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ON GROMOV'S THEOREM AND L^2-HODGE DECOMPOSITION

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Using a functional inequality, the essential spectrum and eigenvalues are estimated for Laplace-type operators on Riemannian vector bundles. Consequently, explicit upper bounds are obtained for the dimension of the corresponding L^2-harmonic sections. In particular, some known results concerning Gromov's theorem and the L^2-Hodge decomposition are considerably improved.

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1. Introduction. Recall that Hodge's decomposition theorem provides a representation of the de Rham cohomology by the space of harmonic forms over a compact Riemannian manifold. A useful consequence of this theorem is that the pth Betti number b_p coincides with the space dimension of harmonic p-forms. This enables one to estimate b_p using analytic approaches. A very famous result in the literature is the following Gromov's theorem [15] (see [5] for extensions to Riemannian vector bundles). Throughout the paper, let M be a connected complete Riemannian manifold of dimension d.

Theorem 1.1 (Gromov's theorem). *If M is compact and oriented with diameter D, then there is a positive constant η depending only on d such that*

$$b_1 \le d \quad \textit{provided } D^2 \operatorname{Ric} \ge -\eta. \tag{1.1}$$

This theorem has already been extended to a Riemannian vector bundle of rank l (cf. [5] and the references therein). Furthermore, an explicit η has been provided by Gallot in [12, 13] for such a theorem to hold. It is not difficult to see that the η given there decays at least exponentially fast in $l^{1/2}$ as $l \to \infty$ (for the first Betti number, it decays exponentially fast in $d^{3/2}$ as $d \to \infty$), see Remark 1.3 for details. In this paper, we provide a more explicit number η of order l^{-2} (see (1.5)).

Let Ω be a Riemannian vector bundle of rank l over the manifold M. Denote by $\mathcal{M}$, $\Gamma(\Omega)$, and $\Gamma_0(\Omega)$, respectively, the measurable (with respect to the dx-complete Borel σ-field), the smooth, and the compactly supported smooth sections of Ω. Let $\langle\cdot,\cdot\rangle_x$ denote the inner product on Ω_x, and for $f \in \mathcal{M}$, let $|f|(x) = |f(x)|_x := \langle f(x), f(x)\rangle_x^{1/2}$.

For $V \in C^2(M)$, we consider the operator

$$\tilde{L} := \Box + \nabla_{\nabla V} - R, \tag{1.2}$$

where $\Box$ denotes the horizontal Laplacian on $\Gamma(\Omega)$, $\nabla_{\nabla V}$ the usual covariant derivative along ∇V, and R a symmetric measurable endomorphism of Ω. Let $\mu(dx) = e^V dx$,

where dx denotes the Riemannian volume element. We assume that $(\tilde{L},\mathscr{D}(\tilde{L}))$ is a bounded-above selfadjoint operator on $L^2_\Omega(\mu) := \{f \in \mathscr{M} : \mu(|f|^2) := \int_M |f|^2 d\mu < \infty\}$ with the usual inner product. In particular, $\tilde{L}$ is bounded above when R is bounded below. Finally, let

$$\underline{R}(x) = \inf\{\langle R\omega,\omega\rangle_x : \omega \in \Omega_x,\ |\omega|_x = 1\}, \quad x \in M. \tag{1.3}$$

Our first result is the following theorem.

Theorem 1.2. *Let M be compact with diameter D. Let $b(\tilde{L})$ denote the space dimension of the $\tilde{L}$-harmonic space* $\ker\tilde{L} := \{f \in \mathscr{D}(\tilde{L}) : \tilde{L}f = 0\}$. *Assume that* $\mathrm{Ric} - \mathrm{Hess}_V \geq -K$ *for some $K \geq 0$. If $\underline{R} \geq -c$ for some $c \in \mathbb{R}$, then*

$$b(\tilde{L}) \leq l \inf_{t>0} \mathrm{e}^{cD^2(t+1)} \left(1 + \frac{1}{4t}\exp\left[\frac{KD^2}{8} + \frac{KD^2}{1-\mathrm{e}^{-KD^2}}\right]\right). \tag{1.4}$$

Consequently, $b(\tilde{L}) \leq l$ provided $D^2 \inf \underline{R} > -\eta$, where

$$\begin{aligned}\eta &:= \sup_{t>0} \frac{1}{1+t} \log \frac{l+1}{l(1+(1/4t)\exp[KD^2(1/8+1/(1-\mathrm{e}^{-KD^2}))])} \\ &\geq \frac{1}{1+l\exp[KD^2(1/8+1/(1-\mathrm{e}^{-KD^2}))]} \log\frac{l+1}{l+1/4}.\end{aligned} \tag{1.5}$$

In particular, when M is oriented, $b_1 \leq d$ if there exists $V \in C^2(M)$ such that

$$D^2(\mathrm{Ric} - \mathrm{Hess}_V) \geq -\frac{\log[(d+1)/(d+1/4)]}{1+d(1+1/d)^{d+9/8}}. \tag{1.6}$$

Remark 1.3. In the case where $V = 0$, it was proved by Bérard et al. [5] that there is $\eta > 0$ depending only on l, KD^2, and d such that $b(\tilde{L}) \leq l$ provided $D^2 \inf \underline{R} > -\eta$. More precisely, [13, Corollary 3.2] provided an explicit $\eta := \varepsilon^2 c(l)^{-2}$, where $\varepsilon \leq 1/2$ is a positive constant depending only on d, and (see [12, page 333] and [13, page 365])

$$c(l) > 2l^{-1/2}\int_0^{l^{1/2}/2} (\cosh t)^{d-1} dt = O\left(l^{-1/2}\exp\left[\frac{(d-1)l^{1/2}}{2}\right]\right) \tag{1.7}$$

for large l. Therefore, if $d > 1$, then $c(l)^{-2}$ is at least exponentially small in $l^{1/2}$. In particular, for the first Betti number, one has $l = d$, and thus the η given in [12, 13] has the main order $\exp[-d^{3/2}]$. On the other hand, Theorem 1.2 provides more explicit η of order l^{-2}. Finally, we mention that there exist examples to show that b_1 can be as big as one likes in the absence of any restriction on D^2K. Also, the above theorem of Gromov does not hold with D^2 being replaced by $[\mathrm{vol}(M)]^{2/d}$ (see, e.g., [3, pages 138–139] for details).

The proof of Theorem 1.2 is based on lower bound estimates of eigenvalues of $-\tilde{L}$. Indeed, when, for example, $\sigma_{\mathrm{ess}}(\tilde{L}) = \emptyset$, letting $\lambda_1 \le \lambda_2 \le \cdots$ denote the eigenvalues of $-\tilde{L}$ counting multiplicity, one has

$$b(\tilde{L}) \le \inf\{n-1 : \lambda_n > 0\}. \tag{1.8}$$

Here and in what follows, $\sigma(\cdot)$ and $\sigma_{\mathrm{ess}}(\cdot)$ denote, respectively, the spectrum and the essential spectrum of a linear operator. This leads us to study the eigenvalue estimation in Section 2. In fact, this study should be interesting in itself.

On the other hand, however, when M is noncompact, it is interesting to study the finiteness of $b(\tilde{L})$. To show that $b(\tilde{L})$ is finite, it suffices to prove $0 \notin \sigma_{\mathrm{ess}}(\tilde{L})$. Moreover, when $-\tilde{L}$ is a weighted Hodge Laplacian on differential forms, the feature that $0 \notin \sigma_{\mathrm{ess}}(\tilde{L})$ implies an L^2-Hodge decomposition (see, e.g., [7, Theorem 5.10, Corollary 5.11]). Therefore, the results obtained in Section 2 also imply the following theorem which improves a result by Ahmed and Stroock [1] who used a different approach. For oriented M, let $\Omega = \Lambda^p := \Lambda^p T^*M$ be the bundle of p-forms (i.e., the exterior p-bundle). Consider $\Delta_\mu^p = d_\mu^* d + d d_\mu^*$, where d_μ^* is the $L^2_\Omega(\mu)$-adjoint of the exterior derivative d. Let $(\Delta_\mu^p, \mathcal{D}(\Delta_\mu^p))$, $((d_\mu^*)^p, \mathcal{D}((d_\mu^*)^p))$, and $(d^p, \mathcal{D}(d^p))$ denote, respectively, the corresponding operators on $L^2_{\Lambda^p}(\mu)$ with domains.

Theorem 1.4. *Let M be noncompact and oriented. Let $\mathcal{R}$ be the curvature term in the Weitzenböck formula on $\Omega := \Lambda^p$. Assume that $\mu(dx) := \mathrm{e}^V dx$ is a finite measure and $\mathcal{R} - \mathrm{Hess}_V$ is bounded below. If there exists a positive function $U \in C^2(M)$ such that $U + V$ is bounded, $\{U \le N\}$ is compact for each $N > 0$, $|\nabla U| \to \infty$ as $U \to \infty$, and*

$$\limsup_{U\to\infty} \frac{\Delta U}{|\nabla U|^2} < 1, \tag{1.9}$$

then $\sigma_{\mathrm{ess}}(\Delta_\mu^p) = \emptyset$. Consequently, $\mathrm{im}\, d^{p-1}$ is closed and

$$\begin{aligned} &L^2_{\Lambda^p}(\mu) = \mathrm{im}\,(d_\mu^*)^{p+1}|_{\mathcal{D}((d_\mu^*)^{p+1})} \oplus \mathrm{im}\, d^{p-1}|_{\mathcal{D}(d^{p-1})} \oplus \ker \Delta_\mu^p|_{\mathcal{D}(\Delta_\mu^p)}, \\ &\alpha := \inf\Big\{\langle \phi, \Delta_\mu^p \phi\rangle_{L^2_{\Lambda^p}(\mu)} : \phi \perp \ker \Delta_\mu^p|_{\mathcal{D}(\Delta_\mu^p)},\ \|\phi\|_{L^2_{\Lambda^p}(\mu)} = 1\Big\} > 0. \end{aligned} \tag{1.10}$$

Remark 1.5. Ahmed and Stroock have proved (1.10) under some stronger conditions (cf. [1, Theorem 5.1]). Indeed, their conditions (e.g., (1.1) and the second part of (2.8) in [1]) imply that $\limsup_{U\to\infty}(\Delta U/|\nabla U|^2) \le 0$ which is stronger than (1.9). Moreover, their conditions also imply the ultracontractivity of the semigroup generated by $\Delta + \nabla V$ on M, which is rather restrictive so that some important models are excluded. For instance, Theorem 1.4 applies to $V = -|x|^2$ on $M = \mathbb{R}^n$, but [1, Theorem 5.1] does not since it is well known that the Ornstein-Uhlenbeck semigroup is not ultracontractive (see, e.g., [21] and the references therein). On the other hand, however, the Gaussian measure is crucial in infinite-dimensional analysis; in particular, it plays a role as the Riemann-Lebesgue measure does in finite dimensions, see [16, 22] for details.

In Section 2, by virtue of semigroup domination and the super Poincaré inequality introduced in [28], estimates of eigenvalues obtained in [29] are extended to the present setting. Indeed, we are able to establish analogous results on Hilbert bundles which are

included in the appendix at the end of the paper. For readers who do not care about Hilbert bundles, the appendix may be ignored since the account for vector bundles is self-contained. Nevertheless, the study of Hilbert bundles possesses its own interest from the perspective of functional analysis and operator algebra (cf. [24]). The proofs of Theorems 1.2 and 1.4 are presented in Section 3.

2. Spectrum estimates on Riemannian vector bundles. Let $\{X_i\}$ be a locally normal frame and ∇_{X_i} the usual covariant derivative along X_i. Then the horizontal Laplacian reads $\Box = \sum_{i=1}^d \nabla_{X_i}^2$ which is naturally defined on $\Gamma(\Omega)$. Let $\mu(dx) = \mathrm{e}^{V(x)}\,dx$ for some $V \in C^2(M)$, where dx denotes the Riemannian volume element.

Consider the operator

$$\tilde{L} = \Box + \nabla_{\nabla V} - R, \tag{2.1}$$

where R is a symmetric measurable endomorphism of Ω such that $(\tilde{L},\Gamma_0(\Omega))$ is essentially selfadjoint on $L^2_\Omega(\mu)$ and is bounded above, that is, $-\tilde{\mathscr{E}}(f,f) := \mu(\langle f,\tilde{L}f\rangle) \le C\mu(|f|^2)$ for some $C \in \mathbb{R}$ and all $f \in \Gamma_0(\Omega)$. Recall that $\langle f,g\rangle_x := \langle f(x),g(x)\rangle_x$ and $\mu(u) := \int_M u\mathrm{d}\mu$ for any $x \in M$, $f,g \in \Omega$, and any $u \in L^1(\mu)$. Let $(\tilde{L},\mathscr{D}(\tilde{L}))$ be the unique selfadjoint extension of $(\tilde{L},\Gamma_0(\Omega))$ which is also bounded above. Let $\underline{R}(x) = \inf\{\langle R\omega,\omega\rangle_x : \omega \in \Omega_x,\ |\omega| = 1\}$, $x \in M$. We assume that $\underline{R} \in L^1_{\mathrm{loc}}(\mathrm{d}x)$ and there exists $C \ge 0$ such that $\mu(|\nabla u|^2) + \mu(\underline{R}u^2) \ge -C\mu(u^2)$ for all $u \in C_0^\infty(M)$. Then the following form is closable and let $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ denote its closure (see, e.g., [17, Corollary VI.1.28]):

$$\mathscr{E}(u,v) := \mu(\langle\nabla u,\nabla v\rangle) + \mu(\underline{R}uv), \quad u,v \in C_0^\infty(M). \tag{2.2}$$

Note that the boundedness from below of $(\mathscr{E},C_0^\infty(M))$ does not imply that of $\underline{R}$, see, for example, [19, Remark 2.4]. Let $(L,\mathscr{D}(L))$ be the smallest closed extension (i.e., the Friedrichs extension) of $(\Delta + \nabla V - \underline{R},C_0^\infty(M))$ (which is selfadjoint by [17, Theorem VI.2.6]) and $P_t^{\underline{R}}$ the corresponding strongly continuous semigroup. Below, we will write $L = \Delta + \nabla V - \underline{R}$ for simplicity.

To study the essential spectrum of $\tilde{L}$, we follow the line of [28] to use the following.

Donnelly-Li's decomposition principle. *If $\underline{R}$ is bounded below, then $\sigma_{\mathrm{ess}}(\tilde{L}) = \sigma_{\mathrm{ess}}(\tilde{L}|_{B^c})$ for any compact domain B, where $\tilde{L}|_{B^c}$ denotes the restriction of $\tilde{L}$ on B^c with Dirichlet boundary conditions.*

Although the principle in [10] was given for the Laplacian on functions, its proof indeed works also for our present case. To see this, let $\{f_n\}_{n=1}^\infty \subset \mathscr{D}(\tilde{L})$ be such that $\mu(|f_n|^2) = 1$ and $\tilde{\mathscr{E}}(f_n,f_n) \le c_1$ for some $c_1 > 0$ and all $n \ge 1$; we have $\sup_n \mu(|\nabla f_n|^2) < \infty$ since $\underline{R}$ is bounded from below. Moreover, since V is locally bounded, we obtain $\sup_n \int_B |\nabla f_n|^2\mathrm{d}x < \infty$. Therefore, by Sobolev embedding theorem, $\{1_B f_n\}$ is relatively compact on $L^2_\Omega(\mu)$ (recall that $\operatorname{supp} f_n \subset B$ and V is bounded on B). Hence Donnelly-Li's argument applies.

In order to study the spectrum of $\tilde{L}$, we compare $\tilde{\mathscr{E}}(f,f)$ with $\mathscr{E}(|f|,|f|)$ and then use known results for functions. A convenient way to do so is to compare $\tilde{P}_t$ with a semigroup on $L^2(\mu)$. This trick has been widely used in spectral geometry, especially

in the study of the Hodge Laplacian on differential forms over compact manifolds, see, for instance, [3, 4] and the references therein.

Theorem 2.1. (1) *For any $t > 0$,*

$$\left|\tilde{P}_t f\right| \le P_t^{\underline{R}}|f|, \quad f \in L^2_\Omega(\mu). \tag{2.3}$$

(2) *For any $t > 0$, $\tilde{P}_t$ and $P_t^{\underline{R}}$ have smooth transition densities with respect to μ denoted by $\tilde{p}_t(x,y)$ and $p_t^{\underline{R}}(x,y)$, respectively, which satisfy $\|\tilde{p}_t(x,y)\|_{\mathrm{op}} \le p_t^{\underline{R}}(x,y)$, $x,y \in M$, where $\|\tilde{p}_t\|_{\mathrm{op}}$ denotes the operator norm of the linear operator $\tilde{p}_t(x,y) : \Omega_y \to \Omega_x$.*

(3) *For any $f \in \Gamma_0(\Omega)$, one has $\tilde{\mathscr{E}}(f,f) \ge \mathscr{E}(|f|,|f|)$.*

Proof. (1) By [3, Theorem 16] (cf. Theorem A.5 below), it suffices to show that for any $f,g \in \Gamma_0(\Omega)$ satisfying $\langle f,g\rangle_{L^2_\Omega(\mu)} = \mu(|f|\cdot|g|)$, one has

$$\left|\tilde{\mathscr{E}}(f,g)\right| \ge \mathscr{E}(|f|,|g|). \tag{2.4}$$

Since $\langle f,g\rangle \le |f|\cdot|g|$ but $\langle f,g\rangle_{L^2_\Omega(\mu)} = \mu(|f|\cdot|g|)$, we have $\langle f,g\rangle = |f|\cdot|g|$, μ-a.e. and hence pointwise since f and g are continuous. Thus, $f = |f|g/|g|$ on $\{|g| > 0\}$. By Kato's inequality, we have $|\nabla|g|| \le |\nabla g|$, μ-a.e. for all $g \in \Gamma_0(\Omega)$ (cf. [3, Lemma VI.31] and its proof). Moreover, since any order derivatives of g are zero on $\{|g| = 0\}$, we obtain

$$\begin{aligned}
\tilde{\mathscr{E}}(f,g) &= -\mu(\langle f,\tilde{L}g\rangle) \ge -\mu(\langle f,(\square+\nabla_{\nabla V})g\rangle) + \mu(\underline{R}|f|\cdot|g|)\\
&= \mu\left(1_{\{|g|>0\}}\left\langle \nabla\frac{|f|g}{|g|},\nabla g\right\rangle\right) + \mu(\underline{R}|f|\cdot|g|)\\
&= \mu\left(1_{\{|g|>0\}}\frac{|f|}{|g|}|\nabla g|^2\right)\\
&\quad + \frac{1}{2}\mu(1_{\{|g|>0\}}\nabla_{\nabla(|f|/|g|)}\langle g,g\rangle) + \mu(\underline{R}|f|\cdot|g|)\\
&\ge \mu\left(1_{\{|g|>0\}}\frac{|f|\cdot|\nabla|g||^2}{|g|}\right)\\
&\quad + \mu\left(|g|\left\langle \nabla\frac{|f|}{|g|},\nabla|g|\right\rangle\right) + \mu(\underline{R}|f|\cdot|g|)\\
&= \mu(\langle\nabla|f|,\nabla|g|\rangle) + \mu(\underline{R}|f|\cdot|g|)\\
&= \mathscr{E}(|f|,|g|).
\end{aligned} \tag{2.5}$$

Therefore (2.4) holds.

(2) Since $\cap_{n=1}^\infty \mathscr{D}(\tilde{L}^n) \subset \Gamma(\Omega)$ and $\cap_{n=1}^\infty \mathscr{D}((\Delta+\nabla V-\underline{R})^n) \subset C^\infty(M)$, by the argument in the proof of [9, Theorem 5.2.1], we conclude that $\tilde{P}_t$ and $P_t^{\underline{R}}$ have smooth transition densities. Moreover, for any $x,y \in M$ and any $\omega \in \Omega_y$ with $|\omega|_y = 1$, let $f \in \mathscr{M}$ be such that $f(y) = \omega$ and $|f| = 1$. Then $\tilde{p}_t(x,\cdot)f(\cdot)$ and $p_t^{\underline{R}}(x,\cdot)$ are bounded and continuous in a neighborhood N_y of y. Let $\{h_n\}$ be a sequence of nonnegative continuous functions

with supports contained in N_y such that $h_n\mu \to \delta_y$ weakly as $n \to \infty$. By (2.3), we obtain

$$\begin{aligned}\Big|\int_M [\tilde{p}_t(x,z)f(z)]h_n(z)\mu(\mathrm{d}z)\Big|_x \\ = |\tilde{P}_t(h_nf)(x)|_x \le P_t^R h_n(x) = \int_M p_t^R(x,z)h_n(z)\mu(\mathrm{d}z).\end{aligned} \tag{2.6}$$

By letting $n \to \infty$, we arrive at $|\tilde{p}_t(x,y)\omega|_x \le p_t^R(x,y)$. Therefore, $\|\tilde{p}(x,y)\|_{\mathrm{op}} \le p_t^R(x,y)$.

(3) Recall that $\mu(\langle f,\tilde{P}_t f\rangle) = \mu(\langle \tilde{P}_{t/2}f,\tilde{P}_{t/2}f\rangle)$ for any $f \in \Gamma_0(\Omega)$; it follows from (2.3) that

$$\begin{aligned}\tilde{\mathscr{E}}(f,f) &= -\frac{1}{2}\lim_{t\downarrow 0}\frac{\mu\big(|\tilde{P}_t f|^2\big)-\mu(|f|^2)}{t} \\ &\ge -\frac{1}{2}\lim_{t\downarrow 0}\frac{\mu\big((P_t^R|f|)^2\big)-\mu(|f|^2)}{t} \\ &= \mathscr{E}(f,f).\end{aligned} \tag{2.7}$$

□

From now on, we let P_t^0 denote the strongly continuous semigroup on $L^2(\mu)$ generated by $\Delta+\nabla V$, and $p_t^0(x,y)$ its transition density with respect to μ which is positive since M is connected. Let $\bar{\lambda} = \inf\sigma_{\mathrm{ess}}(-\tilde{L})$, where we put $\inf\varnothing = \infty$ as usual. When $\bar{\lambda} > \lambda_1 := \inf\sigma(-\tilde{L})$, let $\lambda_1 \le \lambda_2 \le \cdots$ be all the eigenvalues (counting multiplicity) of $-\tilde{L}$ contained in $[\lambda_1,\bar{\lambda})$.

To estimate the number $b(\tilde{L})$ by using Theorem 2.1, we need the following lemma.

Lemma 2.2. *Let $\{f_i\}$ be an orthonormal family in $\mathscr{D}(\tilde{L})$ such that $\tilde{L}f_i = -\delta_i f_i$ with $\{\delta_i\}_{i=1}^n$ satisfying $\delta_1 \le \delta_2 \le \cdots \le \delta_n$. Then*

$$l\int_M \|\tilde{p}_{t/2}(\cdot,y)\|_{\mathrm{op}}^2\mu(\mathrm{d}y) \ge \mathrm{e}^{-\delta_n t}\sum_{i=1}^n |f_i|^2. \tag{2.8}$$

Proof. For any $1 \le j \le l$ and $x \in M$, let

$$g_j(y) = \sum_{i=1}^n \langle f_i(x),e_j(x)\rangle_x f_i(y), \quad y \in M, \tag{2.9}$$

where $\{e_j\}_{1\le j\le l}$ is an orthonormal basis on Ω_x. We have

$$\begin{aligned}\mathrm{e}^{-\delta_n t/2}\sum_{i=1}^n |f_i|^2(x) &\le \sum_{j=1}^l \langle \tilde{P}_{t/2}g_j,e_j\rangle(x) \\ &= \sum_{j=1}^l \int_M \langle \tilde{p}_{t/2}(x,y)g_j(y),e_j(x)\rangle_x\mu(\mathrm{d}y) \\ &\le \Big(l\int_M \|\tilde{p}_{t/2}(x,y)\|_{\mathrm{op}}^2\mu(\mathrm{d}y)\Big)^{1/2}\Big(\sum_{j=1}^l \|g_j\|_{L^2_{\tilde{\Omega}}(\mu)}^2\Big)^{1/2}.\end{aligned} \tag{2.10}$$

The proof is completed by noting that

$$\sum_{j=1}^{l} ||g_j||^2_{L^2_\Omega(\mu)} = \sum_{j=1}^{l}\sum_{i=1}^{n} \langle f_i, e_j\rangle^2(x) = \sum_{i=1}^{n} |f_i|^2(x). \tag{2.11}$$

□

Theorem 2.3. (1) *If* $\inf\sigma(\underline{R}-\Delta-\nabla V)>0$, *then* $b(\tilde{L})=0$.

(2) *Assume that* μ *is a probability measure. If* $\underline{R}\ge 0$, *then* $b(\tilde{L})\le l$. *Moreover, for any* $f\in\ker\tilde{L}$, $|f|$ *is constant.*

(3) *Assume that* $\underline{R}$ *is bounded from below. Then* $\bar{\lambda}\ge\inf\sigma_{\mathrm{ess}}(-(\Delta+\nabla V)+\underline{R})$. *Consequently, let* $\rho(x)$ *be the Riemannian distance between* x *and a fixed point* $o\in M$, *if* $\delta:=\underline{\lim}_{\rho\to\infty}\underline{R}+\inf\sigma_{\mathrm{ess}}(-\Delta-\nabla V)>0$, *then* $\bar{\lambda}\ge\delta>0$ *and hence* $b(\tilde{L})<\infty$.

Proof. (1) Let $f\in\ker(\tilde{L})$. If $a:=\inf\sigma(-\Delta-\nabla V+\underline{R})>0$, then

$$\mu(|f|^2)=\mu(|\tilde{P}_t f|^2)\le\mu\Big((P_t^{\underline{R}}|f|)^2\Big)\le \mathrm{e}^{-2at}\mu(|f|^2)\longrightarrow 0 \tag{2.12}$$

as $t\to\infty$, hence $f=0$, μ-a.e.

(2) For fixed $n\le b(\tilde{L})$, let $\{f_1,\dots,f_n\}\subset\ker\tilde{L}$ be an orthonormal family. If $\underline{R}\ge 0$, by the proof of Theorem 2.1 we have $|\tilde{P}_t f|\le P_t^0|f|$ for all $f\in L^2_\Omega(\mu)$ and hence $\|\tilde{p}_t(x,y)\|_{\mathrm{op}}\le p_t^0(x,y)$. Then by Lemma 2.2 (note that $\delta_n=\lambda_n=0$ in the present case), for any compact set $B\subset M$,

$$\sum_{i=1}^{n}\int_B |f_i|^2\mathrm{d}\mu\le l\int_B p_t^0(x,x)\mu(\mathrm{d}x) \tag{2.13}$$

holds for all $t>0$ and any compact set $B\subset M$. We now intend to show that $p_t^0(x,x)\downarrow 1$ as $t\uparrow\infty$ for all $x\in M$. Observing that

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t^0(x,x)=-\int_M|\nabla p_{t/2}^0(x,\cdot)|^2(y)\mu(\mathrm{d}y)\le 0, \tag{2.14}$$

then $p_t^0(x,x)$ is decreasing in t. Next, noting that the Dirichlet form for $\Delta+\nabla V$ is irreducible since M is connected, we have $\|P_t^0u-\mu(u)\|_{L^2(\mu)}\to 0$ as $t\to\infty$ for any $u\in L^2(\mu)$ (see, e.g., the appendix in [2]). For fixed $x\in M$, letting $u(y)=p_1^0(x,y)$, we obtain

$$||P_t^0u-\mu(u)||^2_{L^2(\mu)}=\mu\Big((p_{t+1}^0(\cdot,x)-1)^2\Big)=p_{2(t+1)}^0(x,x)-1. \tag{2.15}$$

Therefore, $p_t^0(x,x)\to 1$ as $t\to\infty$. Now by first letting $t\to\infty$ and then $B\to M$, we obtain from (2.13) that $n\le l$, hence $b(\tilde{L})\le l$ since n is arbitrary. Moreover, for $f\in\ker\tilde{L}$, we have (note that $|f|$ is continuous)

$$|f|(x)^2=|\tilde{P}_t f|(x)^2\le p_{2t}^0(x,x)\mu(|f|^2). \tag{2.16}$$

By letting $t\to\infty$, we obtain $|f|(x)^2\le\mu(|f|^2)$ and hence $|f|^2=\mu(|f|^2)$ pointwise since $|f|$ is continuous.

(3) By Donnelly-Li's decomposition principle mentioned above, we have

$$\inf \sigma_{\rm ess}\big(-(\Delta+\nabla V)+\underline{R}\big) = \lim_{n\to\infty} \inf \sigma\Big(\big[-(\Delta+\nabla V)+\underline{R}\big]\big|_{B_o(n)^c}\Big) \tag{2.17}$$

and the same formula holds for $\tilde{L}$ in place of $-(\Delta+\nabla V)+\underline{R}$. Then the proof is completed by Theorem 2.1(3) and by noting that $\inf\sigma([-(\Delta+\nabla V)+\underline{R}]|_{B_o(n)^c}) \ge \inf_{B_o(n)^c}\underline{R} + \inf\sigma(-(\Delta+\nabla V)|_{B_o(n)^c})$. □

We remark that Theorem 2.3(1) has already been known by Elworthy and Rosenberg [11] for differential forms. Moreover, as is well known in Hodge's theory, Theorem 2.3(2) is optimal in the sense that there exist examples such that $b(\tilde{L}) = l$ and $\underline{R} \ge 0$; for instance, the Betti numbers on torus (see, e.g., [15]).

We are now ready to estimate λ_n and then use the basic estimate (2.18) to obtain more estimates of $b(\tilde{L})$.

Theorem 2.4. *Assume that μ is a probability measure and $\underline{R} \ge -c$ for some $c \in \mathbb{R}$. If $p_t^0(x,x)$ is integrable with respect to μ for some $t>0$, then $\bar{\lambda}=\infty$, $\sigma_{\rm ess}(\tilde{L})=\varnothing$, and*

$$\lambda_n \ge \sup_{t>0} \frac{1}{t}\log\frac{n\,{\rm e}^{-ct}}{l\int_M p_t^0(x,x)\mu({\rm d}x)}, \quad n\ge 1. \tag{2.18}$$

In the case that $p_t^0(x,x)$ is not integrable, let $\delta_{s,t}=\mu(\{x: p_t^0(x,x)>s\})$. If there exists some positive β defined on $(0,\infty)$ such that

$$\mu(u^2) \le r\mu(|\nabla u|^2) + \beta(r)\mu(|u|)^2, \quad r>0,\ u\in C_0^\infty(M), \tag{2.19}$$

then $\bar{\lambda}=\infty$ and

$$\lambda_n \ge \sup\left\{\left(\frac{1}{t}\log\frac{n\varepsilon\,{\rm e}^{-ct}}{sl}\right)\wedge \sup_{r>0}\frac{1}{r}\big[1-cr-2\beta(r)(\varepsilon+\delta_{s,t})\big] : \varepsilon\in(0,1),\ s,t>0\right\}. \tag{2.20}$$

Especially, if (2.19) holds for $\beta(r)=\exp[\alpha(1+r^{-1/\delta})]$ for some $\alpha>0$ and $\delta>1$, one has $\lambda_n \ge \lambda([\log n-\theta]^+)^\delta$ for some $\lambda,\theta>0$ and all $n\ge 1$.

Proof. If $p_t^0(x,x)$ is integrable with respect to μ, then P_t^0 (and hence $P_t^{\underline{R}}$) is uniformly integrable in $L^2(\mu)$. Recall that a linear operator P on $L^2(\mu)$ is called uniformly integrable if $\sup_{\|u\|_2\le 1}\mu(|Pu|^2 1_{\{|Pu|>r\}})\to 0$ as $r\to\infty$. Therefore, by [29, Theorems 2.2 and 3.1] (see also [14]), we have $\sigma_{\rm ess}(-(\Delta+\nabla V)+\underline{R})=\varnothing$. Thus, $\bar{\lambda}=\infty$ according to Theorem 2.3(3). Next, let $\tilde{L}^c = \square + \nabla_{\nabla V} + c$. For the same reason, we have $\sigma_{\rm ess}(\tilde{L}^c)=\varnothing$. Let $\lambda_1^c \le \lambda_2^c \le \cdots$ denote all eigenvalues of $-\tilde{L}^c$ counting multiplicity. Since $-\tilde{L}\ge -\tilde{L}^c$, it follows from the max-min principle that $\lambda_n \ge \lambda_n^c$ for all $n\ge 1$ (see, e.g., [20, problem 1, page 364]). Therefore, it suffices to prove (2.18) for $\underline{R}\equiv -c$. In this case, if f_i are the L^2-unit eigenvectors for λ_i, then, by Lemma 2.2,

$$l\int_M \|\tilde{p}_{t/2}(x,y)\|_{\rm op}^2\mu({\rm d}y) \ge {\rm e}^{-\lambda_n t}\sum_{i=1}^n |f_i|^2(x), \quad x\in M. \tag{2.21}$$

Then Theorem 2.3(2) yields that

$$n\,\mathrm{e}^{-\lambda_n t} \le l\,\mathrm{e}^{ct}\int_M p_t^0(x,x)\mu(\mathrm{d}x). \tag{2.22}$$

This proves (2.18).

If (2.19) holds for $\beta(r) = \exp[\alpha(1+r^{-1/\delta})]$ for some $\alpha > 0$ and $\delta > 1$, by [28, Corollary 5.2] we have $p_t^0(x,x) \le \exp[\lambda(1+t^{-1/(\delta-1)})]$ for some $\lambda > 0$ and all $t > 0$. Then by (2.18),

$$\lambda_n \ge \sup_{t>0}\frac{1}{t}\log\frac{n\,\mathrm{e}^{-ct}}{l\exp\left[\lambda\left(1+t^{-1/(\delta-1)}\right)\right]}. \tag{2.23}$$

Taking $t = \varepsilon^{-1}(\log n)^{1-\delta}$ for small $\varepsilon > 0$ and $n \ge 2$, we obtain

$$\lambda_n \ge \varepsilon\left(1-\lambda\varepsilon^{1/(\delta-1)}\right)(\log n)^{1-\delta} - c - \varepsilon(\log l+\lambda)(\log n)^{\delta-1}, \quad n \ge 2. \tag{2.24}$$

Therefore, the last assertion follows by taking $\varepsilon \in (0,\lambda^{1-\delta})$.

It remains to prove (2.20). The proof is similar to that of [29, Theorem 3.2]. Let $A_{s,t} = \{x : p_t^0(x,x) \le s\}$. By Theorem 2.1(2) and inequality (2.21), we obtain

$$\mathrm{e}^{-\lambda_n t}\sum_{i=1}^n 1_{A_{s,t}}|f_i|^2 \le l\,\mathrm{e}^{ct}\,p_t^0(\cdot,\cdot)1_{A_{s,t}} \le l\,\mathrm{e}^{ct}\,s. \tag{2.25}$$

This implies that

$$\lambda_n \ge \frac{1}{t}\log\left[\frac{1}{ls\,\mathrm{e}^{ct}}\sum_{i=1}^n \mu\left(1_{A_{s,t}}|f_i|^2\right)\right] \ge \frac{1}{t}\log\frac{n\varepsilon}{ls\,\mathrm{e}^{ct}} \tag{2.26}$$

provided $\mu(1_{A_{s,t}}|f_i|^2) \ge \varepsilon$ for all $1 \le i \le n$. On the other hand, if there exists i such that $\mu(1_{A_{s,t}}|f_i|^2) < \varepsilon$, we have

$$\mu(|f_i|)^2 = \left(\mu(1_{A_{s,t}}|f_i|) + \mu\left(1_{A_{s,t}^c}|f_i|\right)\right)^2 \le 2(\varepsilon+\delta_{s,t}). \tag{2.27}$$

Combining this with (2.19), we obtain

$$\begin{aligned} 1 = \mu\left(|f_i|^2\right) &\le r\mu(|\nabla|f_i||^2) + \beta(r)\mu(|f_i|)^2 \\ &\le r\tilde{\mathscr{E}}(f_i,f_i) + cr + 2\beta(r)(\varepsilon+\delta_{s,t}) \\ &= r\lambda_i + cr + 2\beta(r)(\varepsilon+\delta_{s,t}), \quad r > 0. \end{aligned} \tag{2.28}$$

Therefore,

$$\lambda_n \ge \lambda_i \ge \sup_{r>0}\frac{1}{r}\left[1-cr-2\beta(r)(\varepsilon+\delta_{s,t})\right]. \tag{2.29}$$

Combining this with (2.26), we prove (2.20). □

To estimate $p_t^0(x,x)$, we assume that

$$(\mathrm{Ric}-\mathrm{Hess}_V)(X,X) \ge -K|X|^2, \quad X \in TM, \tag{2.30}$$

for some $K \ge 0$. By the dimension-free Harnack inequality obtained in [25], we have (cf. [29, page 277])

$$p_t^0(x,x) \le \frac{1}{\mu(B_o(r))} \exp\left[\frac{K(\rho(o,x)+r)^2}{1-\mathrm{e}^{-Kt}}\right], \quad r>0,\ t>0,\ o,x \in M. \tag{2.31}$$

Therefore, there exist $c_1, c_2 > 0$ such that $A_{s,1} = \{x : p_1^0(x,x) \le s\} \supset \{x : \rho(x) \le c_1\sqrt{\log s} - c_2\}$ for all $s > 1$.

Corollary 2.5. *Assume that μ is a probability measure and (2.30) holds. If $\underline{R} \ge -c$ and (2.19) holds, then*

$$\lambda_n \ge \sup_{\varepsilon\in(0,1),s>1} \left\{\left(\log\frac{n\varepsilon\mathrm{e}^{-c}}{sl}\right) \wedge \sup_{r>0}\frac{1}{r}\left[1-cr-2\beta(r)\left(\varepsilon+\mu\left(\rho > c_1\sqrt{\log s}-c_2\right)\right)\right]\right\} \tag{2.32}$$

for some $c_1, c_2 > 0$ and all $n \ge 1$. Consequently, if, in addition, (2.19) holds for $\beta(r) = \exp[\alpha(1+r^{-1/\delta})]$ for some $\alpha, \delta > 0$, then $\lambda_n \ge \lambda([\log n - \theta]^+)^\delta$ for some $\lambda, \theta > 0$ and all $n \ge 1$.

Proof. By Theorem 2.4, it suffices to check the second assertion for $\delta \in (0,1]$. By [28, Corollary 6.3], there exists $\alpha_1, \alpha_2 > 0$ such that $\mu(\rho \ge c_1\sqrt{\log s} - c_2) \le \alpha_1 \exp[-\alpha_2(\log s)^{1/(2-\delta)}]$ for all $s > 1$. Then the proof is completed by some simple calculations. □

Now we come back to estimate $b(\tilde{L})$ by using (2.18).

Corollary 2.6. *Assume that μ is a probability measure and $\underline{R} \ge -c$ for some $c \in \mathbb{R}$. Then*

$$b(\tilde{L}) \le \inf_{t>0} \mathrm{e}^{ct} l \int_M p_t^0(x,x)\mu(\mathrm{d}x). \tag{2.33}$$

If (2.30) holds, then

$$b(\tilde{L}) \le \inf_{t>0,r>0} \frac{l\mathrm{e}^{ct}}{\mu(B_o(r))} \int_M \exp\left[\frac{K(\rho(x)+r)^2}{1-\mathrm{e}^{-Kt}}\right]\mu(dx). \tag{2.34}$$

If, in particular, M is compact, then

$$b(\tilde{L}) \le l \inf_{t>0} \exp\left[ct + \frac{KD^2}{1-\mathrm{e}^{-Kt}}\right]. \tag{2.35}$$

Proof. Assume that the right-hand side of (2.33) is finite; by Theorem 2.3 one has $\bar{\lambda} = \infty$ and hence $b(\tilde{L}) < \infty$. Let $n = b(\tilde{L})$, we have $\lambda_n = 0$. Then (2.33) follows from (2.18). Moreover, (2.34) follows from (2.33) and (2.31). Finally, by (2.31) with $r = D$ and $o = x$, we obtain

$$p_t^0(x,x) \le \exp\left[\frac{KD^2}{1-\mathrm{e}^{-Kt}}\right], \quad x \in M. \tag{2.36}$$

Then (2.35) follows from (2.33). □

To conclude this section, we consider the following two examples on noncompact manifolds.

Example 2.7. Let M be noncompact and $\mu = \mathrm{e}^V \mathrm{d}x$ a probability measure. If there exists $\alpha > 0$ such that $\overline{\lim}_{\rho\to\infty}(\Delta + \nabla V)\rho \le -\alpha$, where the limit is taken outside of the cut locus of o, and $\underline{\lim}_{\rho\to\infty}\underline{R} > -\alpha^2/4$, then $0 \notin \sigma_{\mathrm{ess}}(\tilde{L})$ and hence $b(\tilde{L}) < \infty$. Indeed (see, e.g., [27, (2.8)]), we have $\lim_{n\to\infty}\inf\sigma(-(\Delta+\nabla V)|_{B_o(n)^c}) \ge \alpha^2/4$. By Donnelly-Li's decomposition principle [10], we have $\inf\sigma_{\mathrm{ess}}(-(\Delta+\nabla V)) \ge \alpha^2/4$. Therefore, by Theorem 2.3(3), $0 \notin \sigma_{\mathrm{ess}}(\tilde{L})$ provided $\underline{\lim}_{\rho\to\infty}\underline{R} > -\alpha^2/4$.

Example 2.8. Let M be noncompact with Ricci curvature bounded from below and $\delta > 1$ a constant. Let $V = -\alpha\tilde{\rho}$ for some $\alpha > 0$ with $\tilde{\rho} \in C^\infty(M)$ such that $\rho^\delta - \tilde{\rho}$ is bounded and μ is a probability measure, where ρ is as above. The existence of $\tilde{\rho}$ is guaranteed by a classical approximation theorem and the volume comparison theorem. By [28, Corollary 2.5], (2.19) holds with $\beta(r) = \exp[c_1(1+r^{-\delta/[2(\delta-1)]})]$ for some $c_1 > 0$. Therefore by Theorem 2.4, if $\underline{R}$ is bounded from below, then there exist $\lambda, \theta > 0$ such that

$$\lambda_n \ge \lambda([\log n - \theta]^+)^{2(\delta-1)/\delta}, \quad n \ge 1, \tag{2.37}$$

provided $\delta > 2$. If, in addition, (2.30) holds, then (2.37) holds for all $\delta > 1$.

3. Proofs of Theorems 1.2 and 1.4

Proof of Theorem 1.2. Obviously, we may assume that μ is a probability measure since it is finite. Let $x \in M$ be fixed. For any $y \in M$, let $u(y) = p_{D^2}^0(x,y)$. We have $p_{D^2(1+t)}^0(y,x) = P_{D^2t}^0 u(y)$. Since, by (2.36), $\|u\|_\infty \le \exp[KD^2/(1-\mathrm{e}^{-KD^2})]$, applying [26, Theorem 4.4] with $\lambda = 0$ we obtain

$$\begin{aligned}\left|\nabla p_{D^2(t+1)}^0(\cdot,x)\right|(y) &\le \frac{1}{4D^2t}\exp\left[\frac{KD^2}{1-\mathrm{e}^{-KD^2}}\right]\int_0^D \exp\left[\frac{Kr^2}{8}\right]dr \\ &\le \frac{1}{4Dt}\exp\left[KD^2\left(\frac{1}{8}+\frac{1}{1-\mathrm{e}^{-KD^2}}\right)\right].\end{aligned} \tag{3.1}$$

Since $\int_M p^0_{D^2(t+1)}(y,x)\mu(\mathrm{d}y) = 1$, there exists $y \in M$ such that $p^0_{D^2(t+1)}(y,x) \le 1$. We obtain

$$p^0_{D^2(t+1)}(x,x) \le 1 + \frac{1}{4t}\exp\left[KD^2\left(\frac{1}{8} + \frac{1}{1-\mathrm{e}^{-KD^2}}\right)\right]. \tag{3.2}$$

Then (1.4) follows from (2.33).

If $D^2 \inf \underline{R} > -\eta$, then there exist $t_0 > 0$ and $\varepsilon > 0$ such that

$$D^2 \inf \underline{R} \ge \varepsilon - \frac{1}{t_0+1}\log\frac{l+1}{l(1+(1/4t_0)\exp[KD^2(1/8+1/(1-\mathrm{e}^{-KD^2}))])}. \tag{3.3}$$

Therefore, we may apply (1.4) when $t = t_0$ and

$$c := -\frac{\varepsilon}{D^2} + \frac{1}{D^2(t_0+1)}\log\frac{l+1}{l(1+(1/4t_0)\exp[KD^2/8+KD^2/(1-\mathrm{e}^{-KD^2})])} \tag{3.4}$$

to obtain

$$b(\tilde{L}) \le l\,\mathrm{e}^{-\varepsilon(1+t_0)}\frac{l+1}{l} < l+1. \tag{3.5}$$

This implies $b(\tilde{L}) \le l$ since $b(\tilde{L}) \in \mathbb{Z}_+$. Moreover, taking $t = l\exp[KD^2/8 + KD^2/(1-\mathrm{e}^{KD^2})]$, we obtain the desired lower bound of η from its definition.

Finally, let $\Omega = \Lambda^1$ and $\tilde{L} = -\Delta^1_\mu$; we have $R = \mathrm{Ric} - \mathrm{Hess}_V$ (cf. (3.7) and (3.10)) and $l = d$. Let η be defined by (1.5); we have $\eta < \log((d+1)/d)$. If $-KD^2 := D^2 \inf \underline{R} \ge -\eta$, then $KD^2 < \log((d+1)/d)$ and hence

$$\begin{aligned}\eta &> \frac{\log[(d+1)/(d+1/4)]}{1+d\exp[(\log((d+1)/d))(1/8+1+d)]} \\ &= \frac{\log[(d+1)/(d+1/4)]}{1+d(1+d^{-1})^{d+9/8}} =: \eta'.\end{aligned} \tag{3.6}$$

Therefore, if $D^2 \inf \underline{R} \ge -\eta'$, then $D^2 \inf \underline{R} > -\eta$ and hence $b(-\Delta^1_\mu) \le d$. Then the proof is completed by noting that $b_1 = b(-\Delta^1_\mu)$ since M is compact (see [7, Theorem 5.12]). □

To prove Theorem 1.4, we fix $p \in [0,d] \cap \mathbb{Z}_+$, and let $\Omega = \Lambda^p$ be the exterior p-bundle over the oriented manifold M. We have $l = d!/(p!(d-p)!)$. Consider $-\tilde{L} = \Delta^p_\mu := \mathrm{d}^*_\mu \mathrm{d} + \mathrm{d}\mathrm{d}^*_\mu$. We have (see [7]) $\mathrm{d}^*_\mu = \delta - \mathrm{i}_{\nabla V}$ and hence

$$\Delta^p_\mu = \Delta^p - \mathrm{L}_{\nabla V} = -\Box - \mathrm{L}_{\nabla V} + \mathcal{R}, \tag{3.7}$$

where $\Delta^p := \delta\mathrm{d}+\mathrm{d}\delta$ and $\mathcal{R}$ denote, respectively, the usual Hodge Laplacian and the curvature term involved in the Weitzenböck formula on p-forms, and $\mathrm{L}_X = \mathrm{d}\mathrm{i}_X + \mathrm{i}_X\mathrm{d}$ is the Lie differentiation in the direction X. Moreover, $(\Delta^p_\mu, \Gamma_0(\Lambda^p))$ is essentially selfadjoint, see, for example, [7, page 692].

Let $\{E_j\}_{j=1}^d$ be a locally normal frame with dual $\{\omega_j\} \subset \Lambda^1$. We have, for any differential form ϕ,

$$\begin{aligned}
\mathrm{i}_{\nabla V}\mathrm{d}\phi &= \sum_{i,j=1}^d \mathrm{i}_{\langle\nabla V, E_i\rangle E_i}(\omega_j \wedge \nabla_{E_j}\phi) = \nabla_{\nabla V}\phi - \sum_{i,j=1}^d \langle \nabla V, E_i\rangle \omega_j \wedge (\mathrm{i}_{E_i}\nabla_{E_j}\phi),\\
\mathrm{d}\mathrm{i}_{\nabla V}\phi &= \sum_{i,j=1}^d \mathrm{Hess}_V(E_i,E_j)\omega_i \wedge (\mathrm{i}_{E_j}\phi) + \sum_{i,j=1}^d \langle \nabla V, E_j\rangle \omega_i \wedge (\mathrm{i}_{E_j}\nabla_{E_i}\phi).
\end{aligned} \tag{3.8}$$

Therefore,

$$\mathrm{L}_{\nabla V} = \sum_{i,j=1}^d \mathrm{Hess}_V(E_i,E_j)\omega_i \wedge \mathrm{i}_{E_j} + \nabla_{\nabla V} := \mathrm{Hess}_V + \nabla_{\nabla V}. \tag{3.9}$$

Combining this with (3.7), we obtain

$$\Delta^p_\mu = -\Box - \nabla_{\nabla V} + \mathcal{R} - \mathrm{Hess}_V. \tag{3.10}$$

Let $C^{\infty,p}_\mu := \cap_{n=1}^\infty \mathcal{D}((\Delta^p_\mu)^n)$. By [7, Theorem 5.3], $C^{\infty,p}_\mu \subset \Gamma(\Lambda^p)$. Let $\mathrm{d}|_{C^{\infty,p}_\mu}$ (resp., $\mathrm{d}^*_\mu|_{C^{\infty,p}_\mu}$) denote the restriction of d (resp., d^*_μ) on $C^{\infty,p}_\mu$. We have the following general result.

Theorem 3.1. *Let $R = \mathcal{R} - \mathrm{Hess}_V$. If either*

$$\inf \sigma(-\Delta - \nabla V + \underline{R}) > 0 \quad \textit{or} \quad \underline{\lim}_{\rho\to\infty}\underline{R} > \sup\sigma_{\mathrm{ess}}(\Delta + \nabla V), \tag{3.11}$$

where we put $\sup\varnothing = -\infty$ as usual, then $\mathrm{im}\,\mathrm{d}^{p-1}$ is closed and

$$\begin{aligned}
L^2_{\Lambda^p}(\mu) &= \mathrm{im}\,(\mathrm{d}^*_\mu)^{p+1}|_{\mathcal{D}((\mathrm{d}^*_\mu)^{p+1})} \oplus \mathrm{im}\,\mathrm{d}^{p-1}|_{\mathcal{D}(\mathrm{d}^{p-1})} \oplus \ker\Delta^p_\mu|_{\mathcal{D}(\Delta^p_\mu)},\\
C^{\infty,p}_\mu &= \mathrm{im}\,\mathrm{d}^*_\mu|_{C^{\infty,p+1}_\mu} \oplus \mathrm{im}\,\mathrm{d}|_{C^{\infty,p-1}_\mu} \oplus \ker\Delta^p_\mu|_{\mathcal{D}(\Delta^p_\mu)},\\
\ker\Delta^p_\mu|_{\mathcal{D}(\Delta^p_\mu)} &\cong \frac{\ker\mathrm{d}^p|_{\mathcal{D}(\mathrm{d}^p)}}{\mathrm{im}\,\mathrm{d}^{p-1}|_{\mathcal{D}(\mathrm{d}^{p-1})}} \cong \frac{\ker\mathrm{d}|_{C^{\infty,p}_\mu}}{\mathrm{im}\,\mathrm{d}|_{C^{\infty,p-1}_\mu}}.
\end{aligned} \tag{3.12}$$

Proof. By Theorems 2.1(1) and 2.3(3), each condition in Theorem 3.1 implies $0 \notin \sigma_{\mathrm{ess}}(\Delta^p_\mu)$. Then the proof is completed by some classical results (see, e.g., [7, Theorem 5.10, Corollary 5.11] and [8, Corollary 10]). □

Proof of Theorem 1.4. We have

$$(\Delta - \nabla U)\,\mathrm{e}^{\varepsilon U} = \varepsilon[\Delta U - |\nabla U|^2 + \varepsilon|\nabla U|^2]\,\mathrm{e}^{\varepsilon U}, \quad \varepsilon > 0. \tag{3.13}$$

If (1.9) holds, then there exists $\varepsilon \in (0,1)$ such that $\Delta U - (1-\varepsilon)|\nabla U|^2 \to -\infty$ as $U \to \infty$. By the proof of Theorem 1.2 in [25] (or the paragraph after it), we have

$$\begin{aligned} &\inf\left\{\frac{\int_M |\nabla u|^2 \mathrm{e}^{-U}\,\mathrm{d}x}{\int_M u^2 \mathrm{e}^{-U}\,\mathrm{d}x} : 0 \neq u \in C_0^\infty(M),\ u = 0 \text{ on } \{U \le n\}\right\} \\ &\qquad \ge \varepsilon \inf_{U \ge n}\left[(1-\varepsilon)|\nabla U|^2 - \Delta U\right] \longrightarrow \infty \end{aligned} \tag{3.14}$$

as $n \to \infty$. Since $U + V$ is bounded, we have $\inf \sigma(-(\Delta + \nabla V)|_{\{U>n\}}) \to \infty$ as $n \to \infty$. Therefore, $\sigma_{\mathrm{ess}}(\Delta + \nabla V) = \varnothing$ by Donnelly-Li's decomposition principle, and hence $\alpha > 0$. The proof is completed by Theorem 3.1. □

Finally, we would like to introduce one more example to check Theorem 3.1, for which Theorem 3.1 applies but Theorem 1.4 (hence [1, Theorem 5.1]) does not.

Example 3.2. Let M be oriented noncompact with a pole o, and ρ the Riemannian distance function to o. Assume that the Ricci curvature is bounded below by $-k$ for some $k \ge 0$, and the sectional curvatures are nonpositive. Take $V \in C^2(M)$ such that $V = -c_1\rho^\delta + c_2$ outside a neighborhood of o, where $c_1 > 0$, $\delta \ge 1$, and let $c_2 \in \mathbb{R}$ be such that μ is a probability measure. By Hessian comparison theorem, we have $\limsup_{\rho\to\infty} \mathrm{Hess}_V \le 0$. Moreover, by Laplacian comparison theorem, one has $\limsup_{\rho\to\infty} L\rho \le \sqrt{k(d-1)} - c_1$ for $\delta = 1$ but $\limsup_{\rho\to\infty} L\rho = -\infty$ for $\delta > 1$, where $L := \Delta + \nabla V$. Then, by Example 2.7 and Theorem 3.1, we have $0 \notin \sigma_{\mathrm{ess}}(\Delta_\mu^p)$ and hence the decompositions in (3.12) hold provided at least one of the following is fulfilled:

(1) $\delta > 1$ and $\mathcal{R}$ is bounded below.
(2) $\delta = 1$ and $\liminf_{\rho\to\infty} \underline{\mathcal{R}} > -(1/4)(c_1 - \sqrt{k(d-1)})^{+2}$.

Appendix

Spectrum estimates on Hilbert bundles. Let $(E, \mathcal{F}, \mu)$ be a complete measure space and $H := \{(H_x, \langle\cdot,\cdot\rangle_x) : x \in E\}$ a family of separable real Hilbert spaces (i.e., a Hilbert bundle over E). Assume that there is a (possibly finite) sequence $\{e_j\} \subset \prod_{x\in E} H_x$ such that for μ-a.e. $x \in E$, $\{e_j(x)\}$ is an orthonormal basis in H_x. Set

$$\mathcal{M} = \left\{f \in \prod_{x\in E} H_x : \langle f, \mathrm{e}_j\rangle \text{ is } \mathcal{F}\text{-measurable for all } j\right\}, \tag{A.1}$$

where $\langle f, \mathrm{e}_j\rangle(x) := \langle f(x), e_j(x)\rangle_x$. We call $\mathcal{M}$ the space of $\mathcal{F}$-measurable sections of H.

For $p \ge 1$, let $L_H^p(\mu) = \{f \in \mathcal{M} : |f| \in L^p(\mu)\}$, where $L^p(\mu)$ denotes the L^p-space of real-valued functions. As usual, we denote $\mu(u) = \int_E u\,\mathrm{d}\mu$ for $u \in L^1(\mu)$ and regard $f = g$ in $L_H^p(\mu)$ provided $f = g$ μ-a.e. Then $L_H^2(\mu)$ is a Hilbert space with the inner product $\langle f, g\rangle_{L_H^2(\mu)} := \mu(\langle f, g\rangle)$. We refer to [24] for more information on Hilbert bundles.

Recall that $A \subset L_H^p(\mu)$ is said to be L^p-uniformly integrable if

$$\lim_{r\to\infty} \sup\{\mu(|f|^p 1_{\{|f|>r\}}) : f \in A\} = 0. \tag{A.2}$$

A linear operator P on $L^p_H(\mu)$ is called L^p-uniformly integrable if $\{Pf : \mu(|f|^p) \le 1\}$ is so. Moreover, denote by $\sigma(P)$ and $\sigma_{\mathrm{ess}}(P)$, respectively, the spectrum and the essential spectrum of a linear operator P.

Let $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ be a positive-definite symmetric closed form on $L^2_H(\mu)$, and let $\tilde{P}_t$ and $(\tilde{L}, \mathscr{D}(\tilde{L}))$ denote, respectively, the associated contraction semigroup and its generator. It is well known that $\tilde{L}$ is selfadjoint and negative defined on $L^2_H(\mu)$, and (cf. [17] or [18])

$$\begin{aligned} &\tilde{\mathscr{E}}(f,g) = -\mu(\langle f, \tilde{L}g\rangle), \quad f \in \mathscr{D}(\tilde{\mathscr{E}}),\ g \in \mathscr{D}(\tilde{L}), \\ &\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}_t f = \tilde{L}\tilde{P}_t f = \tilde{P}_t \tilde{L} f, \quad t \ge 0,\ f \in \mathscr{D}(\tilde{L}). \end{aligned} \tag{A.3}$$

We will study the essential spectrum of $\tilde{L}$ by using the following Poincaré-type inequality:

$$\mu(|f|^2) \le r\tilde{\mathscr{E}}(f,f) + \beta(r)\mu(|f|)^2, \quad f \in \mathscr{D}(\tilde{\mathscr{E}}),\ r > r_0, \tag{A.4}$$

where $r_0 \ge 0$ is a constant and β is a positive function defined on (r_0, ∞). We may assume that β in (A.4) is decreasing since the inequality remains true with β replaced by $\bar{\beta}(r) := \inf\{\beta(s) : s \in (r_0, r]\}$ for $r > r_0$. A key step of the study is the following lemma, which extends Lemma 3.1 in [14] and hence an earlier result due to Wu [30, 31].

Lemma A.1. *Assume that μ is a probability measure and $p \ge 1$ is fixed. Let P be a bounded linear operator on $L^p_H(\mu)$ with transition density $p(x,y)$, that is, for μ-a.e. $x, y \in E$, $p(x,y) : H_y \to H_x$ is a bounded linear operator such that*

$$Pf(x) = \int_E p(x,y) f(y) \mu(\mathrm{d}y), \quad f \in L^p_H(\mu). \tag{A.5}$$

Suppose that for μ-a.e. $x \in E$,

$$\sum_j \left(\int_E |p(x,y)^* e_j(x)|_y \mu(\mathrm{d}y) \right)^2 < \infty, \tag{A.6}$$

where $p(x,y)^*$ is the adjoint operator of $p(x,y)$. If $\{Pf : \|f\|_\infty \le 1\}$ is L^p-uniformly integrable, then $P(A) := \{Pf : f \in A\}$ is relatively compact in $L^p_H(\mu)$, for any L^p-uniformly integrable set $A \subset L^p_H(\mu)$.

Proof. We will use the following Bourbaki theorem (see [6, page 112]). *"A bounded set in the dual space B' of a separable Banach space B is compact and metrisable with respect to the weak topology $\sigma(B', B)$."* If $P(A)$ is not relatively compact in $L^p_H(\mu)$, then there exist $\varepsilon > 0$ and a sequence $\{f_n\}_{n=1}^\infty \subset A$ such that $\|Pf_n - Pf_m\|_{L^p_H(\mu)} \ge \varepsilon$, $n \ne m$. Since P is bounded and A is L^p-uniformly integrable, we may take $K > 0$ such that

$$\|Pf_{n,K} - Pf_{m,K}\|_{L^p_H(\mu)} \ge \varepsilon/2, \quad n \ne m, \tag{A.7}$$

where $f_{n,K} = f_n 1_{\{|f_n|\le K\}}$. We fix a version of f_n for each n and let $\mathscr{F}'$ be the μ-completion of the σ-field $\sigma(\{\langle f_{n,K}, \mathrm{e}_j\rangle : n, j \ge 1\})$ which is μ-separable. Let

$$\mathscr{M}' = \left\{ f \in \prod_{x\in E} H_x : \langle f, \mathrm{e}_j\rangle \text{ is } \mathscr{F}'\text{-measurable for all } j \ge 1 \right\} \subset \mathscr{M}. \tag{A.8}$$

Let $L_H^p(\mu)'$ be defined as $L_H^p(\mu)$ for $\mathscr{F}'$ and $\mathscr{M}'$ in place of $\mathscr{F}$ and $\mathscr{M}$, respectively, which is separable since $\mathscr{F}'$ is μ-separable. For $f \in L_H^1(\mu)$, let $\mu(f|\mathscr{F}') := \sum_j \mu(\langle f, e_j\rangle|\mathscr{F}')e_j$, where $\mu(\cdot|\mathscr{F}')$ is the conditional expectation with respect to μ under $\mathscr{F}'$. By Bourbaki theorem with $B = L_H^1(\mu)'$, there exists $f \in L_H^\infty(\mu)'$ such that $f_{n_i,K} \to f$ weakly for some $n_i \uparrow \infty$, that is, for any $g \in L_H^1(\mu)'$, one has $\mu(\langle f_{n_i,K}, g\rangle) \to \mu(\langle f, g\rangle)$. Noting that for any $g \in L_H^1(\mu)$ and any $f' \in L_H^\infty(\mu)'$, one has $\mu(\langle f', g\rangle) = \mu(\langle \mu(g|\mathscr{F}'), f'\rangle)$, we obtain $\mu(\langle f_{n_i,K}, g\rangle) \to \mu(\langle f, g\rangle)$ for all $g \in L_H^1(\mu)$. Then for μ-a.e. x,

$$\begin{aligned}
&|Pf(x) - Pf_{n_i,K}(x)|_x^2 \\
&\quad = \sum_j \langle P_{n_i,K} f(x) - Pf(x), e_j(x)\rangle_x^2 \\
&\quad = \sum_j \left(\int_E \langle f_{n_i,K}(y) - f(y), p(x,y)^* e_j(x)\rangle_y \mu(\mathrm{d}y) \right)^2 \\
&\quad \le (K + \|f\|_\infty)^2 \sum_j \left(\int_E |p(x,y)^* e_j(x)|_y \mu(\mathrm{d}y) \right)^2.
\end{aligned} \tag{A.9}$$

By (A.6) and the dominated convergence theorem, we obtain, for μ-a.e. x,

$$\lim_{n_i\to\infty} |Pf(x) - Pf_{n_i,K}(x)|_x^2 = \sum_j \lim_{n_i\to\infty} \mu(\langle f_{n_i,K} - f, p(x,\cdot)^* e_j(x)\rangle)^2 = 0. \tag{A.10}$$

Since $\{|Pf_{n_i,K}|^p : i \ge 1\}$ is uniformly integrable and μ is a probability measure, $Pf_{n_i,K} \to Pf$ in $L_H^p(\mu)$. This is a contradiction to (A.7). □

Directly following an argument in [14] (see also [29, Theorem 3.1]), we obtain the following result.

Theorem A.2. *Assume that μ is a probability measure. If $\sigma_{\mathrm{ess}}(-\tilde{L}) \subset [r_0^{-1}, \infty)$ for some $r_0 \ge 0$, then (A.4) holds for some $\beta \in C(r_0, \infty)$. Conversely, if $\tilde{P}_t$ has transition density satisfying (A.6) for each $t > 0$, then (A.4) implies $\sigma_{\mathrm{ess}}(-\tilde{L}) \subset [r_0^{-1}, \infty)$.*

Next, we turn to estimate eigenvalues of $\tilde{L}$. Let $\bar{\lambda} = \inf \sigma_{\mathrm{ess}}(-\tilde{L})$, where we put $\inf \varnothing = \infty$. Assume that $\sigma(-\tilde{L}) \cap [0, \bar{\lambda}) \ne \varnothing$, where $\sigma(-\tilde{L})$ denotes the spectrum of $-\tilde{L}$. We list all eigenvalues of $-\tilde{L}$ (counting multiplicity) in $[0, \bar{\lambda})$ as follows: $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$. We will follow the line of [29] where the eigenvalues estimation was studied on real-valued function spaces.

Lemma A.3. *Let $\bar{\lambda}$ and $\{\lambda_n\}$ be as above. Assume that $\tilde{P}_t$ has transition density $\tilde{p}_t(x,y)$ and there exists $l \in \mathbb{N}$ such that $\dim H_x = l$, μ-a.e. x. Let $\{f_i\}$ be the family of normalized*

eigenvectors for $\{\lambda_i\}$. *Then*

$$l\int_E \|\tilde{p}_{t/2}(\cdot,y)\|_{\mathrm{op}}^2\mu(\mathrm{d}y) \geq \mathrm{e}^{-\lambda_n t}\sum_{i=1}^{n}|f_i|^2 \tag{A.11}$$

for any $n \leq \#\{i : \lambda_i < \bar{\lambda}\}$.

Proof. Let $x \in E$ be such that $l = \dim H_x$ and $\{e_j(x)\}$ is an orthonormal basis in H_x. The proof is then similar to that of Lemma 2.2. □

Theorem A.4. *In the situation of Lemma A.3, assume that* μ *is a probability measure. If there exists* $t > 0$ *such that*

$$C(t) := \int_{E\times E}\|\tilde{p}_{t/2}(x,y)\|_{\mathrm{op}}^2\mu(\mathrm{d}x)\mu(\mathrm{d}y) < \infty, \tag{A.12}$$

then $\sigma_{\mathrm{ess}}(\tilde{L}) = \emptyset$ *and hence* $\bar{\lambda} = \infty$; *equivalently, (A.4) holds for* $r_0 = 0$ *and some* $\beta \in C(0,\infty)$ *by Theorem A.2. Moreover,*

$$\lambda_n \geq \frac{1}{t}\log\frac{n}{lC(t)}, \quad n \geq 1. \tag{A.13}$$

Consequently, $\#\{i : \lambda_i \leq \lambda\} \leq lC(t)\,\mathrm{e}^{\lambda t}$ *for each* $\lambda \geq 0$.

Proof. If $C(t) < \infty$, then $\tilde{P}_{t/2}$ is L^2-uniformly integrable. By Lemma A.1, $\tilde{P}_{t/2}$ is compact and hence $\sigma_{\mathrm{ess}}(\tilde{L}) = \emptyset$. Next, by (A.11) we obtain $lC(t) \geq n\,\mathrm{e}^{-\lambda_n t}$. □

A natural way to apply the above results is to compare $\tilde{\mathscr{E}}$ and $\tilde{p}_t$ with the correspondences on the L^2-space of functions, so that known results on functions can be used. The next result is implied by [3, Theorem 16] (see also [23] for related results), and the final one is a result on the existence of transition density for operators on $L^2_H(\mu)$.

Theorem A.5. *Let* $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ *be a symmetric closed form on* $L^2(\mu)$ *which is bounded from below. Assume that the associated semigroup* P_t *is positivity-preserving. Then the following two statements are equivalent:*

(1) $|\tilde{P}_t f| \leq P_t|f|$ *for all* $t > 0$ *and* $f \in L^2_H(\mu)$;
(2) *if* $f \in \mathscr{D}(\tilde{\mathscr{E}})$, *one has* $|f| \in \mathscr{D}(\mathscr{E})$ *and* $\mathscr{E}(|f|,|g|) \leq |\tilde{\mathscr{E}}(f,g)|$ *for all* f, g *in a core of* $\tilde{L}$ *such that* $\langle f,g\rangle_{L^2_H(\mu)} = \mu(|f|\cdot|g|)$.

Proposition A.6. *Assume that* μ *is a probability measure. Let* $\tilde{P}$ *be a bounded linear operator on* $L^2_H(\mu)$ *and* p *a nonnegative measurable function on* $E\times E$ *such that*

$$Pu = \int p(\cdot,y)u(y)\mu(\mathrm{d}y) \tag{A.14}$$

provides a bounded linear operator P *on* $L^2(\mu)$. *If* $l := \#\{e_j\} < \infty$ *and* $|\tilde{P}f| \leq P|f|$ *for any* $f \in L^2_H(\mu)$, *then* $\tilde{P}$ *has transition density* $\tilde{p}$ *satisfying* $\|\tilde{p}(x,y)\|_{\mathrm{op}} \leq lp(x,y)$.

If, in addition, E *is a metric space and for each* $x \in E$, $\tilde{p}(x,\cdot)$ *and* $p(x,\cdot)$ *are continuous and locally bounded on the support of* μ, *that is, for any* y *in the support of* μ *and any unit* $\omega \in H_y$, *there exists* $e \in \mathscr{M}$ *with* $e(y) = \omega$ *such that, in a neighborhood of* y, *one has* $|e| = 1$ *and* $\tilde{p}(x,\cdot)e(\cdot)$, $p(x,\cdot)$ *are bounded and continuous, then* $\|\tilde{p}(x,y)\|_{\mathrm{op}} \leq p(x,y)$.

Proof. For any $i,j \in \mathbb{N}$ with $i,j \le l$, let μ^{ij} be a set function on $\mathcal{F}\times\mathcal{F}$ defined by

$$\mu^{ij}(A) = \int_{A_1} \langle e_i(x), \tilde{P}(1_{A(x)}e_j)(x)\rangle_x \mu(\mathrm{d}x), \quad A \in \mathcal{F}\times\mathcal{F}, \tag{A.15}$$

where $A_1 = \{x \in E : \text{there exists } y \in E \text{ such that } (x,y) \in A\}$ and $A(x) = \{y \in E : (x,y) \in A\}$. Since $\tilde{P}$ is bounded, it is easy to check that μ^{ij} is a signed measure. Moreover, by Jordan's decomposition theorem and that $|\tilde{P}f| \le P|f|$ for any $f \in L^2(\mu)$, we have $|\mu^{ij}| := (\mu^{ij})^+ + (\mu^{ij})^- \le p\mu\times\mu$. Then μ^{ij} is absolutely continuous with respect to $\mu\times\mu$ with density p^{ij} satisfying $|p^{ij}| \le p$. Define $\tilde{p}(x,y) : H_y \to H_x$ by

$$\tilde{p}(x,y)\omega = \sum_{i,j=1}^{l} p^{ij}(x,y)\langle \omega, e_j(y)\rangle_y e_i(x), \quad \omega \in H_y. \tag{A.16}$$

It is easy to check that $\|\tilde{p}(x,y)\|_{\mathrm{op}} \le lp(x,y)$ for any $x,y \in E$ and $\tilde{p}$ is a transition density of $\tilde{P}$. Indeed, for any $f,g \in L^2_H(\mu)$, we have

$$\begin{aligned}
\langle g, \tilde{P}f\rangle_{L^2_H(\mu)} &= \sum_{i,j=1}^{l} \int_E \langle \tilde{P}(\langle f, e_j\rangle e_j), \langle g, e_i\rangle e_i\rangle \mathrm{d}\mu \\
&= \sum_{i,j=1}^{l} \int_{E\times E} \langle f, e_j\rangle(y)\langle g, e_i\rangle(x)\mu^{ij}(\mathrm{d}x\mathrm{d}y) \\
&= \sum_{i,j=1}^{l} \int_{E\times E} p^{ij}(x,y)\langle f, e_j\rangle(y)\langle g, e_i\rangle(x)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\
&= \left\langle g, \int_E \tilde{p}(\cdot,y)f(y)\mu(\mathrm{d}y)\right\rangle_{L^2_H(\mu)}.
\end{aligned} \tag{A.17}$$

Next, let the additional conditions hold. For $x \in E$, y in the support of μ, and any $\omega \in H_y$ with $|\omega|_y = 1$, let $e \in \mathcal{M}$ be such that $e(y) = \omega$, $|e| = 1$, and $\tilde{p}(x,\cdot)e(\cdot)$ and $p(x,\cdot)$ are bounded and continuous in a neighborhood N_y of y. Let $\{f_n\}$ be a sequence of nonnegative continuous functions with supports contained in N_y such that $f_n\mu \to \delta_y$ weakly as $n \to \infty$. We have

$$\begin{aligned}
\left|\int_E [\tilde{p}(x,z)e(z)]f_n(z)\mu(\mathrm{d}z)\right|_x &= |\tilde{P}(f_n e)(x)|_x \le Pf_n(x) \\
&= \int_E p(x,z)f_n(z)\mu(\mathrm{d}z).
\end{aligned} \tag{A.18}$$

By letting $n \to \infty$, we obtain $|\tilde{p}(x,y)\omega|_x \le p(x,y)$. Therefore, $\|\tilde{p}\|_{\mathrm{op}} \le p$ since we may take $\tilde{p}(x,\cdot) = 0$ outside of the support of μ. □

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