PROPERTIES OF SOME $\ast$-DENSE-IN-ITSELF SUBSETS

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$\mathcal{J}$-open sets were introduced and studied by Janković and Hamlett (1990) to generalize the well-known Banach category theorem. Quasi-$\mathcal{J}$-openness was introduced and studied by Abd El-Monsef et al. (2000). These are $\ast$-dense-in-itself sets of the ideal spaces. In this note, properties of these sets are further investigated and characterizations of these sets are given. Also, their relation with $\mathcal{J}$-dense sets and $\mathcal{J}$-locally closed sets is discussed. Characterizations of completely codense ideals are given in terms of semi-preopen sets.

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1. Introduction and preliminaries. The subject of ideals in topological spaces has been studied by Kuratowski [12] and Vaidyanathaswamy [20]. An ideal $\mathcal{J}$ on a topological space $(X,\tau)$ is a collection of subsets of $X$ which satisfies that (i) $A \in \mathcal{J}$ and $B \subseteq A$ implies $B \in \mathcal{J}$ and (ii) $A \in \mathcal{J}$ and $B \in \mathcal{J}$ implies $A \cup B \in \mathcal{J}$. Given a topological space $(X,\tau)$ with an ideal $\mathcal{J}$ on $X$ and if $\varphi(X)$ is the set of all subsets of $X$, a set operator $(\cdot)^\ast : \varphi(X) \to \varphi(X)$, called a local function [12] of $A$ with respect to $\mathcal{J}$ and $\tau$, is defined as follows: for $A \subseteq X$, $A^\ast(\mathcal{J},\tau) = \{x \in X \mid U \cap A \notin \mathcal{J} \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts concerning the local functions [10, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $\text{cl}^\ast(\cdot)$ for a topology $\tau^\ast(\mathcal{J},\tau)$, called the $\ast$-topology, finer than $\tau$, is defined by $\text{cl}^\ast(A) = A \cup A^\ast(\mathcal{J},\tau)$ [19]. When there is no chance for confusion, we will simply write $A^\ast$ for $A^\ast(\mathcal{J},\tau)$ and $\tau^\ast$ or $\tau^\ast(\mathcal{J})$ for $\tau^\ast(\mathcal{J},\tau)$. If $\mathcal{J}$ is an ideal on $X$, then $(X,\tau,\mathcal{J})$ is called an ideal space. By a space, we always mean a topological space $(X,\tau)$ with no separation properties assumed. If $A \subseteq X$, $\text{cl}(A)$ and $\text{int}(A)$ will denote the closure and interior of $A$ in $(X,\tau)$, respectively, and $\text{cl}^\ast(A)$ and $\text{int}^\ast(A)$ will denote the closure and interior of $A$ in $(X,\tau^\ast)$, respectively. A subset $A$ of a space $(X,\tau)$ is semiopen [13] if there exists an open set $G$ such that $G \subseteq A \subseteq \text{cl}(G)$ or, equivalently, $A \subseteq \text{cl}(\text{int}(A))$. The complement of a semiopen set is semiclosed. The smallest semiclosed set containing $A$ is called the semiclosure of $A$ and is denoted by $\text{scl}(A)$. Also, $\text{scl}(A) = A \cup \text{int}(\text{cl}(A))$ [4, Theorem 1.5(a)]. The largest semiopen set contained in $A$ is called the semi-interior of $A$ and is denoted by $\text{sint}(A)$. A subset $A$ of a space $(X,\tau)$ is an $\alpha$-set [15] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$. The family of all $\alpha$-sets in $(X,\tau)$ is denoted by $\tau^\alpha$. $\tau^\alpha$ is a topology on $X$ which is finer than $\tau$. The complement of an $\alpha$-set is called an $\alpha$-closed set. The closure and interior of $A$ in $(X,\tau^\alpha)$ are denoted by $\text{cl}_\alpha(A)$ and $\text{int}_\alpha(A)$, respectively. If $N$ is the ideal of all nowhere dense subsets in $(X,\tau)$, then $\tau^\ast(N,\tau) = \tau^\alpha$ and $\text{cl}_\alpha(A) = A \cup A^\ast(N)$ [10]. An open subset $A$ of a space $(X,\tau)$ is said to be regular open
if $A = \text{int}(\text{cl}(A))$. The complement of a regular open set is regular closed. A subset $A$ of a space $(X, \tau)$ is said to be preopen [14] if $A \subset \text{int}(\text{cl}(A))$. The family of all preopen sets is denoted by $\text{PO}(X, \tau)$ or simply $\text{PO}(X)$. The largest preopen set contained in $A$ is called the $\text{preinterior}$ of $A$ and is denoted by $\text{pint}(A)$ and $\text{pint}(A) = A \cap \text{int}(\text{cl}(A))$ [4]. $A$ is preopen if and only if there is a regular open set $G$ such that $A \subset G$ and $\text{cl}(A) = \text{cl}(G)$ [7, Proposition 2.1]. A subset $A$ of a space $(X, \tau)$ is semi-preopen [4] if there exists a preopen set $G$ such that $G \subset A \subset \text{cl}(G)$. The family of all semi-preopen sets in $(X, \tau)$ is denoted by $\text{SPO}(X, \tau)$ or simply $\text{SPO}(X)$. The complement of a semi-preopen set is called semi-preclosed. The largest semi-preopen set contained in $A$ is called the semi-preinterior of $A$ and is denoted by $\text{spint}(A)$. Also, $\text{spint}(A) = A \cap \text{cl}(\text{int}(\text{cl}(A)))$ for every $A$ of $X$ [4]. Given a space $(X, \tau)$ and ideals $\mathcal{J}$ and $\mathcal{I}$ on $X$, the extension of $\mathcal{J}$ via $\mathcal{I}$ [11], denoted by $\mathcal{J} \ast \mathcal{I}$, is the ideal given by $\mathcal{J} \ast \mathcal{I} = \{ A \subset X \mid A^* (\mathcal{J}) \in \mathcal{I} \}$. In particular, $\mathcal{J} \ast \mathcal{N} = \{ A \subset X \mid \text{int}(A^* (\mathcal{J})) = \phi \}$ is an ideal containing both $\mathcal{J}$ and $\mathcal{N}$ and $\mathcal{J} \ast \mathcal{N}$ is usually denoted by $\mathcal{H}$. The following lemmas will be useful in the sequel.

**Lemma 1.1.** Let $(X, \tau, \mathcal{J})$ be an ideal space and $A \subset X$. If $A \subset A^*$, then
(a) $A^* = \text{cl}(A) = A^* (A)$,
(b) $A^* (\mathcal{J}) = A^* (\mathcal{N})$.

**Proof.** Clearly, for every subset $A$ of $X$, $A^* (A) \subset \text{cl}(A)$. Let $x \notin A^* (A)$. Then there exists a $\tau^*$-open set $G$ containing $x$ such that $G \cap A = \phi$. There exists $V \in \tau$ and $I \in \mathcal{J}$ such that $x \in V - I \subset G, G \cap A = \phi \Rightarrow (V - I) \cap A = \phi \Rightarrow (V \cap A) - I = \phi \Rightarrow \{(V \cap A) - I\}^* = \phi \Rightarrow (V \cap A)^* - I^* = \phi \Rightarrow (V \cap A)^* = \phi \Rightarrow V \cap A^* = \phi \Rightarrow V \cap A = \phi$. Since $V$ is an open set containing $x$, $x \notin \text{cl}(A)$ and so $\text{cl}(A) \subset A^* (A)$. Hence $\text{cl}(A) = A^* (A)$. Since $A \subset A^* \subset \text{cl}(A)$, $\text{cl}(A) = A^*$. This proves (a).

(b) By [11, Theorem 3.2], $A^* (\mathcal{J}) = \text{cl}(\text{int}(A^*))$ and so by (a), $A^* (\mathcal{J}) = \text{cl}(\text{int}(\text{cl}(A))) = A^* (\mathcal{N})$.

**Lemma 1.2.** Let $(X, \tau)$ be a space and $A \subset X$. If $A$ is semiopen, then $\text{cl}(A) = \text{cl}_{\alpha}(A)$ and if $A$ is semiclosed, then $\text{int}(A) = \text{int}_{\alpha}(A)$ [18, Lemma 2.1].

**Lemma 1.3.** If $(X, \tau, \mathcal{J})$ is an ideal space, then the following are equivalent.
(a) For every $A \subset \tau$, $A \subset A^*$.
(b) For every $A \in \text{SO}(X, \tau)$, $A \subset A^*$.

**Proof.** Since $\tau \subset \text{SO}(X, \tau)$, it is enough to prove that (a)$\Rightarrow$(b). Suppose $A \in \text{SO}(X, \tau)$. Then there exists an open set $H$ such that $H \subset A \subset \text{cl}(H)$. Since $H$ is open, $H \subset H^*$ and so, by Lemma 1.1, $A \subset \text{cl}(H) = H^* \subset A^*$. Hence $A \subset A^*$.

2. Completely codense ideal. An ideal $\mathcal{J}$ on a space $(X, \tau)$ is said to be codense [6] if $\tau \cap \mathcal{J} = \{ \phi \}$ or, equivalently, $X = X^*$ [10]. $\mathcal{J}$ is said to be completely codense [6] if $\text{PO}(X) \cap \mathcal{J} = \{ \phi \}$ or, equivalently, $\mathcal{J} \subset \mathcal{N}$ [6, Theorem 4.13]. Every completely codense ideal is codense. The converse implication is not true, since in $\mathbb{R}$, the set of all real numbers with the usual topology, the ideal $\ell$ of all countable subsets is codense but not completely codense [6]. The following theorem characterizes completely codense ideals.
**Theorem 2.1.** Let \((X, \tau, \mathcal{J})\) be an ideal space. Then the following are equivalent.

(a) \(\mathcal{J}\) is completely codense.
(b) \(\text{SPO}(X) \cap \mathcal{J} = \{\phi\}\).
(c) \(A \subset A^*\) for every \(A \in \text{SPO}(X)\).
(d) \(\text{spint}(A) = \phi\) for every \(A \in \mathcal{J}\).

**Proof.** (a)⇒(b). Suppose \(A \in \text{SPO}(X) \cap \mathcal{J}\). \(A \in \mathcal{J} \Rightarrow A \in \mathcal{K}\) and so \((\text{cl}(A)) = \phi\). \(A \in \text{SPO}(X) \Rightarrow A \subset \text{cl}(\text{int}(A)) \Rightarrow A = \phi\). Therefore, \(\text{SPO}(X) \cap \mathcal{J} = \{\phi\}\).

(b)⇒(c). Suppose \(A \in \text{SPO}(X)\) and \(x \notin A^*\). Then there exists an open set \(G\) containing \(x\) such that \(G \cap A \in \mathcal{J}\). Since \(A \in \text{SPO}(X)\), \(G \cap A \in \text{SPO}(X)\), by [4, Theorem 2.7] and so by hypothesis, \(G \cap A = \phi\) which implies that \(x \notin A\).

(c)⇒(d). Let \(A \in \mathcal{J}\) such that \(\text{spint}(A) \neq \phi\). Then there exists a nonempty semi-preopen set \(G\) such that \(G \subset A\) and so \(G^* \subset A^* = \phi\). Since \(G \subset G^*, G = \phi\) which is a contradiction. Therefore, \(\text{spint}(A) = \phi\).

(d)⇒(a). Let \(A \in \text{PO}(X) \cap \mathcal{J}\). Then \(A \in \text{PO}(X) \Rightarrow A \subset \text{int}(\text{cl}(A)) \subset \text{cl}(\text{int}(\text{cl}(A))))\). \(A \in \mathcal{J} \Rightarrow \text{spint}(A) = \phi \Rightarrow A \cap \text{cl}(\text{int}(\text{cl}(A))) = \phi \Rightarrow A = \phi\). □

**Corollary 2.2.** Let \((X, \tau, \mathcal{J})\) be an ideal space with a completely codense ideal \(\mathcal{J}\).

(a) If \(A \in \text{SPO}(X)\), then \(A^* (\mathcal{J})\) is regular closed, \(A^* (\mathcal{J}) = A^* (\mathcal{K})\), and \(\text{cl}(A) = \text{cl}^*(A) = \text{cl}_{\alpha}(A)\).

(b) If \(A\) is semi-preclosed, then \(\text{int}(A) = \text{int}^*(A) = \text{int}_{\alpha}(A)\).

**Proof.** (a) If \(A \in \text{SPO}(X)\), by Theorem 2.1(c), \(A \subset A^* \subset \text{cl}(A)\) and so \(A^* = \text{cl}(A)\) which implies that \(A^*\) is regular closed, since the closure of a semi-preopen set is regular closed [4, Theorem 2.4]. Therefore, \(A^* = \text{cl}(\text{int}(A^*)) = \text{cl}(\text{int}(\text{cl}(A))) = A^* (\mathcal{K})\). \(\text{cl}(A) = \text{cl}^*(A)\) by Theorem 2.1(c) and Lemma 1.1. Also, \(\text{cl}^*(A) = A \cup A^* (\mathcal{J}) = A \cup A^* (\mathcal{K}) = \text{cl}_{\alpha}(A)\). This proves (a).

(b) The proof follows from (a). □

3. \(\mathcal{J}\)-open sets. A subset \(A\) of an ideal space \((X, \tau, \mathcal{J})\) is \(\tau^*\)-closed [10] (resp., \(\ast\)-dense in itself [9], \(\ast\)-perfect [9]) if \(A^* \subset A\) (resp., \(A \subset A^*, A = A^*\)). Clearly, \(A\) is \(\ast\)-perfect if and only if \(A\) is \(\tau^*\)-closed and \(\ast\)-dense in itself. The following Theorem 3.1 is useful in the sequel.

**Theorem 3.1.** Let \((X, \tau, \mathcal{J})\) be an ideal space and let \(U\) and \(A\) be subsets of \(X\) such that \(A \subset U \subset A^*\). Then \(U\) is \(\ast\)-dense in itself, and \(U^*\) and \(A^*\) are \(\ast\)-perfect.

**Proof.** \(A \subset U \subset A^*\) implies that \(U^* = A^*\) and so \(U\) is \(\ast\)-dense in itself. Since \((A^*)^* \subset A^*, A \subset A^*\) implies that \(A^*\) is \(\ast\)-perfect and so \(U^*\) is \(\ast\)-perfect. □

A subset \(A\) of an ideal space \((X, \tau, \mathcal{J})\) is \(\mathcal{J}\)-locally closed, [5] if \(A = G \cap A^*\), where \(G\) is open and \(V\) is \(\ast\)-perfect. Clearly, every \(\ast\)-perfect set is \(\mathcal{J}\)-locally closed. The following theorem gives a characterization of \(\mathcal{J}\)-locally closed sets.

**Theorem 3.2.** Let \((X, \tau, \mathcal{J})\) be an ideal space. A subset \(A\) of \(X\) is \(\mathcal{J}\)-locally closed if and only if \(A = G \cap A^*\) for some open set \(G\).
Suppose $A$ is $\mathcal{I}$-locally closed. Then $A = G \cap V$ where $G$ is open and $V$ is $\ast$-perfect. Now $A^* = (G \cap V)^* \supset G \cap V^* = G \cap V = A$. Also, $A \subseteq V$ implies that $A^* \subseteq V^* = V$. Therefore, $G \cap A^* = G \cap (A^* \cap V) = (G \cap V) \cap A^* = A \cap A^* = A$. Conversely, if $A = G \cap A^*$ where $G$ is open, then $A \subseteq A^*$ and so by Theorem 3.1, $A^*$ is $\ast$-perfect and so $A$ is $\mathcal{I}$-locally closed.

The following corollary follows from [10, Theorems 2.1 and 2.2 and Theorem 6.1(d)].

**Corollary 3.3.** Let $(X, \tau, \mathcal{I})$ be an ideal space.
(a) Every $\mathcal{I}$-locally closed set is $\ast$-dense in itself.
(b) Every open, $\ast$-dense-in-itself subset of $X$ is $\mathcal{I}$-locally closed.
(c) If $\mathcal{I}$ is codense, then every open set is $\mathcal{I}$-locally closed.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-open [11] if $A \subseteq \operatorname{int}(A^*)$. The family of all $\mathcal{I}$-open sets is denoted by $\operatorname{IO}(X, \tau, \mathcal{I})$, $\operatorname{IO}(X, \tau)$, or $\operatorname{IO}(X)$. The complement of an $\mathcal{I}$-open set is said to be $\mathcal{I}$-closed. The largest $\mathcal{I}$-open set contained in $A$ is called the $\mathcal{I}$-interior of $A$ and is denoted by $\operatorname{lint}(A)$ and $\operatorname{lint}(A) = A \cap \operatorname{int}(A^*)$ [11, Theorem 4.1(3)]. The following theorem gives some properties of $\mathcal{I}$-open sets.

**Theorem 3.4.** If $A$ is an $\mathcal{I}$-open subset of an ideal space $(X, \tau, \mathcal{I})$, then
(a) $A$ is $\ast$-dense in itself,
(b) $A^* = \operatorname{cl}(A) = \operatorname{cl}^*(A)$ and $\operatorname{cl}(A)$ and $A^*$ are regular closed,
(c) $A^*$ is $\ast$-perfect and $\mathcal{I}$-locally closed,
(d) $\operatorname{int}(A^*)$ is $\ast$-dense in itself and $\mathcal{I}$-locally closed,
(e) $\operatorname{cl}(\operatorname{int}(A^*)) = A^*(\mathcal{I})$ is $\ast$-dense in itself,
(f) $A^* = (\operatorname{int}(A^*))^* = (\operatorname{cl}(\operatorname{int}(A^*))^*) = (A^*(\mathcal{I}))^*(\mathcal{I})$,
(g) $(\operatorname{int}(A^*))^*$ and $(\operatorname{cl}(\operatorname{int}(A^*))^*)$ are $\mathcal{I}$-locally closed,
(h) $\operatorname{int}(A^*)$ is $\mathcal{I}$-open.

**Proof.** (a) follows from the definition. (b) follows from (a), Lemma 1.1, and the fact that every $\mathcal{I}$-open set is preopen [1] and the closure of a preopen set is regular closed [7, Proposition 2.1(ii)]. (c) follows from Theorem 3.1 and from the fact that every $\ast$-perfect set is $\mathcal{I}$-locally closed. (d) follows from Theorem 3.1 and Corollary 3.3(b). (e) $\operatorname{cl}(\operatorname{int}(A^*)) = A^*(\mathcal{I})$ by [11, Theorem 3.2] and since $A \subseteq \operatorname{int}(A^*) \subseteq \operatorname{cl}(\operatorname{int}(A^*)) \subseteq A^*$, by Theorem 3.1, $\operatorname{cl}(\operatorname{int}(A^*))$ is $\ast$-dense in itself. (f) From the inequality in the proof of (e), we have $A^* = (\operatorname{int}(A^*))^* = (\operatorname{cl}(\operatorname{int}(A^*))^*)^*$. Each is equal to $(A^*(\mathcal{I}))^*(\mathcal{I})$ by (e). (g) and (h) follow from (c) and (f), respectively.

**Theorem 3.5.** Let $(X, \tau, \mathcal{I})$ be an ideal space. If $A$ is $\mathcal{I}$-open and $V$ is semiopen, then
(a) $V \cap A$ is $\ast$-dense in itself;
(b) $(V \cap A)^*$ is $\ast$-perfect and $\mathcal{I}$-locally closed,
(c) $\operatorname{cl}(V) \cap A$ is $\ast$-dense in itself;
(d) $(\operatorname{cl}(V) \cap A)^*$ is $\ast$-perfect and $\mathcal{I}$-locally closed.

**Proof.** Since $V \cap A \subseteq \operatorname{cl}(V) \cap A \subseteq (V \cap A)^*$ by [1, Theorem 2.10], $V \cap A$ is $\ast$-dense in itself and by Theorem 3.1, $\operatorname{cl}(V) \cap A$ is $\ast$-dense in itself and so by Theorem 3.1, $(V \cap A)^*$ and $(\operatorname{cl}(V) \cap A)^*$ are $\ast$-perfect and so are $\mathcal{I}$-locally closed.
The following theorem shows that \((X, \tau)\) and \((X, \tau^\alpha)\) have the same collection of \(\mathcal{F}\)-open sets.

**Theorem 3.6.** If \((X, \tau, \mathcal{F})\) is an ideal space, then \(IO(X, \tau, \mathcal{F}) = IO(X, \tau^\alpha, \mathcal{F})\).

**Proof.** \( A \in IO(X, \tau) \) if and only if \( A \subset \text{int}(A^*) \) if and only if \( A \subset \text{int}_\alpha(A^*) \), by Lemma 1.2 if and only if \( A \in IO(X, \tau^\alpha) \). \( \square \)

**Corollary 3.7.** If \((X, \tau, \mathcal{F})\) is an ideal space where \( \mathcal{F} \) is completely codense, then \( IO(X, \tau) = IO(X, \tau^\alpha) = IO(X, \tau^*) \).

**Proof.** Follows from Corollary 2.2(b). \( \square \)

The following theorem and corollary are generalizations of [1, Theorem 2.6(iii) and Corollary 2.1(ii)], respectively.

**Theorem 3.8.** Let \((X, \tau, \mathcal{F})\) be an ideal space. If \( A \in IO(X) \) and \( B \in \tau^\alpha \), then \( A \cap B \in IO(X) \).

**Proof.** \( A \in IO(X, \tau) \Rightarrow A \in IO(X, \tau^\alpha) \) and so by [1, Theorem 2.6(ii)], \( A \cap B \in IO(X, \tau^\alpha) \) which implies that \( A \cap B \in IO(X, \tau) \). \( \square \)

**Corollary 3.9.** Let \((X, \tau, \mathcal{F})\) be an ideal space. If \( A \) is \( \mathcal{F} \)-closed and \( B \) is \( \alpha \)-closed, then \( A \cup B \) is \( \mathcal{F} \)-closed.

Every \( \mathcal{F} \)-open set is preopen but the converse need not be true [1, Example 2.3]. The following theorem characterizes \( \mathcal{F} \)-open sets in terms of preopen sets.

**Theorem 3.10.** Let \((X, \tau, \mathcal{F})\) be an ideal space and \( A \subset X \). Then the following are equivalent.

(a) \( A \) is \( \mathcal{F} \)-open.
(b) \( A \subset A^* \) and \( \text{scl}(A) = \text{int}(\text{cl}(A)) \).
(c) \( A \subset A^* \) and \( A \) is preopen.

**Proof.** \( A \in IO(X) \) if and only if \( A \subset A^* \) and \( A \subset \text{int}(A^*) \) if and only if \( A \subset A^* \) and \( A \subset \text{int}(\text{cl}(A)) \), since \( \text{cl}(A) = A^* \) if and only if \( A \subset A^* \) and \( A \cup \text{int}(\text{cl}(A)) = \text{int}(\text{cl}(A)) \) if and only if \( A \subset A^* \) and \( \text{scl}(A) = \text{int}(\text{cl}(A)) \). Therefore, (a) and (b) are equivalent. It is clear that (a) and (c) are equivalent. \( \square \)

**Corollary 3.11.** Let \((X, \tau, \mathcal{F})\) be an ideal space and \( A \subset X \).

(a) If \( A \) is semiclosed and \( \mathcal{F} \)-open, then \( A \) is regular open.
(b) If \( A \) is semiopen and \( \mathcal{F} \)-closed, then \( A \) is regular closed.
(c) If \( A \) is \( \mathcal{F} \)-open, then \( \text{sint}(\text{scl}(A)) = \text{int}(\text{scl}(A)) = \text{int}(\text{cl}(A)) \).

For subsets of any ideal space \((X, \tau, \mathcal{F})\), openness and \( \mathcal{F} \)-openness are independent concepts [1, Examples 2.1 and 2.2]. The following Theorem 3.12 shows that the two concepts coincide for \( * \)-perfect sets. Corollary 3.13 follows from the fact that every \( \tau^* \)-closed, \( \mathcal{F} \)-open set is \( * \)-perfect.

**Theorem 3.12.** Let \((X, \tau, \mathcal{F})\) be an ideal space and \( A \subset X \).

(a) If \( A \) is \( * \)-dense in itself, then \( \text{lint}(A^*) = \text{int}(A^*) \).
(b) If $A$ is $*$-perfect, then $\text{lint}(A) = \text{int}(A)$ and so, for $*$-perfect sets, the concepts open and $\mathcal{J}$-open coincide.

**Proof.** Since $A$ is $*$-dense in itself, $A^*$ is $*$-perfect, by Theorem 3.1. Now $\text{lint}(A^*) = A^* \cap \text{int}((A^*)^*) = A^* \cap \text{int}(A^*) = \text{int}(A^*)$. This proves (a). (b) follows from (a). \qed

**Corollary 3.13.** Let $(X, \tau, \mathcal{J})$ be an ideal space and $A \subset X$. If $A$ is $\tau^*$-closed and $\mathcal{J}$-open, then $A$ is open.

In [17, Remark 4], it was stated that $\mathcal{J}$ is codense if and only if $\tau \subset \text{IO}(X)$. The following Theorem 3.14(a) follows from the above result. Theorem 3.14(b) follows from Theorem 3.6 and the fact that $\text{SO}(X) \cap \mathcal{J} = \{\phi\}$ if and only if $\tau \cap \mathcal{J} = \{\phi\}$. Theorem 3.15 is a characterization of completely codense ideals.

**Theorem 3.14.** Let $(X, \tau, \mathcal{J})$ be an ideal space.

(a) If $\text{SO}(X) \subset \text{IO}(X)$, then $\mathcal{J}$ is codense.

(b) $\mathcal{J}$ is codense if and only if $\tau^* \subset \text{IO}(X)$.

**Theorem 3.15.** Let $(X, \tau, \mathcal{J})$ be an ideal space. Then $\mathcal{J}$ is completely codense if and only if $\text{PO}(X) = \text{IO}(X)$.

**Proof.** Suppose $\mathcal{J}$ is completely codense and $G \in \text{PO}(X)$. Then $G \subset G^*$, by Theorem 2.1(c) and so $\text{cl}(G) = G^*$. $G \in \text{PO}(X)$ implies $G \subset \text{int}(\text{cl}(G)) = \text{int}(G^*)$ and so $G \in \text{IO}(X)$. Therefore, $\text{PO}(X) \subset \text{IO}(X)$. Clearly, $\text{IO}(X) \subset \text{PO}(X)$. Conversely, if $G \in \text{SPO}(X)$, then there exists $V \in \text{PO}(X)$ such that $V \subset G \subset \text{cl}(V)$ and by hypothesis, $V \subset V^*$ and so by Lemma 1.1, $\text{cl}(V) = V^*$. Hence by Theorem 3.1, $G$ is $*$-dense in itself and so by Theorem 2.1, $\mathcal{J}$ is completely codense. \qed

In the following Theorem 3.16, we show that if $A$ is $\mathcal{J}$-open, then $\text{sint}(A^*)$ is regular closed.

**Theorem 3.16.** Let $(X, \tau, \mathcal{J})$ be an ideal space and $A \subset X$.

(a) For every subset $A$ of $X$, $\text{cl}(\text{lint}(A)) = \text{cl}(\text{int}(A^*)) = \text{sint}(A^*)$.

(b) If $A$ is $\mathcal{J}$-open, then $A^* = \text{cl}(A) = \text{cl}(\text{int}(A^*)) = \text{sint}(A^*)$ and so $\text{sint}(A^*)$ is regular closed.

**Proof.** If $A$ is a subset of $X$, then $\text{sint}(A^*) = A^* \cap \text{cl}(\text{int}(A^*)) = \text{cl}(\text{int}(A^*))$. To prove the other equality, since $\text{lint}(A) = A \cap \text{int}(A^*)$, $\text{cl}(\text{lint}(A)) = \text{cl}(A \cap \text{int}(A^*)) = \text{cl}(A) \cap \text{int}(A^*) = \text{int}(A^*)$ and so $\text{cl}(\text{lint}(A)) \supset \text{cl}(\text{int}(A^*))$. To prove the reverse direction, note that $\text{lint}(A) \subset \text{int}(A^*)$ and so $\text{cl}(\text{lint}(A)) \subset \text{cl}(\text{int}(A^*))$. This completes the proof of (a). (b) follows from (a) and Theorem 3.4(b). \qed

A subset $A$ of an ideal space $(X, \tau, \mathcal{J})$ is $\mathcal{J}$-dense [6] if $A^* = X$. Clearly, every $\mathcal{J}$-dense set is dense but the converse is not true. If $G$ is any proper dense subset of an ideal space $(X, \tau, \mathcal{J})$ where $\mathcal{J}$ is the maximal ideal $\mathcal{J}(X)$, then $G$ is not $\mathcal{J}$-dense. In particular, if $\mathcal{J}$ is not codense, then $X$ is not $\mathcal{J}$-dense and hence no subset of $X$ is $\mathcal{J}$-dense [6]. Therefore, the existence of an $\mathcal{J}$-dense set implies that the ideal is codense. The following theorem characterizes $\mathcal{J}$-open sets in terms of $\mathcal{J}$-dense sets.
THEOREM 3.17. Let \((X, \tau, \mathcal{J})\) be an ideal space with a codense ideal \(\mathcal{J}\) and \(A \subset X\). Then the following are equivalent.

(a) \(A\) is \(\mathcal{J}\)-open.
(b) There is a regular open set \(G\) such that \(A \subset G\) and \(A^* = G^*\).
(c) \(A = G \cap D\) where \(G\) is regular open and \(D\) is \(\mathcal{J}\)-dense.
(d) \(A = G \cap D\) where \(G\) is open and \(D\) is \(\mathcal{J}\)-dense.

PROOF. (a)⇒(b). That \(A\) is \(\mathcal{J}\)-open implies \(A \subset \text{int}(A^*) \subset A^*\). Let \(G = \text{int}(A^*)\). Then \(A \subset G\) and \(\text{int}(\text{cl}(G)) = \text{int}(\text{cl}(\text{int}(A^*))) = \text{int}(\text{cl}(\text{int}(A))) = \text{int}(\text{cl}(A)) = \text{int}(A^*) = G\) and so \(G\) is regular open. \(G^* = (\text{int}(A^*))^* = A^*\), by Theorem 3.4(f).

(b)⇒(c). Let \(G\) be a regular open set such that \(A \subset G\) and \(A^* = G^*\). Let \(D = A \cup (X - G)\). Then \(A = G \cap D\) where \(G\) is regular open. Now \(D^* = (A \cup (X - G))^* = A^* \cup (X - G)^* = G^* \cup (X - G)^* = (G \cup (X - G))^* = X^* = X\), since \(\mathcal{J}\) is codense. Therefore, \(D\) is \(\mathcal{J}\)-dense which proves (c).

(c)⇒(d) is clear.

(d)⇒(a). Suppose \(A = G \cap D\) where \(G\) is open and \(D\) is \(\mathcal{J}\)-dense. Now \(G = G \cap X = G \cap D^* \subset (G \cap D)^*\) and so \(G \subset \text{int}((G \cap D)^*) = \text{int}(A^*)\). Therefore, \(A \subset G \subset \text{int}(A^*)\) which implies that \(A\) is \(\mathcal{J}\)-open.

The following theorem is a generalization of [1, Theorem 2.14(ii)].

THEOREM 3.18. Let \((X, \tau, \mathcal{J})\) be an ideal space and \(A \subset X\). If \(A\) is \(\mathcal{J}\)-closed and \(\alpha\)-open, then \(A = \text{cl}(A) = \text{int}(\text{cl}(A)) = \text{cl}(\text{int}(A))\) and so \(A\) is both regular open and regular closed.

PROOF. \(A\) is \(\mathcal{J}\)-closed ⇒ \(X - A\) is \(\mathcal{J}\)-open ⇒ \(X - A \subset \text{int}((X - A)^*) \Rightarrow X - A \subset \text{int}(\text{cl}(X - A)) \Rightarrow X - A \subset X - \text{cl}(\text{int}(A)) \Rightarrow \text{cl}(\text{int}(A)) \subset A\). \(A\) is \(\alpha\)-open ⇒ \(A\) is semiopen and preopen [16] ⇒ \(\text{cl}(A) = \text{cl}(\text{int}(A))\) and \(A \subset \text{int}(\text{cl}(A))\). Therefore, \(\text{int}(\text{cl}(A)) \subset \text{cl}(A) = \text{cl}(\text{int}(A)) \subset A \subset \text{int}(\text{cl}(A))\) and so \(A = \text{cl}(A) = \text{cl}(\text{int}(A)) = \text{int}(\text{cl}(A))\).

4. Quasi-\(\mathcal{J}\)-open sets. A subset \(A\) of an ideal space \((X, \tau, \mathcal{J})\) is quasi-\(\mathcal{J}\)-open [2] if \(A \subset \text{cl}(\text{int}(A^*))\). Every \(\mathcal{J}\)-open set is quasi-\(\mathcal{J}\)-open and every quasi-\(\mathcal{J}\)-open set is semi-preopen but the converse implications need not be true [2, Examples 1 and 2]. Also, quasi-\(\mathcal{J}\)-openness and semiopenness (resp., preopenness) are independent concepts [2, Examples 1 and 2]. The family of all quasi-\(\mathcal{J}\)-open sets is denoted by \(Q\mathcal{J}O(X, \tau)\). The following theorem gives some of the properties of quasi-\(\mathcal{J}\)-open sets, the proof of which is similar to the proof of Theorem 3.4.

THEOREM 4.1. Let \((X, \tau, \mathcal{J})\) be an ideal space and \(A\) a quasi-\(\mathcal{J}\)-open subset of \(X\). Then

(a) \(A\) is \(*\)-dense in itself,
(b) \(A^* = \text{cl}(A) = \text{cl}^*(A)\),
(c) \(A^*\) is \(*\)-perfect, regular closed, and \(\mathcal{J}\)-locally closed,
(d) \(\text{cl}(\text{int}(A^*)) = A^*(\mathcal{J})\) is \(*\)-dense in itself,
(e) \(A^* = (\text{cl}(\text{int}(A^*)))^* = (A^*(\mathcal{J}))^*(\mathcal{J})\),
(f) \((\text{cl}(\text{int}(A^*)))^*\) is \(*\)-perfect and \(\mathcal{J}\)-locally closed.
**Corollary 4.2.** Let \((X, \tau, \mathcal{J})\) be an ideal space. A subset \(A\) of \(X\) is quasi-\(\mathcal{J}\)-open if and only if \(A \subset A^* (\mathcal{J})\) [2, Theorem 3].

**Theorem 4.3.** Let \((X, \tau, \mathcal{J})\) be an ideal space and let \(U\) and \(A\) be subsets of \(X\) such that \(A \subset U \subset A^*\). Then \(U^*\) is \(*\)-perfect, and if \(A\) is quasi-\(\mathcal{J}\)-open, then \(U\) is quasi-\(\mathcal{J}\)-open and so \(\text{cl}(\text{int}(A^*))\) is quasi-\(\mathcal{J}\)-open.

**Proof.** By Theorem 3.1, \(U^* = A^*\) and \(U^*\) is \(*\)-perfect. \(A\) is quasi-\(\mathcal{J}\)-open \(\Rightarrow\) \(A \subset \text{cl}(\text{int}(A^*)) = \text{cl}(\text{int}(U^*))\). Now \(U \subset A^* \Rightarrow U \subset (\text{cl}(\text{int}(U^*))^* \Rightarrow U \subset \text{cl}(\text{int}(U^*)) = \text{cl}(\text{int}(U^*))\). Therefore, \(U\) is quasi-\(\mathcal{J}\)-open. Since \(A \subset \text{cl}(\text{int}(A^*)) \subset A^*, \text{cl}(\text{int}(A^*))\) is quasi-\(\mathcal{J}\)-open. \(\square\)

Every quasi-\(\mathcal{J}\)-open set is semi-preopen but the converse is not true [2]. [2, Proposition 3(iii)] says that every semiopen set which is \(*\)-dense in itself is quasi-\(\mathcal{J}\)-open. The following Theorem 4.4 is a generalization of this result and shows that for \(*\)-dense in itself, the concepts quasi-\(\mathcal{J}\)-open and semi-preopen are equivalent. Theorem 4.5(a) gives a characterization of codense ideals and Theorem 4.5(b) gives a characterization of completely codense ideals.

**Theorem 4.4.** Let \((X, \tau, \mathcal{J})\) be an ideal space. If \(A\) is semi-preopen and \(*\)-dense in itself, then \(A\) is quasi-\(\mathcal{J}\)-open.

**Proof.** \(A \subset A^* \Rightarrow \text{cl}(A) = A^*,\) by Lemma 1.1. \(A\) is semi-preopen \(\Rightarrow\) \(A \subset \text{cl}(\text{int}(A^*)) = \text{cl}(\text{int}(A^*))\) and so \(A\) is quasi-\(\mathcal{J}\)-open. \(\square\)

**Theorem 4.5.** Let \((X, \tau, \mathcal{J})\) be an ideal space. Then
(a) \(\mathcal{J}\) is codense if and only if \(SO(X) \subset Q\mathcal{J}O(X)\),
(b) \(\mathcal{J}\) is completely codense if and only if \(SPO(X) = Q\mathcal{J}O(X)\).

**Proof.** (a) Suppose \(\mathcal{J}\) is codense. Let \(G \in SO(X)\). By [10, Theorem 6.1] and Lemma 1.3, \(G\) is \(*\)-dense in itself and so by [2, Proposition 3(iii)], \(G \subset Q\mathcal{J}O(X)\). Conversely, suppose that \(SO(X) \subset Q\mathcal{J}O(X)\). If \(G \in SO(X)\), then \(G \subset Q\mathcal{J}O(X)\) and so \(G \subset G^*\). Therefore, \(\mathcal{J}\) is codense by [10, Theorem 6.1] and Lemma 1.3.

(b) Suppose \(\mathcal{J}\) is completely codense and \(G \in SPO(X)\). Then \(G \subset G^*\), by Theorem 2.1(c) and so \(\text{cl}(G) = G^*\). \(G \in SPO(X) \Rightarrow G \subset \text{cl}(\text{int}(G)) = \text{cl}(\text{int}(G^*))\) and so \(G \subset Q\mathcal{J}O(X)\). Therefore, \(SPO(X) \subset Q\mathcal{J}O(X)\). Clearly, \(Q\mathcal{J}O(X) \subset SPO(X)\). Conversely, if \(G \in SPO(X)\), then \(G \subset Q\mathcal{J}O(X)\), by hypothesis, and so \(G \subset G^*\), and so by Theorem 2.1(c), \(\mathcal{J}\) is completely codense. \(\square\)

In [2], it was established that the intersection of a quasi-\(\mathcal{J}\)-open set with an \(\alpha\)-set is semi-preopen. The following theorem is a generalization of the above result.

**Theorem 4.6.** Let \((X, \tau, \mathcal{J})\) be an ideal space. Then (a) \(Q\mathcal{J}O(X, \tau) = Q\mathcal{J}O(X, \tau^\alpha)\) and (b) \(A \in Q\mathcal{J}O(X, \tau)\) and \(B \in \tau^\alpha\) implies \(A \cap B \in Q\mathcal{J}O(X, \tau)\).

**Proof.** \(A \in Q\mathcal{J}O(X, \tau)\) if and only if \(A \subset \text{cl}(\text{int}(A^*))\) if and only if \(A \subset \text{cl}(\text{int}_\alpha(A^*))\) [3] if and only if \(A \in Q\mathcal{J}O(X, \tau^\alpha)\) which proves (a). \(A \in Q\mathcal{J}O(X, \tau)\) and \(B \in \tau^\alpha \Rightarrow A \in Q\mathcal{J}O(X, \tau^\alpha)\) and \(B \in \tau^\alpha \Rightarrow A \cap B \in Q\mathcal{J}O(X, \tau^\alpha)\); by [2, Proposition 2] implies \(A \cap B \in Q\mathcal{J}O(X, \tau)\). \(\square\)
[2, Lemma 2] states that $W^{\ast} (\mathcal{N}) \subset W$ for every subset $W$ of $X$ in the ideal space $(X, \tau, \mathcal{N})$. That is, every subset of $X$ is $\tau^{\ast}$-closed and so $\tau^{\ast}$ is the discrete topology. This is not always the case. For example, if we consider $\mathbb{R}$ with the usual topology $\tau$ and the ideal $\mathcal{N}$ of nowhere dense subsets of $\mathbb{R}$, then $Q^{\ast} = \mathbb{R}$ and so $Q$ is not $\tau^{\ast}$-closed. Therefore, [2, Proposition 4] is no longer valid. Also, it was established that every $\tau^{\ast}$-closed, quasi-$\mathcal{I}$-open set is semiopen [2, Proposition 3(iii)]. The following Theorem 4.7(a) is a generalization of the above result and also shows that the condition preclosed is not necessary in [2, Proposition 5(i)], and Theorem 4.7(b) shows that [2, Proposition 3(iii)] is also true if we replace the condition $\tau^{\ast}$-closed by semiopen.

**Theorem 4.7.** Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$.

(a) If $A$ is $\tau^{\ast}$-closed and quasi-$\mathcal{I}$-open, then $A$ is regular closed.

(b) If $A$ is semiopen and quasi-$\mathcal{I}$-open, then $A$ is semiopen and $A^{\ast} = A^{\ast} (\mathcal{N})$.

**Proof.** (a) That $A$ is $\tau^{\ast}$-closed and quasi-$\mathcal{I}$-open implies $A = A^{\ast}$. Also, $A \in Q_{\mathcal{I}} O (X)$ $\Rightarrow$ $A \subset cl (int (A^{\ast})) \Rightarrow$ $int (A^{\ast}) \subset A^{\ast} \subset cl (int (A^{\ast})) \Rightarrow$ $cl (int (A^{\ast})) \subset A^{\ast} \subset cl (int (A^{\ast}))$. Therefore, $A = A^{\ast} = cl (int (A^{\ast})) = cl (int (A))$ and so $A$ and $A^{\ast}$ are regular closed. (b) $A$ is semiopen $\Rightarrow$ int$(A)$ = int$(cl (A))$ by [8, Proposition 1]. That $A$ is quasi-$\mathcal{I}$-open implies $A \subset cl (int (A^{\ast})) = cl (int (cl (A))) = cl (int (A))$ and so $A$ is semiopen. By Theorem 4.1(b), $cl (A) = A^{\ast}$. Since $int (cl (A)) \subset A \subset cl (int (A^{\ast})) = cl (int (cl (A)))$, $cl (int (cl (A))) \subset cl (cl (A)) = cl (int (A))$ and so $A^{\ast} = cl (A) = cl (int (A^{\ast})) = A^{\ast} (\mathcal{N})$. 

The following theorem gives a characterization of quasi-$\mathcal{I}$-open sets.

**Theorem 4.8.** Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. $A$ is quasi-$\mathcal{I}$-open if and only if $A \subset A^{\ast}$ and $\alpha (A) = cl (int (A^{\ast}))$.

**Proof.** Suppose $A \in Q_{\mathcal{I}} O (X)$. Then $A \subset A^{\ast}$ and $cl (A) = A^{\ast}$. Also $A \subset cl (int (A^{\ast})) \Rightarrow A \subset cl (int (cl (A))) \Rightarrow A \cup cl (int (cl (A))) = cl (int (cl (A))) \Rightarrow cl (A) = cl (int (A^{\ast}))$, since $\alpha (A) = A \cup cl (int (cl (A)))$ [3]. Conversely, suppose the conditions hold. Then $\alpha (A) = cl (int (cl (A)))$ and so $A \subset cl (int (cl (A))) = cl (int (A^{\ast}))$. Therefore, $A$ is quasi-$\mathcal{I}$-open.

The quasi-$\mathcal{I}$-interior of a subset $A$ in an ideal space $(X, \tau, \mathcal{I})$ is the largest quasi-$\mathcal{I}$-open set contained in $A$ and is denoted by $qint (A)$. The following theorem deals with the properties of the quasi-$\mathcal{I}$-interior of subsets of ideal spaces. In [11], it was established that $int (A) = \phi$ if and only if $A \in \mathcal{I}$. Theorem 4.9(c) is a partial generalization of this result.

**Theorem 4.9.** Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. Then

(a) $qint (A) = A \cap cl (int (A^{\ast}))$ for every subset $A$ of $X$,

(b) if $A$ is $\alpha$-closed, then $qint (A) = cl (int (A^{\ast}))$ and the converse holds if $A \subset A^{\ast}$,

(c) $qint (A) = \phi$ if and only if $A \in \mathcal{I}$.

**Proof.** (a) $A \cap cl (int (A^{\ast})) = cl (int (int (A^{\ast}))) = cl (int (A^{\ast} \cap (int A^{\ast}))) \subset cl (int ((A \cap int (A^{\ast}))) = cl (int ((A \cap cl (int (A^{\ast})))^{\ast})))$. Therefore, $A \cap cl (int (A^{\ast}))$ is a quasi-$\mathcal{I}$-open set contained in $A$ and so $A \cap cl (int (A^{\ast})) \subset qint (A)$. Since $qint (A)$ is
quasi-$\delta$-open, $\text{qlint}(A) \subset \text{cl}(\text{int}(\text{qlint}(A))) \subset \text{cl}(\text{int}(A))$ and so $A \cap \text{qlint}(A) \subset A \cap \text{cl}(\text{int}(A))$ which implies that $\text{qlint}(A) \subset A \cap \text{cl}(\text{int}(A))$. Hence $\text{qlint}(A) = A \cap \text{cl}(\text{int}(A))$.

(b) $A$ is $\alpha$-closed $\Rightarrow$ $\text{cl}(\text{int}(A)) \subset A$ $\Rightarrow$ $\text{cl}(\text{int}(A^*)) \subset A$ $\Rightarrow$ $\text{qlint}(A) = \text{cl}(\text{int}(A^*))$. Conversely, if $A \subset A^*$, then $A^* = \text{cl}(A)$. $\text{qlint}(A) = \text{cl}(\text{int}(A^*)) \Rightarrow \text{cl}(\text{int}(A^*)) \subset A$ and so $\text{cl}(\text{int}(A)) \subset A$ and so $A$ is $\alpha$-closed.

(c) $\text{qlint}(A) = \phi$ $\Rightarrow$ $A \cap \text{cl}(\text{int}(A^*)) = \phi$ $\Rightarrow$ $A \cap \text{int}(A^*) = \phi$ $\Rightarrow$ $A = \phi \in \mathcal{J}$. Conversely, $A \in \mathcal{J}$ $\Rightarrow$ $\text{int}(A^*) = \phi$ $\Rightarrow$ $\text{cl}(\text{int}(A^*)) = \phi$ $\Rightarrow$ $\text{qlint}(A) = \phi$.

\[ \square \]

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