A REMARK ON FOUR-DIMENSIONAL ALMOST KÄHLER-EINSTEIN MANIFOLDS WITH NEGATIVE SCALAR CURVATURE

R. S. LEMENCE, T. OGURO, and K. SEKIGAWA

Received 12 October 2003

Concerning the Goldberg conjecture, we will prove a result obtained by applying the result of Iton in terms of $L^2$-norm of the scalar curvature.

2000 Mathematics Subject Classification: 53C25, 53C55.

1. Introduction. An almost Hermitian manifold $M$ is called an almost Kähler manifold if the corresponding Kähler form is a closed 2-form. It is well known that an almost Kähler manifold with integrable almost-complex structure is Kählerian. Concerning the integrability of almost Kähler manifold, the following conjecture by Goldberg is known (see [2]).

**Conjecture 1.1.** A compact almost Kähler-Einstein manifold is Kählerian.

Sekigawa [8] proved that the conjecture is true if the scalar curvature $\tau$ of $M$ is nonnegative. But the conjecture is still open in the case where $\tau$ is negative. Recently, applying the Seiberg-Witten theory, Itoh [4] obtained the following integrability condition for certain almost Kähler-Einstein 4-manifolds in terms of the $L^2$-norm of the scalar curvature.

**Theorem 1.2** [4]. Let $M$ be a four-dimensional compact almost Kähler-Einstein manifold with negative scalar curvature. If $M$ satisfies

$$\int_M \tau^2 dV = 32\pi^2 (2\chi(M) + p_1(M)), \quad (1.1)$$

then it must be Kähler-Einstein. Here, $\chi(M)$ and $p_1(M)$ are the Euler characteristic and the first Pontrjagin number of $M$, respectively.

As a corollary, he also proved the following.

**Corollary 1.3** [4]. Let $M$ be a four-dimensional compact almost Kähler-Einstein manifold with negative scalar curvature. If $M$ satisfies

$$\int_M \tau^2 dV \leq 24 \int_M \|W^+\|^2 dV, \quad (1.2)$$

or, more strictly, if $|\tau| \leq 2\sqrt{6}\|W^+\|$ at each point of $M$, then $M$ must be Kähler-Einstein. Here, $W^+$ is the self-dual Weyl curvature operator of the metric $g$. 
In this paper, concerning the Goldberg conjecture, we will prove a result obtained by using Corollary 1.3 (see Theorem 2.2).

2. Preliminaries and the result. Let $M = (M, J, g)$ be a four-dimensional almost Kähler-Einstein manifold with the almost-complex structure $J$ and the Hermitian metric $g$. We denote by $\Omega$ the Kähler form of $M$ defined by $\Omega(X, Y) = g(X, JY)$ for $X, Y \in \mathfrak{X}(M)$, the set of all smooth vector fields on $M$. We assume that $M$ is oriented by the volume form $dV = \Omega^2/2$. We denote by $\nabla$, $R$, $\rho$, and $\tau$ the Riemannian connection, the curvature tensor, the Ricci tensor, and the scalar curvature of $M$, respectively. We assume that the curvature tensor is defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for $X, Y, Z \in \mathfrak{X}(M)$. We denote by $\rho^*$ the Ricci $\ast$-tensor of $M$ defined by

$$\rho^*(x, y) = \frac{1}{2} \text{trace of } (z \mapsto R(x, Jy)Jz) \tag{2.1}$$

for $x, y, z \in T_pM$, the tangent space of $M$ at $p \in M$. The Ricci $\ast$-tensor satisfies $\rho^*(x, y) = \rho^*(Jy, Jx)$ for any $x, y \in T_pM, p \in M$. We note that if $M$ is Kählerian, the Ricci tensor and the Ricci $\ast$-tensor coincide on $M$. The $\ast$-scalar curvature $\tau^*$ of $M$ is the trace of the linear endomorphism $Q^*$ defined by $g(Q^*x, y) = \rho^*(x, y)$ for $x, y \in T_pM, p \in M$. Since $||\nabla J||^2 = 2(\tau^* - \tau)$, $M$ is a Kähler manifold if and only if $\tau^* - \tau = 0$ on $M$. An almost Hermitian manifold $M$ is called a weakly $\ast$-Einstein manifold if $\rho^* = \lambda^* g$ ($\lambda^* = \tau^*/4$) and a $\ast$-Einstein if $M$ is weakly $\ast$-Einstein with constant $\ast$-scalar curvature. The following identity holds for any four-dimensional almost Hermitian Einstein manifold:

$$\frac{1}{2} \{\rho^*(x, y) + \rho^*(y, x)\} = \frac{\tau^*}{4} g(x, y) \tag{2.2}$$

for $x, y \in T_pM, p \in M$.

Now, let $\wedge^2 M$ be the vector bundle of all real 2-forms on $M$. The bundle $\wedge^2 M$ inherits a natural inner product $g$ and we have an orthogonal decomposition

$$\wedge^2 M = \mathbb{R} \Omega \oplus LM \oplus \wedge^1_0 M, \tag{2.3}$$

where $LM$ (resp., $\wedge^1_0 M$) is the bundle of $J$-skew-invariant (resp., $J$-invariant) 2-forms on $M$ perpendicular to $\Omega$. We can identify the subbundle $\mathbb{R} \Omega \oplus LM$ (resp., $\wedge^1_0 M$) with the bundle $\wedge^2_\perp M$ (resp., $\wedge^2 M$) of self-dual (resp., anti-self-dual) 2-forms on $M$. Since $M$ is Einstein, it is well known that the curvature operator $\mathcal{R}: \wedge^2 M \to \wedge^2 M$ preserves $\wedge^2_\perp M$ and that the Weyl curvature operator $\mathcal{W}: \wedge^2 M \to \wedge^2 M$ is given by

$$\mathcal{W}(\iota(X) \wedge \iota(Y)) = \mathcal{R}(\iota(X) \wedge \iota(Y)) - \frac{\tau}{12} \iota(X) \wedge \iota(Y), \tag{2.4}$$

where $\iota$ is the duality between the tangent bundle and the cotangent bundle of $M$ by means of the metric $g$. Let $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ be a (local) unitary frame field and put $e^i = \iota(e_i)$. Then, the Kähler form is represented by $\Omega = -e^1 \wedge e^2 - e^3 \wedge e^4$. Further,
we see that
\[
\{\Phi, J\Phi\} = \left\{ \frac{1}{\sqrt{2}} (e^1 \wedge e^3 - e^2 \wedge e^4), \frac{1}{\sqrt{2}} (e^1 \wedge e^4 + e^2 \wedge e^3) \right\},
\]
\[
\{\Psi_1, \Psi_2, \Psi_3\} = \left\{ \frac{1}{\sqrt{2}} (e^1 \wedge e^2 - e^3 \wedge e^4), \frac{1}{\sqrt{2}} (e^1 \wedge e^3 + e^2 \wedge e^4), \frac{1}{\sqrt{2}} (e^1 \wedge e^4 - e^2 \wedge e^3) \right\}
\] are (local) orthonormal bases of $LM$ and $\wedge^1_0 M = \wedge^2 M$, respectively.

In this paper, for any orthonormal basis (resp., any local orthonormal frame field) \(\{e_1, e_2, e_3, e_4\}\) of a point $p \in M$ (resp., on a neighborhood of $p$), we will adopt the following notational convention:

\[
J_{ij} = g(J e_i, e_j), \quad \Gamma_{ijk} = g(\nabla_{e_i} e_j, e_k),
\]
\[
R_{ijkl} = g(R(e_i, e_j) e_k, e_l), \ldots, R_{ijkl} = g(R(J e_i, J e_j) J e_k, J e_l),
\]
\[
\rho_{ij} = \rho(e_i, e_j), \ldots, \rho_{ij} = \rho(e_i, e_j),
\]
\[
\rho^{\ast}_{ij} = \rho^{\ast}(e_i, e_j), \ldots, \rho^{\ast}_{ij} = \rho^{\ast}(e_i, e_j),
\]
\[
\nabla_{i} J_{jk} = g((\nabla_{e_i} J) e_j, e_k), \ldots, \nabla_{i} J_{jk} = g((\nabla_{e_i} J) e_j, e_k),
\]
and so on, where the Latin indices run over the range 1, 2, 3, 4. We define functions $A, B, C, D, G,$ and $K$ on $M$ by

\[
A = \sum_{i,j,k,l,a=1}^{4} (\nabla_{a} J_{ij}) R_{ijkl}(\nabla_{a} J_{kl}),
\]
\[
B = \sum_{i,j,k,l,a=1}^{4} (\nabla_{a} J_{ij})(\nabla_{a} J_{kl})(\nabla_{b} J_{ij})(\nabla_{b} J_{kl}),
\]
\[
C = \sum_{i,j,k,l=1}^{4} R_{ijkl} R^{\ast}_{ijkl}, \quad D = \sum_{i,j,k,l=1}^{4} (R_{ijkl} - R^{\ast}_{ijkl})^2,
\]
\[
G = \sum_{i,j=1}^{4} (\rho^{\ast}_{ij} - \rho_{ij})^2, \quad K = (u - v)^2 + 4w^2,
\]
where $u = g(\mathcal{R}(\Phi), \Phi), v = g(\mathcal{R}(J \Phi), J \Phi),$ and $w = g(\mathcal{R}(\Phi), J \Phi).$ First, we will prove the following.

**Lemma 2.1.** The norm of the self-dual Weyl operator $\mathcal{W}^+$ is given by

\[
||\mathcal{W}^+||^2 = \frac{1}{16} \left( G + D + (\tau^*)^2 - \frac{\tau^2}{3} \right).
\]

**Proof.** Let \(\{e_1, e_2 = J e_1, e_3, e_4 = J e_3\}\) be any (local) unitary frame field on $M$ and we put $\Omega_0 = -\Omega/\sqrt{2} = (e^1 \wedge e^2 + e^3 \wedge e^4)/\sqrt{2},$ $\Phi = (e^1 \wedge e^3 - e^2 \wedge e^4)/\sqrt{2},$ and $J \Phi = (e^1 \wedge e^4 + e^2 \wedge e^3)/\sqrt{2}.$ Then, $\{\Omega_0, J \Phi, \Phi\}$ is an orthonormal basis of $\wedge^2 M.$ Thus, we have

\[
||\mathcal{W}^+||^2 = g(\mathcal{W}^+(\Omega_0), \Omega_0)^2 + g(\mathcal{W}^+(\Omega_0), \Phi)^2 + g(\mathcal{W}^+(\Omega_0), J \Phi)^2 + g(\mathcal{W}^+(\Phi), \Omega_0)^2 + g(\mathcal{W}^+(\Phi), \Phi)^2 + g(\mathcal{W}^+(\Phi), J \Phi)^2 + g(\mathcal{W}^+(J \Phi), \Omega_0)^2 + g(\mathcal{W}^+(J \Phi), \Phi)^2 + g(\mathcal{W}^+(J \Phi), J \Phi)^2.
\]
Taking account of (2.4), we have

\[ g(W^+(\Omega_0), \Omega_0) = \frac{1}{2} \left( -R_{1212} - 2R_{1234} - R_{3434} - \frac{\tau}{6} \right) = \frac{1}{12} (3\tau^* - \tau), \]
\[ g(W^+(\Omega_0), \Phi) = \frac{1}{2} \left( -R_{1213} - R_{1224} - R_{3413} - R_{3424} \right) = -\frac{1}{2} (\rho_1^* - \rho_4^*), \]
\[ g(W^+(\Omega_0), J\Phi) = \frac{1}{2} \left( -R_{1214} - R_{1223} - R_{3414} - R_{3423} \right) = \frac{1}{2} (\rho_{13}^* - \rho_{31}^*), \]
\[ g(W^+ (\Phi), \Phi) = \frac{1}{2} \left( -R_{1313} + 2R_{1324} - R_{2424} - \frac{\tau}{6} \right) = -(R_{1313} - R_{1324}) - \frac{\tau}{12}, \]
\[ g(W^+ (\Phi), J\Phi) = \frac{1}{2} \left( -R_{1314} - R_{1323} + R_{2414} + R_{2423} \right) = -(R_{1314} + R_{1323}), \]
\[ g(W^+ (J\Phi), J\Phi) = \frac{1}{2} \left( -R_{1414} - 2R_{1423} - R_{2323} - \frac{\tau}{6} \right) = -(R_{1414} + R_{1423}) - \frac{\tau}{12}. \]

Thus, we have

\[
||W^+||^2 = \frac{1}{12^2} (3\tau^* - \tau)^2 + \frac{2\tau^*}{12^2} + \frac{1}{2} (\rho_{13}^* - \rho_{31}^*)^2 + \frac{1}{2} (\rho_{14}^* - \rho_{41}^*)^2 \\
+ (R_{1313} - R_{1324})^2 + (R_{1314} + R_{1323})^2 + (R_{1314} + R_{1323})^2 \\
+ (R_{1414} + R_{1423})^2 + \frac{\tau}{6} (R_{1313} - R_{1324} + R_{1414} + R_{1423}) \\
= \frac{1}{12^2} (3\tau^* - \tau)^2 + \frac{2\tau^*}{12^2} + \frac{G}{8} \\
+ \frac{1}{4} \sum_{i<j, k<l} (R_{ijkl} - R_{jikl})^2 - \frac{1}{4} \sum_{k<l} (R_{12kl} - R_{12kl})^2 \\
- \frac{1}{4} \sum_{k<l} (R_{34kl} - R_{34kl})^2 + \frac{\tau}{6} \left( -\frac{3}{4} - R_{1212} - R_{1324} \right) \\
= \frac{1}{12^2} (3\tau^* - \tau)^2 + \frac{2\tau^*}{12^2} + \frac{G}{8} + \frac{D}{16} - \frac{G}{32} - \frac{G}{32} + \frac{\tau}{6} \left( -\frac{3}{4} + \frac{\tau^*}{4} \right) \\
= \frac{D}{16} + \frac{G}{16} + \frac{(\tau^*)^2}{16} - \frac{\tau^2}{48}.
\]

The lemma follows. \( \square \)

Next, we recall the following equalities established in [6]:

\[ A = \frac{1}{4} B = \frac{(\tau^* - \tau)^2}{2}, \]
\[ C = -2K + \frac{(\tau^* - \tau)^2}{8}, \]
\[ G = 4||\rho^*||^2 - (\tau^*)^2 = 16 \left\{ (\rho_{13}^*)^2 + (\rho_{14}^*)^2 \right\}, \]
\[ K = (u + v)^2 + 4(w^2 - uv) = \frac{(\tau^* - \tau)^2}{16} - 4 \det \mathcal{R}^\prime_{LM}, \]
\[ ||\mathcal{R}_{LM}||^2 = \frac{1}{16}D, \quad ||\mathcal{R}^\prime_{LM}||^2 = \frac{1}{16}(D - G), \]
\[ (2.10) \]
where $\mathcal{R}_{LM}$ is the restriction of $\mathcal{R}$ to $LM$ and $\mathcal{R}'_{LM} = P_{LM} \circ \mathcal{R}_{LM}$, the composition of $\mathcal{R}_{LM}$ and the natural projection $P_{LM} : \Lambda^2 M \to LM$. We define a vector field $\eta = (\eta_a)$ on $M$ by $\eta_a = \sum_{i,j=1}^4 (\nabla_a J_{ij}) \rho^i_a \bar{\rho}^j_a$, then we obtain the following (see [6, (2.23)]):

$$\Delta \tau^* = \frac{G}{2} + 4K + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} - 4 \text{div} \eta. \quad (2.13)$$

Further, from (2.12) and the curvature identity

$$R_{ijkl} - R_{i j k l} - R_{i j k l} + R_{i j k l} + R_{i j k l} + R_{i j k l} = 2 \sum_{a=1}^4 (\nabla_a J_{ij}) \nabla_a J_{kl} \quad (2.14)$$

by Gray [3] for almost Kähler manifold, we have

$$A = \frac{1}{2} \sum R_{ijkl} (R_{ijkl} - R_{ij kl} - R_{ij kl} + R_{ij kl} + R_{ij kl} + R_{ij kl})$$

$$= \frac{1}{4} \sum (R_{ijkl} - R_{ij kl})^2 - \frac{1}{4} \sum (R_{ijkl} - R_{ij kl}) (R_{ijkl} - R_{ij kl}) + 2 \sum R_{ijkl} R_{ij kl}$$

$$= \frac{D}{4} - \frac{1}{4} \{ -16 ||\mathcal{R}'_{LM}||^2 + \sum (R_{ij12} + R_{ij34} - R_{ij34} - R_{ij34})^2 \} + 2C \quad (2.15)$$

$$= \frac{D}{4} + 4 ||\mathcal{R}'_{LM}||^2 - \frac{G}{4} + 2C$$

$$= \frac{D}{2} - \frac{G}{2} - 4K + \frac{(\tau^* - \tau)^2}{4}.$$  

Thus, from (2.12) and this equality, we obtain

$$\frac{D}{2} - \frac{G}{2} - 4K - \frac{(\tau^* - \tau)^2}{4} = 0. \quad (2.16)$$

Now, we are ready to prove the following.

**Theorem 2.2.** Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler-Einstein manifold with negative scalar curvature. If $M$ satisfies

$$\int_M \{ G + \tau (\tau^* - \tau) \} dV \geq 0, \quad (2.17)$$

or, more strictly, if $\tau^* - \tau \leq -G/\tau$ at each point of $M$, then $M$ is Kähler-Einstein.

**Proof.** From (2.8), we have

$$24 \int_M ||\mathcal{W}^*||^2 dV - \int_M \tau^2 dV = \frac{3}{2} \int_M \{ G + D + (\tau^* - \tau)(\tau^* + \tau) \} dV. \quad (2.18)$$

On one hand, from (2.13) and (2.16), we have

$$0 = \int_M \left\{ \frac{G}{2} + 4K + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} \right\} dV = \int_M \left\{ \frac{D}{2} + \frac{\tau^*(\tau^* - \tau)}{2} \right\} dV. \quad (2.19)$$
Thus, from (2.18) and (2.19), we obtain
\[
24 \int_M \|W^+\|^2 dV - \int_M \tau^2 dV = \frac{2}{3} \int_M \{G + \tau (\tau^* - \tau)\} dV. \tag{2.20}
\]
Therefore, from Corollary 1.3, the assertion of the theorem immediately follows.

**Remark 2.3.** The above theorem is concerned with the following facts.

1. For a compact four-dimensional almost Kähler-Einstein manifold, the function \(\tau^* - \tau\) vanishes at some point of \(M\) (see [1, 5]).

2. A four-dimensional compact almost Kähler-Einstein and weakly \(*\)-Einstein manifold \((G \equiv 0)\) is a Kähler manifold (see [7]).

3. Let \(M\) be a four-dimensional compact strictly almost Kähler-Einstein, but not weakly \(*\)-Einstein manifold. Then, we see that \(G > 0\) on \(M_0 = \{p \in M \mid \tau^* - \tau > 0\}\), and hence \(\tau^* - \tau = 0\) at which \(G = 0\) (see [5]).

**References**


R. S. Lemence: Department of Mathematical Science, Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan

*E-mail address: f02n406n@mail.cc.niigata-u.ac.jp*

T. Oguro: Department of Mathematical Sciences, School of Science and Engineering, Tokyo Denki University, Saitama 350-0394, Japan

*E-mail address: oguro@r.dendai.ac.jp*

K. Sekigawa: Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181, Japan

*E-mail address: sekigawa@sc.niigata-u.ac.jp*
Submit your manuscripts at
http://www.hindawi.com