FLAT SEMIMODULES

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To my dearest friend Najla Ali

We introduce and investigate flat semimodules and k -flat semimodules. We hope these concepts will have the same importance in semimodule theory as in the theory of rings and modules.

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1. Introduction. We introduce the notion of flat and k -flat. In [Section 2](#), we study the structure ensuing from these notions. [Proposition 2.4](#) asserts that V is flat if and only if $(V \otimes_R -)$ preserves the exactness of all right-regular short exact sequences. [Proposition 2.5](#) gives necessary and sufficient conditions for a projective semimodule to be k -flat. In [Section 3](#), [Proposition 3.3](#) gives the relation between flatness and injectivity. In [Section 4](#), [Proposition 4.1](#) characterizes the k -flat cancellable semimodules with the left ideals. [Proposition 4.4](#) describes the relationship between the notions of projectivity and flatness for a certain restricted class of semirings and semimodules. Throughout, R will denote a semiring with identity 1. All semimodules M will be left R -semimodules, except at cited places, and in all cases are unitary semimodules, that is, $1 \cdot m = m$ for all $m \in M$ ($m \cdot 1 = m$ for all $m \in M$) for all left R -semimodules ${}_R M$ (resp., for all right R -semimodule M_R).

We recall here (cf. [\[1, 2, 4, 7, 8\]](#)) the following facts.

(a) A semiring R is said to satisfy the left cancellation law if and only if for all $a, b, c \in R$, $a + b = a + c \Rightarrow b = c$. A semimodule M is said to satisfy the left cancellation law if for all $m, m', m'' \in M$, $m + m' = m + m'' \Rightarrow m' = m''$.

(b) We say that a nonempty subset N of a left semimodule M is subtractive if and only if for all $m, m' \in M$, $m, m + m'$ in N imply m' in N .

(c) A semiring R is called completely subtractive if ${}_R R$ is a completely subtractive semimodule; and a left R -semimodule M is called completely subtractive if and only if for every subsemimodule N of M , N is subtractive.

(d) A semimodule M is said to be free R -semimodule if M has a basis over R .

(e) A semimodule C is said to be semicogenerated by U when there is a homomorphism $\varphi : M \rightarrow \prod_A C$ such that $\ker \theta = 0$. A semimodule C is said to be a semicogenerator when C semicogenerates every left R -semimodule M .

(f) Let $\alpha : M \rightarrow N$ be a homomorphism of semimodules. The subsemimodule $\text{Im } \alpha$ of N is defined as follows: $\text{Im } \alpha = \{n \in N : n + \alpha(m') = \alpha(m) \text{ for some } m, m' \in M\}$. Also α is

said to be a semimonomorphism if $\ker \alpha = 0$, to be a semi-isomorphism if α is surjective and $\ker \alpha = 0$, to be an isomorphism if α is injective and surjective, to be i -regular if $\alpha(M) = \text{Im } \alpha$, to be k -regular if for $a, a' \in A$, $\alpha(a) = \alpha(a')$ implying $a + k = a' + k'$ for some $k, k' \in \ker \alpha$, and to be regular if it is both i -regular and k -regular.

(g) An R -semimodule M is said to be k -regular if there exist a free R -semimodule F and a surjective R -homomorphism $\alpha : F \rightarrow M$ such that α is k -regular.

(h) The sequence $K \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is called an exact sequence if $\ker \beta = \text{Im } \alpha$, and proper exact if $\ker \beta = \alpha(K)$.

(i) A short sequence $0 \rightarrow K \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$ is said to be left k -regular right regular if α is k -regular and β is right regular.

(j) For any two R -semimodules N, M , $\text{Hom}_R(N, M) := \{\alpha : N \rightarrow M \mid \alpha \text{ is an } R\text{-homomorphism of semimodules}\}$ is a semigroup under addition. If M, N , and U are R -semimodules and $\alpha : M \rightarrow N$ is a homomorphism, then $\text{Hom}(\alpha, I_U) : \text{Hom}_R(N, U) \rightarrow \text{Hom}_R(M, U)$ is given by $\text{Hom}(\alpha, I_U)\gamma = \gamma\alpha$, where I_U is the identity on U .

(k) If M is a right R -semimodule, N is a left R -semimodule, and T is an $\mathbf{N}$ -semimodule, then a function $\theta : M \times N \rightarrow T$ is R -balanced if and only if, for all $m, m' \in M$, for all $n, n' \in N$, and for all $r \in R$, we have

- (1) $\theta(m + m', n) = \theta(m, n) + \theta(m', n)$,
- (2) $\theta(m, n + n') = \theta(m, n) + \theta(m, n')$,
- (3) $\theta(mr, n) = \theta(m, rn)$.

Let R be a semiring, let M be a right R -semimodule, and let N be a left R -semimodule. Let A be the set $M \times N$, and let U be the $\mathbf{N}$ -semimodule $\oplus_A \mathbf{N} \times \oplus_A \mathbf{N}$. Let W be the subset of U consisting of all elements of the following forms:

- (1) $(\alpha[m + m', n], \alpha[m, n] + \alpha[m', n])$,
- (2) $(\alpha[m, n] + \alpha[m', n], \alpha[m + m', n])$,
- (3) $(\alpha[m, n + n'], \alpha[m, n] + \alpha[m, n'])$,
- (4) $(\alpha[m, n] + \alpha[m, n'], \alpha[m, n + n'])$,
- (5) $(\alpha[mr, n], \alpha[m, rn])$,
- (6) $(\alpha[m, rn], \alpha[mr, n])$,

for m and m' in M , n and n' in N , and r in R , and where $\alpha[m, n]$ is the function from $M \times N$ to $\mathbf{N}$ which sends (m, n) to 1 and sends every other element of $M \times N$ to 0. Let U' be the $\mathbf{N}$ -subsemimodule of U generated by W . Define $\mathbf{N}$ congruence relation $\equiv$ on $\oplus_A \mathbf{N}$ by setting $\alpha \equiv \alpha'$ if and only if there exists an element $(\beta, \gamma) \in U'$ such that $\alpha + \beta = \alpha' + \gamma$. The factor $\mathbf{N}$ -semimodule $\oplus_A \mathbf{N} / \equiv$ will be denoted by $M \otimes_R N$, and is called the tensor product of M and N over R .

(A) A left R -semimodule P is said to be projective semimodule if and only if for each surjective R -homomorphism $\varphi : M \rightarrow N$, the induced homomorphism $\overline{\varphi} : \text{Hom}_R(P, M) \rightarrow \text{Hom}_R(P, N)$ is surjective.

2. Flat and k -flat semimodules. In this section, we discuss the structure of flat and k -flat semimodules. [Proposition 2.4](#) asserts that V is flat if and only if $(V \otimes_R -)$ preserves the exactness of all left k -regular right regular short sequences. In [Proposition 2.5](#), we give the necessary and sufficient condition for the projective right semimodule to be k -flat relative to a cancellable left semimodule.

DEFINITION 2.1. A semimodule V_R is flat relative to a semimodule ${}_R M$ (or that V is M -flat) if and only if for every subsemimodule $K \leq M$, the sequence $0 \rightarrow V \otimes_R K \xrightarrow{I_V \otimes i_K} V \otimes_R M$ is proper exact (i.e., $\text{Ker}(I_V \otimes_R i_K) = 0$) where $I_V \otimes_R i_K(v \otimes k) = v \otimes i_K(k)$. A semimodule V_R that is flat relative to every left R -semimodule is called a flat right R -semimodule.

DEFINITION 2.2. A semimodule V_R is k -flat relative to a semimodule ${}_R M$ (or that V is Mk -flat) if and only if for every subsemimodule $K \leq M$, the sequence $0 \rightarrow V \otimes_R K \xrightarrow{I_V \otimes_R i_K} V \otimes_R M$ is proper exact and $I_V \otimes i_K$ is k -regular (i.e., $I_V \otimes_R i_K$ is injective). A semimodule V_R that is k -flat relative to every right R -semimodule is called a k -flat right R -semimodule. Thus, if V_R is k -flat relative to ${}_R M$, then V_R is flat relative to ${}_R M$.

Our next result shows that the class of flat and k -flat semimodules is closed under direct sums.

PROPOSITION 2.3. Let $(V_\alpha)_{\alpha \in A}$ be an indexed set of right R -semimodules. Then $\oplus_A V_\alpha$ is M -flat (k -flat) if and only if each V_α is M -flat (k -flat).

PROOF. Let M be a left R -semimodule and K a subsemimodule of M . Consider the following commutative diagram:

$$\begin{array}{ccc}
 V_\alpha \otimes K & \xrightarrow{I_{V_\alpha} \otimes i_K} & V_\alpha \otimes M \\
 \pi_\alpha \uparrow \downarrow i_\alpha & & i_\alpha \uparrow \downarrow \pi_\alpha \\
 \oplus(V_\alpha \otimes K) & \xrightarrow{\theta} & \oplus(V_\alpha \otimes M) \\
 \varphi \uparrow \downarrow & & \varphi' \uparrow \downarrow \\
 (\oplus V_\alpha) \otimes K & \xrightarrow{I_{\oplus V_\alpha} \otimes i_K} & (\oplus V_\alpha) \otimes M,
 \end{array} \tag{2.1}$$

where $\pi_\alpha : \oplus(V_\alpha \otimes K) \rightarrow V_\alpha \otimes K$ and $i_\alpha : V_\alpha \otimes K \rightarrow \oplus(V_\alpha \otimes K)$ are defined respectively by $\pi_\alpha : (v_\alpha \otimes k_\alpha) \mapsto v_\alpha \otimes k_\alpha$ and $i_\alpha : v_\alpha \otimes k_\alpha \mapsto (v_\alpha \otimes k_i)$, where $v_i \otimes k_i = 0$ if $i \neq \alpha$ and $v_i \otimes k_i = v_\alpha \otimes k_\alpha$ if $\alpha = i$; φ and φ' are the isomorphisms of [8, Proposition 5.4] given by $\varphi[(v_\alpha) \otimes k] = (v_\alpha \otimes k)$ and $\theta(v_\alpha \otimes k) = (v_\alpha \otimes i(k))$. Now suppose that $\oplus V_\alpha$ is M -flat (k -flat). If $I_{V_\alpha} \otimes i_K(v_\alpha \otimes k) = 0[I_{V_\alpha} \otimes i_K((v_\alpha \otimes k)) = I_{V_\alpha} \otimes i_K((v'_\alpha \otimes k'))]$, then by the above diagram we have $(v_\alpha) \otimes i_K(k) = 0[(v_\alpha) \otimes i(k) = (v'_\alpha) \otimes i(k')]$. Since $\oplus V_\alpha$ is flat (k -flat), then $(v_\alpha) \otimes k = 0[(v_\alpha) \otimes k = (v'_\alpha) \otimes k']$. Again by (2.1), $(v_\alpha \otimes k) = 0$ whence $v_\alpha \otimes k = 0[(v_\alpha \otimes k) = (v'_\alpha \otimes k')]$, whence $v_\alpha \otimes k = v'_\alpha \otimes k'$. Therefore V_α is flat (k -flat).

Conversely, suppose that V_α is M -flat (k -flat) for each $\alpha \in A$. If $I_{\oplus V_\alpha} \otimes i_K((v_\alpha) \otimes k) = 0[I_{\oplus V_\alpha} \otimes i_K((v'_\alpha) \otimes k')]$, then by the above diagram we have $v_\alpha \otimes i(k) = 0[v_\alpha \otimes i(k) = v'_\alpha \otimes i(k')]$ for each $\alpha \in A$. Since V_α is flat (k -flat), then $v_\alpha \otimes k = 0[v_\alpha \otimes k = v'_\alpha \otimes k']$ for each α . Therefore, $(v_\alpha \otimes k) = 0[(v_\alpha \otimes k) = (v'_\alpha \otimes k')]$. Again by (2.1), $(v_\alpha) \otimes k = 0[(v_\alpha) \otimes k = (v'_\alpha) \otimes k']$. Thus $\oplus V_\alpha$ is flat (k -flat). $\square$

PROPOSITION 2.4. Let M be a left R -semimodule. A right R -semimodule V is M flat if and only if the functor $(V \otimes_R -)$ preserves the exactness of all left k -regular right regular

short exact sequences with middle term M :

$$0 \rightarrow {}_R K \xrightarrow{\alpha} {}_R M \xrightarrow{\beta} {}_R N \rightarrow 0. \quad (2.2)$$

PROOF. “If” part. Let $0 \rightarrow {}_R K \xrightarrow{\alpha} {}_R M \xrightarrow{\beta} {}_R N \rightarrow 0$ be a left k -regular right regular exact sequence. Since V_R is ${}_R M$ -flat, then using [8, Theorem 5.5(2)], the sequence

$$0 \rightarrow V \otimes_R K \xrightarrow{I_V \otimes \alpha} V \otimes_R M \xrightarrow{I_V \otimes \beta} V \otimes_R N \rightarrow 0 \quad (2.3)$$

is exact.

“Only if” part. Let ${}_R K \leq {}_R M$. Consider the following exact sequence:

$$0 \rightarrow K \xrightarrow{i_K} M \xrightarrow{\pi_{\text{Im } i_K}} M / \text{Im } i_K \rightarrow 0. \quad (2.4)$$

By hypothesis, $0 \rightarrow V \otimes_R K \xrightarrow{I_V \otimes i_K} V \otimes_R M$ is an exact sequence. Thus V is M -flat. $\square$

Our next result gives a necessary and sufficient condition for a projective semimodule to be k -flat relative to a cancellable semimodule M .

PROPOSITION 2.5. *Let V_R be projective and ${}_R M$ cancellable. Then, V is Mk -flat if and only if the functor $(V \otimes_R -)$ preserves the exactness of all left k -regular right regular short exact sequences*

$$0 \rightarrow {}_R K \xrightarrow{\alpha} {}_R M \xrightarrow{\beta} {}_R N \rightarrow 0. \quad (2.5)$$

PROOF. “If” part. Let $0 \rightarrow {}_R K \xrightarrow{\alpha} {}_R M \xrightarrow{\beta} {}_R N \rightarrow 0$ be a left k -regular right regular exact sequence. Since V_R is ${}_R M$ k -flat, then V_R is ${}_R M$ -flat. By using Proposition 2.4, the sequence

$$0 \rightarrow V \otimes_R K \xrightarrow{I_V \otimes \alpha} V \otimes_R M \xrightarrow{I_V \otimes \beta} V \otimes_R N \rightarrow 0 \quad (2.6)$$

is exact.

“Only if” part. Let $K \leq M$. Consider the following exact sequence:

$$0 \rightarrow K \xrightarrow{i_K} M \xrightarrow{\pi_{\text{Im } i_K}} M / \text{Im } i_K \rightarrow 0. \quad (2.7)$$

Since V is projective and M is cancellable, then by using [9, Proposition 1.16], $I_V \otimes i_K$ is k -regular. By hypothesis, $0 \rightarrow V \otimes_R K \xrightarrow{I_V \otimes i_K} V \otimes_R M$ is an exact sequence. Thus V is Mk -flat. $\square$

3. Flatness via injectivity. We will discuss the relation between the injectivity and flatness. By $(\cdot)^*$ we mean the functor $\text{Hom}_N(-, C)$, where C is a fixed injective semicogenerator cancellative N -semimodule.

REMARK 3.1. If U is a right R -semimodule, then U^* is a left R -semimodule.

PROOF. Let $\alpha \in \text{Hom}_N(U, C)$ and let $r \in R$. Define $r\alpha(u) = \alpha(ur)$. If $s \in R$, then $s(r\alpha)u = (r\alpha)(us) = \alpha(usr) = (sr)\alpha(u)$. Therefore, U^* is a left R -semimodule. $\square$

We state and prove the following lemma, analogous to the one on modules which is needed in the proof of [Proposition 3.3](#).

LEMMA 3.2. *Let R be a semiring, let M and M' be left R -semimodules, and let U be a right R -semimodule. Let T be a cancellative $\mathbf{N}$ -semimodule. If $\alpha : M' \rightarrow M$ is an R -homomorphism, then there exist $\mathbf{N}$ -isomorphisms φ and φ' such that the following diagram commutes:*

$$\begin{array}{ccc} \text{Hom}_R(M, \text{Hom}_N(U, T)) & \xrightarrow{\text{Hom}_R(\alpha, I_{\text{Hom}_N(U, T)})} & \text{Hom}_R(M', \text{Hom}_N(U, T_N)) \\ \varphi \downarrow & & \downarrow \varphi' \\ \text{Hom}_N((U \otimes_R M), T) & \xrightarrow{\text{Hom}_N((I_U \otimes \alpha), I_T)} & \text{Hom}_N((U \otimes_R M'), T). \end{array} \quad (3.1)$$

PROOF. By [\[7, Proposition 14.15\]](#), there exists an $\mathbf{N}$ -isomorphism

$$\varphi : \text{Hom}_R(M, \text{Hom}_N(U, T)) \rightarrow \text{Hom}_R(M \otimes U, T) \quad (3.2)$$

given by $\varphi(y) : u \otimes m \mapsto y(m)u$. Then with a parallel definition for φ' , we have

$$\begin{aligned} \varphi' \circ \text{Hom}_R(\alpha, I_{\text{Hom}_N(U, T)})(y)(u \otimes m') &= \varphi'(y\alpha)(u \otimes m') = (y\alpha)(m')(u) \\ &= y(\alpha(m'))(u) = \varphi(y)(u \otimes \alpha(m')) \\ &= \varphi(y) \circ \text{Hom}_N(I_U \otimes \alpha)(u \otimes m') \\ &= \text{Hom}_N(I_U \otimes \alpha, I_T)(\varphi(y))(u \otimes m'), \end{aligned} \quad (3.3)$$

and the diagram commutes. $\square$

PROPOSITION 3.3. *Let M be a left R -semimodule.*

- (1) *If the right R -semimodule V is Mk -flat, then V^* is M -injective.*
- (2) *If V^* is M -injective, then V is M -flat.*

PROOF. (1) Let K be a subsemimodule of M . Since V is Mk -flat, then the sequence $0 \rightarrow V \otimes K \xrightarrow{I_V \otimes i_K} V \otimes M$ is proper exact, and $I_V \otimes i_K$ is k -regular. By [Lemma 3.2](#), we have the following commutative diagram:

$$\begin{array}{ccccc} \text{Hom}_R(M, V^*) & \xrightarrow{\text{Hom}_R(i_K, I_{V^*})} & \text{Hom}_R(K, V^*) & \xrightarrow{\quad} & 0 \\ \varphi' \downarrow & & & & \downarrow \varphi \\ (V \otimes M)^* & \xrightarrow{\text{Hom}(I_V \otimes i_K, I_C)} & (V \otimes K)^* & \xrightarrow{\quad} & 0, \end{array} \quad (3.4)$$

where φ' and φ are $\mathbf{N}$ -isomorphisms. It follows that the top row is proper exact if and only if the bottom row is proper exact, whence by [\[6, Proposition 3.1\]](#), V^* is injective.

(2) If V^* is injective, then

$$\text{Hom}(M, V^*) \xrightarrow{\text{Hom}(i_K, I_{V^*})} \text{Hom}(K, V^*) \rightarrow 0 \quad (3.5)$$

is proper exact. Again by the above diagram,

$$(V \otimes M)^* \xrightarrow{\text{Hom}(I_V \otimes i_K, I_C)} (V \otimes K)^* \longrightarrow 0 \quad (3.6)$$

is proper exact. Hence, the sequence is exact. Since C is a semicogenerator, then by [3, Proposition 4.1], the sequence $0 \rightarrow V \otimes K \rightarrow V \otimes M$ is an exact sequence. Hence, V is M -flat. $\square$

4. Cancellable semimodules. In this section, we deal with cancellable semimodules. We characterize k -flat cancellable semimodules by means of left ideals.

PROPOSITION 4.1. *The following statements about a cancellable right R -semimodule V are equivalent:*

- (1) V is k -flat relative to ${}_R R$;
- (2) for each (finitely generated) left ideal $I \leq {}_R R$, the surjective $\mathbf{N}$ -homomorphism $\varphi : V \otimes_R I \rightarrow VI$ with $\varphi(v \otimes a) = va$ is a k -regular semimonomorphism.

PROOF. (1) $\Rightarrow$ (2). Since V is cancellable, then by using [7, Proposition 14.16], $V \otimes_R R \simeq V$. Consider the following commutative diagram:

$$\begin{array}{ccc} V \otimes_R I & \xrightarrow{I_V \otimes i_I} & V \otimes_R R \\ \varphi \downarrow & & \downarrow \theta \\ VI & \xrightarrow{i_{VI}} & V, \end{array} \quad (4.1)$$

where θ is the isomorphism of [7, Proposition 14.16]. Since $\psi : V \times I \rightarrow VI$ given by $\psi(v, i) = vi$ is an R -balanced function, then by using [7, Proposition 14.14], there is an exact unique $\mathbf{N}$ -homomorphism $\varphi : V \otimes I \rightarrow VI$ satisfying the condition $\varphi(v \otimes i) = \psi(v, i)$. Since V is k -flat relative to ${}_R R$, then $I_V \otimes_R i_I$ is injective. If $\varphi(\Sigma v_i \otimes a_i) = \varphi(\Sigma v'_i \otimes a'_i)$, then $\theta(I_V \otimes_R i_I)(\Sigma v_i \otimes a_i) = \theta(I_V \otimes_R i_I)(\Sigma v'_i \otimes a'_i)$. Since θ and $I_V \otimes i_I$ are injective, then $\Sigma(v_i \otimes a_i) = (\Sigma v'_i \otimes a'_i)$.

(2) $\Rightarrow$ (1). Again consider the above diagram. Let I be any left ideal of R and let $I_V \otimes_R i_I(\Sigma v_i \otimes a_i) = I_V \otimes_R i_I(\Sigma v'_i \otimes a'_i)$, where $\Sigma v'_i \otimes a'_i, \Sigma v_i \otimes a_i \in V \otimes_R I$. Let $K_1 = \Sigma R a_i$, $K_2 = \Sigma R a'_i$, and $K = K_1 + K_2$. Now $\theta(I_V \otimes i_I)(\Sigma v_i \otimes a_i) = \theta(I_V \otimes i_I)(\Sigma v'_i \otimes a'_i)$, whence $\Sigma v_i a_i = \Sigma v'_i a'_i$. Now consider the following diagram, where $i_K : K \rightarrow I$ is the inclusion map:

$$\begin{array}{ccc} V \otimes K & \xrightarrow{I_V \otimes_R i_K} & V \otimes I \\ \varphi_K \downarrow & & \downarrow \theta \\ VK & \xrightarrow{i_{VK}} & V. \end{array} \quad (4.2)$$

By hypothesis, φ_K is monic. Thus, $\Sigma v_i \otimes a_i = \Sigma v'_i \otimes a'_i$ as an element of $V \otimes K$. Hence, $I_V \otimes_R i_K(\Sigma v_i \otimes a_i) = I_V \otimes_R i_K(\Sigma v'_i \otimes a'_i) \in V \otimes I$, and $\Sigma v_i \otimes a_i = \Sigma v'_i \otimes a'_i$ as an element of $V \otimes I$. Therefore, $I_V \otimes_R i_I$ is monic. Hence, V is k -flat relative to ${}_R R$. $\square$

PROPOSITION 4.2. *Let M be a cancellable left R -semimodule. Then R_R is Mk -flat.*

PROOF. Let $i_K : K \rightarrow M$ be the inclusion homomorphism. By [7, Proposition 14.16], $R \otimes_R K \simeq K$ and $R \otimes_R M \simeq M$. Consider the following commutative diagram:

$$\begin{array}{ccc} R \otimes_R K & \xrightarrow{I_R \otimes i} & R \otimes_R M \\ \simeq \downarrow & & \downarrow \simeq \\ K & \xrightarrow{i_K} & M, \end{array} \quad (4.3)$$

since i_K is injective, then $I \otimes_R i_K$ is injective. $\square$

COROLLARY 4.3. *Let M be a cancellable left R -semimodule. Then every free R -semimodule is Mk -flat.*

PROOF. The proof is immediate from Propositions 2.3 and 4.2. $\square$

In module theory every projective module is flat. Now we see that this is true for certain special semimodules.

PROPOSITION 4.4. *Let M be a cancellable left R -semimodule, where R is a cancellative completely subtractive semiring. Then every k -regular projective R -semimodule P is Mk -flat.*

PROOF. By using [5, Theorem 19], P is isomorphic to a direct summand of a free semimodule F . By Corollary 4.3, F is Mk -flat. Hence, by using Proposition 2.3, P is Mk -flat. $\square$

COROLLARY 4.5. *Let M be a k -regular left R -semimodule and R a cancellative completely subtractive semiring. Then every k -regular projective R -semimodule P is Mk -flat.*

PROOF. We only need to show that M is cancellable. Since M is k -regular, then there exists a free R -semimodule F such that $\varphi : F \rightarrow M$ is surjective. Let $m_1 + m = m_2 + m$, where $m_1, m_2, m \in M$. Since φ is surjective, then $\varphi(a_1) + \varphi(a) = \varphi(a_2) + \varphi(a)$, where $\varphi(a_1) = m_1$, $\varphi(a) = m$, and $\varphi(a_2) = m_2$. Since φ is k -regular, then $a_1 + a + k_1 = a_2 + a + k_2$, where $k_1, k_2 \in \text{Ker } \varphi$. Since F is cancellable, then $a_1 + k_1 = a_2 + k_2$. Hence $\varphi(a_1) = \varphi(a_2)$. $\square$

PROPOSITION 4.6. *Let M be a cancellable left R -semimodule. If V is a free R -semimodule, then the following assertions hold:*

- (a) V is Mk -flat;
- (b) V^* is M -injective.

PROOF. By using Corollary 4.3, V is Mk -flat.

(i) $\Rightarrow$ (ii). The proof is immediate from Proposition 3.3. $\square$

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