2 \\ e_2 R e_1 & e_2 R e_2 \end{pmatrix}. \quad (7)$$

It follows by Lemma 1 that $\begin{pmatrix} e_1 I e_1 & e_1 I e_2 \\ e_2 I e_1 & e_2 I e_2 \end{pmatrix}$ satisfies unit 1-stable range for $\begin{pmatrix} e_1 I e_1 & e_1 I e_2 \\ e_2 I e_1 & e_2 I e_2 \end{pmatrix}$, as required. $\square$

COROLLARY 7. *Let R be a ring. Then the following are equivalent:*

- (1) R satisfies unit 1-stable range;
- (2) there exists a complete set of idempotents $\{e_1, \dots, e_n\}$ such that all $e_i R e_i$ satisfy unit 1-stable range.

PROOF. It is an immediate consequence of Theorem 8. $\square$

Recall that I is a regular ideal of R . In case for any $x \in I$, there exists $y \in R$ such that $x = x y x$. We prove that unit 1-stable range condition for an ideal is right and left symmetric for regular ideals.

THEOREM 8. *Let I be a regular ideal of a ring R . Then the following are equivalent:*

- (1) R satisfies unit 1-stable for I ;
- (2) for any $x, y \in I$, there exists some $a \in U(R)$ such that $1 + x a, y + a \in U(R)$;
- (3) $a x + b = 1$ with $a \in I$, $b \in R$ implying that there exists $u \in U(R)$ such that $a + b u \in U(R)$.

PROOF. (1) $\Rightarrow$ (2) is trivial by Lemma 1.

(2) $\Rightarrow$ (3). Suppose that $a x + b = 1$ with $a \in I$, $b \in R$. By the regularity of I , we have a $c \in R$ such that $a = a c a$. Hence, $a(c a x) + b = 1$. Since $c a x \in I$, it follows by Lemma 1 that there exists a $d \in R$ such that $1 + a d, c a x + d \in U(R)$. Set $u = c a x + d$. Then we have $1 + a d = 1 + a(u - c a x) = a u + b \in U(R)$. Therefore, $a + b u^{-1} \in U(R)$.

(3) $\Rightarrow$ (2). For any $x, y \in I$, we have $x y + (1 - x y) = 1$. By hypothesis, we can find $u \in U(R)$ such that $x + (1 - x y) u \in U(R)$. Hence, $x u^{-1} + 1 - x y \in U(R)$. Set $a = u^{-1} - y$. Then $1 + x a, y + a \in U(R)$, as required. $\square$

COROLLARY 9. *Let I be a regular ideal of a ring R . Then the following are equivalent:*

- (1) *R satisfies unit 1-stable range for I ;*
- (2) *R^{op} satisfies unit 1-stable range for I^{op} .*

PROOF. (1) $\Rightarrow$ (2). For any $x^{\text{op}}, y^{\text{op}} \in I^{\text{op}}$, we have $x, y \in R$. Hence, there exists some $a \in U(R)$ such that $1 + xa, y + a \in U(R)$ by Theorem 8. Clearly, $1 + xa \in U(R)$ if and only if $1 + ax \in U(R)$. So $1^{\text{op}} + x^{\text{op}}a^{\text{op}}, x^{\text{op}} + a^{\text{op}} \in U(R^{\text{op}})$. Therefore, I^{op} is unit 1-stable by Theorem 8 again.

(2) $\Rightarrow$ (1) is proved in the same manner. $\square$

Recall that an ideal I of a ring R has stable rank one in case $aR + bR = R$ with $a \in 1 + I$, $b \in R$ implying that there exists $y \in R$ such that $a + by \in U(R)$. It is well known that an ideal I of a regular ring R has stable range one if and only if eRe is unit-regular for all idempotents $e \in I$.

THEOREM 10. *Let I be an ideal of a regular ring R . If I has stable range one, then the following are equivalent:*

- (1) *R satisfies unit 1-stable range for I ;*
- (2) *if $e \in I$, $f \in 1 + I$ are idempotents such that $eR + fR = R$, then there exist $u, v \in U(R)$ such that $eu + fv = 1$.*

PROOF. (1) $\Rightarrow$ (2). Suppose that $eR + fR = R$ with idempotents $e \in I$, $f \in 1 + I$. Since I has stable rank one, we can find a $y \in R$ such that $ey + f = u \in U(R)$. Hence, $eyu^{-1} + fu^{-1} = 1$. As R satisfies unit 1-stable range for I , there exists $v \in U(R)$ such that $e + fu^{-1}v = w \in U(R)$ by Theorem 8. Hence, $ew^{-1} + fu^{-1}vw^{-1} = 1$, as required.

(2) $\Rightarrow$ (1). Given $ax + b = 1$ with $a \in I$, $x, b \in R$, then $b = 1 - ax \in 1 + I$. Since R is regular, we have $c \in R$ such that $b = bcb$. Clearly, $c \in 1 + I$. As I has stable range one, it follows from $bc + (1 - bc) = 1$ that $b + (1 - bc)y \in U(R)$ for a $y \in R$. Hence, $b + (1 - bc)y = u$, and so $b = bcb = bcu$. Similarly, we have $d \in R$ such that $a = ada$. Since $(a + (1 - ad))d + (1 - ad)(1 - d) = 1$ with $a + (1 - ad) \in 1 + I$, we can find $z \in R$ such that $a + (1 - ad) + z(1 - ad)(1 - d) = v \in U(R)$. This shows that $a = ada = adv$. Let $e = ad$ and $f = bc$. Then $e \in I$ and $f \in 1 + I$. Clearly, $eR + fR = R$. By hypothesis, there are $s, t \in U(R)$ such that $es + ft = 1$. Therefore, $av^{-1}s + bu^{-1}t = 1$, and then $a + bu^{-1}ts^{-1}v \in U(R)$. According to Theorem 8, we obtain the result. $\square$

Let I be a bounded ideal of a regular ring R . As a result, we deduce that R satisfies unit 1-stable range for I if and only if $e \in I$, $f \in 1 + I$ are idempotents such that $eR + fR = R$; then there exist $u, v \in U(R)$ such that $eu + fv = 1$.

COROLLARY 11. *Let R be unit-regular. Then the following are equivalent:*

- (1) *R satisfies unit 1-stable range;*
- (2) *if $e, f \in R$ are idempotents such that $eR + fR = R$, then there exist $u, v \in U(R)$ such that $eu + fv = 1$.*

PROOF. It is clear by Theorem 10. $\square$

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