A NOTE ON THE SPECTRA OF TRIDIAGONAL MATRICES

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Using orthogonal polynomials, we give a different approach to some recent results on tridiagonal matrices.

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1. Introduction and preliminaries. Most of the results on tridiagonal matrices can be seen in the context of orthogonal polynomials. Here, we will see two examples. The aim of this note is to give simplified proofs of some recent results on tridiagonal matrices, using arguments from the theory of orthogonal polynomials.

One of the most important tools in the study of orthogonal polynomials is the spectral theorem for orthogonal polynomials, which states that any orthogonal polynomial sequence (OPS) \( \{P_n\}_{n \geq 0} \) is characterized by a three-term recurrence relation

\[
xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n = 0, 1, 2, \ldots,
\]

with initial conditions \( P_{-1}(x) = 0 \) and \( P_0(x) = \text{const} \neq 0 \), where \( \{\alpha_n\}_{n \geq 0}, \{\beta_n\}_{n \geq 0}, \) and \( \{\gamma_n\}_{n \geq 0} \) are sequences of complex numbers such that \( \alpha_n \gamma_n + 1 \neq 0 \) for all \( n = 0, 1, 2, \ldots \).

The next proposition is known as the separation theorem for zeros and tells us that the zeros of \( P_n \) and \( P_{n+1} \) mutually separate each other.

Theorem 1.1 (see, e.g., [1, page 28]). In (1.1), let \( \beta_n \in \mathbb{R} \) and \( \gamma_{n+1} > 0 \) for all \( n = 0, 1, 2, \ldots \). Then, for each \( n \), \( P_n \) has \( n \) real and distinct zeros, denoted in increasing order by \( x_1 < x_2 < \cdots < x_n \). Furthermore, the interlacing inequalities \( x_{n+1,i} < x_{ni} < x_{n+1,i+1} \) (\( i = 1, \ldots, n \)) hold for every \( n = 1, 2, \ldots \).

In 1961, Wendroff showed that some reciprocal-like of the separation theorem is also true.

Theorem 1.2 [7]. Let \( a < x_1 < x_2 < \cdots < x_n < b \) and \( x_i < y_i < x_{i+1} \), for \( i = 1, \ldots, n-1 \). Then there exists a monic orthogonal polynomial sequence (MOPS) \( \{P_n\}_{n \geq 0} \) such that \( P_n(x) = (x - x_1) \cdots (x - x_n) \) and \( P_{n-1}(x) = (x - y_1) \cdots (x - y_{n-1}) \).

Notice that the three-term recurrence relation (1.1) can be written in matrix form as

\[
x \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} = J_{n+1} \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} + \alpha_n P_{n+1}(x) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \tag{1.2} \]
where $J_{n+1}$ is a tridiagonal Jacobi matrix of order $n+1$, defined by

$$
J_{n+1} := \begin{pmatrix}
\beta_0 & \alpha_0 \\
\gamma_1 & \beta_1 & \alpha_1 \\
\gamma_2 & \ddots & \ddots & \ddots \\
& \ddots & \beta_{n-1} & \alpha_{n-1} \\
& & \gamma_n & \beta_n
\end{pmatrix} \quad (n = 0, 1, 2, \ldots).
$$

(1.3)

It follows that if $\{x_{nj}\}_{j=1}^n$ is the set of zeros of the polynomial $P_n$, then each $x_{nj}$ is an eigenvalue of the corresponding Jacobi matrix $J_n$ of order $n$, and an associated eigenvector is $[P_0(x_{nj}), P_1(x_{nj}), \ldots, P_{n-1}(x_{nj})]^t$.

Given a family of orthogonal polynomials $\{P_n\}_{n \geq 0}$ defined by (1.1) with $\alpha_n = 1$ for all $n$ and $\gamma_n > 0$ for all $n = 1, 2, \ldots$ (so that $\{P_n\}_{n \geq 0}$ is an MOPS), we may define the associated monic polynomials of order $r$ ($r$ a positive integer) $\{P^{(r)}_n\}_{n \geq 0}$, $r = 0, 1, 2, \ldots$, by the shifted recurrence

$$
P^{(r)}_{n+1}(x) = (x - \beta_{n+r})P^{(r)}_n(x) - \gamma_{n+r}P^{(r)}_{n-1}(x), \quad n = 0, 1, 2, \ldots,
$$

(1.4)

with $P^{(r)}_0 = 0$ and $P^{(0)}_1 = 1$. The anti-associated polynomials for the family $\{P_n\}_{n \geq 0}$, denoted by $\{P^{(-r)}_n\}_{n \geq 0}$, are obtained by pushing down a given Jacobi matrix and by introducing, in the empty upper left corner, new coefficients $\beta_{-i}$, $i = r, r-1, \ldots, 1$, on the subdiagonal and new coefficients $\gamma_{-i} > 0$, $i = r-1, r-2, \ldots, 0$, on the lower subdiagonal (see, e.g., [6]). The new Jacobi matrix is then of the form

$$
J^{(-r)}_{n+1} := \begin{pmatrix}
\beta_{-r} & 1 \\
\gamma_{-r+1} & \beta_{-r+1} & 1 \\
\gamma_{-r+2} & \ddots & \ddots & \ddots \\
& \ddots & \beta_{-1} & 1 \\
& & \gamma_0 & \beta_0 & 1 \\
& & \gamma_1 & \ddots & \ddots \\
& & & \ddots & \ddots
\end{pmatrix}.
$$

(1.5)

If $\{Q_n\}_{n \geq 0}$ satisfies $Q_{-1} = 0$, $Q_0 = 1$, and

$$
Q_{n+1}(x) = (x - \beta_{-r+n})Q_n(x) - \gamma_{-r+n}Q_{n-1}(x), \quad n = 0, 1, \ldots, r-1,
$$

(1.6)

then, clearly,

$$
P^{(-r)}_n(x) = Q_n(x), \quad n = 0, 1, \ldots, r.
$$

(1.7)

For $n > r$, the anti-associated polynomials satisfy the three-term recurrence relation

$$
P^{(-r)}_{n+r+1}(x) = (x - \beta_n)P^{(-r)}_{n+r}(x) - \gamma_nP^{(-r)}_{n+r-1}(x), \quad n = 0, 1, 2, \ldots.
$$

(1.8)
The anti-associated polynomial $P_{n+r}^{(-r)}$ was represented in [6] as a linear combination of the original family $P_n$ and the associated polynomials $P_{n-1}^{(1)}$ in the following way:

$$P_{n+r}^{(-r)}(x) = Q_r(x)P_n(x) - \gamma_0 Q_{r-1}(x)P_{n-1}^{(1)}(x), \quad n = 0, 1, 2, \ldots$$  \hfill (1.9)

2. Inverse eigenvalue problems. For an $n \times n$ matrix $A$, we denote by $\sigma(A)$ the spectrum of $A$ and by $A'$ the $(n-1)$th leading principal submatrix of $A$. In [5], Gray and Wilson stated the following theorem.

**Theorem 2.1** [5]. Let $\{\mu_1, \ldots, \mu_n\}$ and $\{\nu_1, \ldots, \nu_{n-1}\}$ be sets of real numbers satisfying

$$\mu_1 < \nu_1 < \mu_2 < \nu_2 < \cdots < \mu_{n-1} < \nu_{n-1} < \mu_n.$$  \hfill (2.1)

Then there exists a unique symmetric tridiagonal $n \times n$ matrix $A$ with positive super- and subdiagonals such that $\sigma(A) = \{\mu_1, \ldots, \mu_n\}$ and $\sigma(A') = \{\nu_1, \ldots, \nu_{n-1}\}$.

This result is a consequence of Theorem 1.2. The positiveness of the two mentioned off-diagonals is a direct consequence of Wendroff’s proof.

One says that a set of numbers $S$ is symmetric if $S = -S$. If $\{P_k\}_{k \geq 0}$ is the MOPS given by Wendroff’s theorem, then it is possible to prove that $\beta_k = 0$ for all $k$ in the three-term recurrence relation (1.1). As a consequence, we have the following proposition.

**Theorem 2.2** [3]. Let $S_1 = \{\mu_1, \ldots, \mu_n\}$ and $S_2 = \{\nu_1, \ldots, \nu_{n-1}\}$ be symmetric sets of real numbers satisfying

$$\mu_1 < \nu_1 < \mu_2 < \nu_2 < \cdots < \mu_{n-1} < \nu_{n-1} < \mu_n.$$  \hfill (2.2)

Then there exists a unique symmetric tridiagonal zero main diagonal $n \times n$ matrix $A$ with positive super- and subdiagonals such that $\sigma(A) = \{\mu_1, \ldots, \mu_n\}$ and $\sigma(A') = \{\nu_1, \ldots, \nu_{n-1}\}$.

3. Antipodal tridiagonal pattern. An $n \times n$ (sign) pattern is a matrix, where each entry is $+$, $-$, or $0$. A pattern $S = (s_{ij})$ defines a pattern class of real matrices

$$Q(S) = \{A = (a_{ij}) \mid \text{sign}(a_{ij}) = s_{ij}, \forall i, j\}.$$  \hfill (3.1)

For $n \geq 2$, we consider the antipodal tridiagonal pattern $T_n$ defined as

$$T_n = \begin{pmatrix}
- & + \\
- & 0 & + \\
& \ddots & \ddots \\
& & 0 & + \\
& & & - & +
\end{pmatrix}_{n \times n}.$$  \hfill (3.2)
In [2], Drew et al. have considered the matrix $A_n \in Q(T_n)$:

$$
A_n = \begin{pmatrix}
-a_0 & 1 & & & \\
-a_1 & 0 & 1 & & \\
& -a_2 & \ddots & \ddots & \\
& & \ddots & 0 & 1 \\
& & & -a_{n-1} & a_n
\end{pmatrix}. \tag{3.3}
$$

They proved that if $\det(zI_n - A_n) = z^n$, then $A_n$ is symmetric about the reverse diagonal.

**Theorem 3.1** [2]. *If there exists $A_n \in Q(T_n)$ such that $\det(\lambda I_n - A_n) = z^n$, then $a_i = a_{n-i}$ for $i = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor$.*

This can be generalized in the following way.

**Theorem 3.2.** *Let $A_n \in Q(T_n)$ and $\phi_n(\lambda) = \det(\lambda I_n - A_n)$. Then $\phi_n(-\lambda) = (-1)^n \phi_n(\lambda)$ if and only if $A_n$ is symmetric about the reverse diagonal.*

Recently, Elsner et al. [4] gave a proof of this theorem. We present a different proof by using anti-associated polynomials as defined in the first section. Let $P^{(-1)}_{n-1}$ be the anti-associated polynomial, with $\gamma_0 = a_1$ and $\beta_{-1} = 0$, for the characteristic polynomial $P_{n-1}$ of the matrix

$$
\begin{pmatrix}
0 & 1 & & & \\
a_2 & \ddots & \ddots & & \\
& \ddots & \ddots & 0 & 1 \\
& & a_{n-1} & 0 &
\end{pmatrix}. \tag{3.4}
$$

Let $\tilde{\phi}_n(\lambda) = \det(\lambda I_n - iA_n)$. Then

$$
\tilde{\phi}_n(\lambda) = P^{(-1)}_n(\lambda) - ia_nP^{(-1)}_{n-1}(\lambda) + i\alpha_0P_{n-1}(\lambda) + a_0a_nP_{n-2}(\lambda). \tag{3.5}
$$

If $\tilde{\phi}_n(-\lambda) = (-1)^n \tilde{\phi}_n(\lambda)$, then

$$
a_nP^{(-1)}_{n-1}(\lambda) = a_0P_{n-1}(\lambda). \tag{3.6}
$$

Since the polynomials are monic, $a_n = a_0$. On the other hand, $P_{n-1}(\lambda) = \lambda P_{n-2}(\lambda) - a_{n-1}P_{n-3}(\lambda)$ and, by (1.9), $P^{(-1)}_{n-1}(\lambda) = \lambda P^{(-1)}_{n-2}(\lambda) - a_1P^{(-1)}_{n-3}(\lambda)$, which implies $a_{n-1} = a_1$. Furthermore, $P_{n-3}(\lambda) = P^{(-1)}_{n-3}(\lambda)$, and now we only have to apply the procedure to the matrix

$$
\begin{pmatrix}
0 & 1 & & & \\
a_3 & \ddots & \ddots & & \\
& \ddots & \ddots & 0 & 1 \\
& & a_{n-2} & 0 &
\end{pmatrix}. \tag{3.7}
$$
Notice that the condition $\phi_n(-\lambda) = (-1)^n \phi_n(\lambda)$ is equivalent to $\tilde{\phi}_n(-\lambda) = (-1)^n \tilde{\phi}_n(\lambda)$. This completes the proof.

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**REFERENCES**


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