SKEW-SYMMETRIC VECTOR FIELDS ON A CR-SUBMANIFOLD OF A PARA-KÄHLERIAN MANIFOLD

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We deal with a CR-submanifold M of a para-Kählerian manifold $\widetilde{M}$, which carries a J -skew-symmetric vector field X . It is shown that X defines a global Hamiltonian of the symplectic form Ω on $M^\top$ and JX is a relative infinitesimal automorphism of Ω . Other geometric properties are given.

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1. Introduction. CR-submanifolds M of some pseudo-Riemannian manifolds $\widetilde{M}$ have been first investigated by Rosca [10], and also studied in [2, 3, 11].

If $\widetilde{M}$ is a para-Kählerian manifold, it has been proved that any coisotropic submanifold M of $\widetilde{M}$ is a CR-submanifold (such CR-submanifolds have been denominated CICR-submanifolds [6]).

In this note, one considers a foliate CICR-submanifold M of a para-Kählerian manifold $\widetilde{M}(J, \widetilde{\Omega}, \widetilde{g})$. It is proved that the necessary and sufficient condition in order that the leaf $M^\top$ of the horizontal distribution $D^\top$ on M carries a J -skew-symmetric vector field X , that is, $\nabla X = X \wedge JX$, is that the vertical distribution $D^\perp$ on M is autoparallel.

In this case, M may be viewed as the local Riemannian product $M = M^\top \times M^\perp$, where $M^\top$ is an invariant totally geodesic submanifold of M and $M^\perp$ is an isotropic totally geodesic submanifold.

Furthermore, if Ω is the symplectic form of $M^\top$, it is shown that X is a global Hamiltonian of Ω and JX is a relative infinitesimal automorphism of Ω (a similar discussion can be made for proper CR-submanifolds of a Kählerian manifold).

2. Preliminaries. Let $\widetilde{M}(J, \widetilde{\Omega}, \widetilde{g})$ be a $2m$ -dimensional para-Kählerian manifold, where, as is well known [7], the triple $(J, \widetilde{\Omega}, \widetilde{g})$ of tensor fields is the *paracomplex operator*, the *symplectic form*, and the *para-Hermitian metric tensor field*, respectively.

If $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$, these manifolds satisfy

$$J^2 = Id, \quad d\widetilde{\Omega} = 0, \quad (\widetilde{\nabla} J)\widetilde{Z} = 0, \quad \widetilde{Z} \in \Gamma T\widetilde{M}. \quad (2.1)$$

Let $x : M \rightarrow \widetilde{M}$ be the immersion of an l -codimensional submanifold M , $l < m$, in $\widetilde{M}$ and let $T_p^\perp M$ and $T_p M$ be the normal space and the tangent space at each point $p \in M$.

If $J(T_p^\perp M) \subset T_p M$, then M is said to be a *coisotropic* submanifold of $\widetilde{M}$ (see [2]). If $\widetilde{W} = \text{vect}\{h_a, h_{a^*}; a = 1, \dots, m, a^* = a + m\}$ is a real Witt vector basis on $\widetilde{M}$, one has

$$\tilde{g}(h_a, h_b) = \tilde{g}(h_{a^*}, h_{b^*}) = \delta_{ab}. \quad (2.2)$$

Next, if $\widetilde{W}^* = \{\omega^a, \omega^{a^*}\}$ denotes the associated cobasis of $\widetilde{W}$, then $\tilde{g}$ and $\tilde{\Omega}$ are expressed by

$$\tilde{g} = 2 \sum \omega^a \otimes \omega^{a^*}, \quad (2.3)$$

$$\tilde{\Omega} = \sum \omega^a \wedge \omega^{a^*}. \quad (2.4)$$

We recall also that $\widetilde{W}$ may split as

$$\widetilde{W} = \tilde{S} + \tilde{S}^*, \quad (2.5)$$

where the pairing $(\tilde{S}, \tilde{S}^*)$ defines an involutive automorphism of square 1, that is,

$$Jh_a = h_{a^*}, \quad Jh_{a^*} = h_a, \quad (2.6)$$

and the local connection forms $\tilde{\theta}_B^A \in \Lambda^1 \widetilde{M}$, $A, B \in \{1, 2, \dots, 2m\}$ satisfy

$$\tilde{\theta}_b^{a^*} = 0, \quad \tilde{\theta}_{b^*}^a = 0, \quad \tilde{\theta}_b^a + \tilde{\theta}_{a^*}^{b^*} = 0. \quad (2.7)$$

It has been proved in [10] that any coisotropic submanifold M of a para-Kählerian manifold $\widetilde{M}$ is a CR-submanifold of $\widetilde{M}$ and such a submanifold has been called a CICR-submanifold [6].

Let $D^\top : p \rightarrow D_p^\top = T_p M \setminus J(T_p^\perp M)$ and $D^\perp : p \rightarrow D_p^\perp = J(T_p^\perp M) \subset T_p M$ be the two complementary differentiable distributions on M . One has

$$JD_p^\top = D_p^\top, \quad JD_p^\perp = T_p^\perp M, \quad (2.8)$$

and $D^\top$ (resp., $D^\perp$) is called the *horizontal* (resp., *vertical*) *distribution* on M .

As in the Kählerian case, the vertical distribution $D^\perp$ is always involutive.

If M is defined by the Pfaffian system

$$\omega^r = 0, \quad r = 2m + 1 - l, \dots, 2m, \quad (2.9)$$

then one has

$$\begin{aligned} D_p^\top &= \{h_i, h_{i^*}, i = 1, \dots, m - l, i^* = i + m\}, \\ D_p^\perp &= \{h_r, r = m - l + 1, \dots, m\}. \end{aligned} \quad (2.10)$$

Further denote by

$$\varphi^\perp = \omega^{m-l+1} \wedge \dots \wedge \omega^m \quad (2.11)$$

the simple unit form which corresponds to $D^\perp$.

Then, in order that the distribution $D^\top$ be also involutive, it is necessary and sufficient that $\varphi^\perp$ be a conformal integral invariant of $D^\top$, that is,

$$\mathcal{L}_{D^\top} \varphi^\perp = f \varphi^\perp \quad (2.12)$$

for a certain scalar function f .

By a standard calculation, one derives that the above equation implies

$$\theta_i^r = 0, \quad (2.13)$$

and in this case, one may write

$$d\varphi^\perp = -\left(\sum \theta_r^r\right) \wedge \varphi^\perp, \quad (2.14)$$

that is, $\varphi^\perp$ is exterior recurrent.

In this case, as is known [2, 10], M is a foliated CR-submanifold of $\widetilde{M}$.

We will investigate now the case when the leaf $M^\top$ of $D^\top$ carries a J -skew-symmetric vector field X , that is,

$$\nabla X = X \wedge JX. \quad (2.15)$$

One may express ∇X as

$$\nabla X = (JX)^\flat \otimes X - X^\flat \otimes JX, \quad (2.16)$$

where

$$X = X^i h_i + X^{i*} h_{i*} = X^i \omega^{i*} + X^{i*} \omega^i. \quad (2.17)$$

Recalling Cartan structure equations [4],

$$\begin{aligned} \nabla h &= \theta \otimes e \in A^1(M, TM), \\ d\omega &= -\theta \wedge \omega, \\ d\theta &= -\theta \wedge \theta + \Theta. \end{aligned} \quad (2.18)$$

In the above equations, θ , respectively Θ , are the local connection forms in the bundle W , respectively the curvature forms on M .

Then making use of Cartan structure equations, one finds by a standard calculation that (2.16) implies that the vertical distribution $D^\perp$ is autoparallel, that is, $\nabla_{Z'} Z'' \in D^\perp$, for all $Z', Z'' \in D^\perp$, which, in terms of connection forms, is expressed by

$$\theta_r^i = 0. \quad (2.19)$$

We agree to call θ_r^i and θ_i^r the *mixed connection forms*.

Taking account of (2.13) and (2.19), one derives from (2.16)

$$dX^\flat = 2(JX)^\flat \wedge X^\flat, \quad (2.20)$$

which agrees with the general equation of skew-symmetric killing vector fields [5, 8].

Next, by (2.1), one has

$$\nabla JX = (JX)^{\flat} \otimes JX - X^{\flat} \otimes X, \quad (2.21)$$

which shows that JX is a gradient vector field.

Hence, we may state the following theorem.

THEOREM 2.1. *Let $x : M \rightarrow \widetilde{M}$ be an improper immersion of a CR-submanifold in a para-Kählerian manifold $\widetilde{M}(J, \widetilde{\Omega}, \widetilde{g})$ and let $D^{\top}$ (resp., $D^{\perp}$) be the horizontal distribution (resp., the vertical distribution) on M . If M is a foliate CR-submanifold, then the necessary and sufficient condition in order that the leaf $M^{\top}$ of $D^{\top}$ carries a J -skew-symmetric vector field X is that $D^{\perp}$ is an autoparallel foliation. In this case, the CR-submanifold M under consideration may be viewed as the local Riemannian product $M = M^{\top} \times M^{\perp}$, where $M^{\top}$ is an invariant totally geodesic submanifold of M and $M^{\perp}$ is an isotropic totally geodesic submanifold. In addition, in this case, JX is a gradient vector field.*

3. Properties. In this section, we will point out some additional properties of X involving the symplectic form Ω of $M^{\top}$ and the exterior covariant differential d^{∇} of ∇X . Operating on (2.16) and (2.21), one derives by a short calculation

$$\begin{aligned} d^{\nabla}(\nabla X) &= \nabla^2 X = 2(X^{\flat} \wedge (JX)^{\flat}) \otimes JX, \\ d^{\nabla}(\nabla JX) &= \nabla^2 JX = 2(X^{\flat} \wedge (JX)^{\flat}) \otimes X, \end{aligned} \quad (3.1)$$

which gives

$$\begin{aligned} \nabla^2(X + JX) &= 2(X^{\flat} \wedge (JX)^{\flat}) \otimes (X + JX), \\ \nabla^2(X - JX) &= -2(X^{\flat} \wedge (JX)^{\flat}) \otimes (X - JX). \end{aligned} \quad (3.2)$$

Therefore, we agree to define $X + JX$ and $X - JX$ as *2-covariant recurrent vector fields*. It should also be noticed that by reference to the general formula

$$\nabla_V(X_1 \wedge \cdots \wedge X_p) = \sum (X_1 \wedge \cdots \wedge \nabla_V X_j \wedge \cdots \wedge X_p), \quad V \in \Gamma TM, \quad (3.3)$$

one finds by (2.15) and (2.21)

$$\nabla_V(X \wedge JX) = 2g(V, JX)(X \wedge JX). \quad (3.4)$$

This shows that the covariant derivative of $X \wedge JX$ with respect to any vector field V is proportional to $X \wedge JX$.

On the other hand, by the general formula

$$\nabla^2 V(Z, Z') = R(Z, Z')V, \quad (3.5)$$

where R denotes the curvature tensor field and V, Z, Z' are vector fields, one has (see also [9])

$$\mathcal{R}(Z, V) = \text{Tr} R(\cdot, Z)V, \quad (3.6)$$

where $\mathcal{R}$ is the Ricci tensor field of ∇ .

Since in the case under consideration one must take in (3.6) the para-Hermitian trace, then setting in (3.6) $Z = V = X$, one finds

$$\mathcal{R}(X, X) = 0, \quad (3.7)$$

that is, the Ricci curvature of X vanishes.

Denote by $\tilde{\Omega}$ the symplectic form of $\tilde{M}$, then $\Omega = \tilde{\Omega}|_{M^\top}$ is a symplectic form of rank equal to the dimension of $M^\top$, that is, in our case, $2(m-l)$.

Then, if ${}^bZ : Z \rightarrow -i_Z\Omega$ is the symplectic isomorphism, by a short calculation and on behalf of (2.4), one gets

$${}^bX = -(JX)^\flat, \quad (3.8)$$

and since JX is a gradient vector field, we conclude according to a known definition (see also [1]) that X is a *global Hamiltonian* of Ω .

In a similar manner, one finds

$${}^b(JX) = X^\flat, \quad (3.9)$$

and by (2.20), it follows that

$$d(\mathcal{L}_{JX}\Omega) = 0, \quad (3.10)$$

which shows that JX is a relative infinitesimal automorphism of Ω [1].

We state the following theorem.

THEOREM 3.1. *Let M be a CR-submanifold of a para-Kählerian manifold $\tilde{M}$ and let Ω be the symplectic form on $M^\top$. If M carries a J -skew-symmetric vector field X , then the following properties hold:*

- (i) *X is a global Hamiltonian of Ω and JX is a relative infinitesimal automorphism of Ω ;*
- (ii) *the Ricci tensor field $\mathcal{R}(X, X)$ vanishes;*
- (iii) *the vector fields $X + JX$ and $X - JX$ are 2-covariant recurrent.*

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