POWERSUM FORMULA FOR DIFFERENTIAL RESOLVENTS

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We will prove that we can specialize the indeterminate α in a linear differential α -resolvent of a univariate polynomial over a differential field of characteristic zero to an integer q to obtain a q -resolvent. We use this idea to obtain a formula, known as the *powersum formula*, for the terms of the α -resolvent. Finally, we use the powersum formula to rediscover Cockle's differential resolvent of a cubic trinomial.

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1. Introduction. It was proved in [4, Theorem 37, page 67] that for any integer q , a polynomial $P(t) \equiv \sum_{k=0}^N (-1)^{N-k} e_{N-k} t^k$ of a single variable t whose coefficients $\{e_{N-k}\}_{k=0}^N$ lie in an ordinary differential ring $\mathbb{R}$ with derivation D possesses an ordinary linear differential α -resolvent and an ordinary linear differential q -resolvent, where α is a constant, transcendental over $\mathbb{R}$. With no loss of generality, we assume that P is monic and has no zero roots. Then the coefficient e_{N-k} is the $(N-k)$ th elementary symmetric function of the roots of P . We assume that $P(t) = \prod_{k=1}^n (t - z_k)^{\pi_k}$ has $n \leq N$ distinct roots and π_k is the multiplicity of the root z_k in P . It was proved in [2] that for each root z_k , there exists a nonzero solution y_k of the logarithmic differential equation $Dy_k/y_k = \alpha \cdot (Dz_k/z_k)$. Obviously, such solutions are unique only up to a constant multiple. We define the notation z_k^α to represent any such solution y_k . Hence, we will call y_k an α -power of z_k . From now on, we will drop the subscript k on z_k and y_k . It will be understood that a different z implies a different y .

In this paper, we present the powersum formula as a new method for computing resolvents, although it remains a conjecture whether the powersum formula always yields a (nonzero) resolvent rather than an identically zero equation. It was proved in [5, Theorem 4.1, page 726] that if all the distinct roots of a polynomial are differentially independent over constants, then the powersum formula yields a resolvent. It was shown in [6, Section 11, pages 344–345], how the solution of the Riccati nonlinear differential equation is related to the resolvent of a quadratic polynomial.

2. Notation. Let $\mathbb{N}$ denote the set of positive integers. Let $\mathbb{N}_0$ denote the set of non-negative integers. Let $\mathbb{Z}^\#$ denote the set of nonzero integers. The following notation has been slightly modified from Kolchin's notation in [2] and Macdonald's notation in [3]. Let $\mathbb{Z}\{e\}$ denote the differential ring generated by the integers $\mathbb{Z}$ and the N coefficients $e \equiv \{e_k\}_{k=1}^N$ of t in P . Let $\mathbb{Q}\langle e \rangle$ denote the differential field generated by the rational numbers $\mathbb{Q}$ and e . For each $m \in \mathbb{N}$, let $\mathbb{Z}\{e\}_m$ denote the ordinary (nondifferential) ring

generated by $\mathbb{Z}$, e , and the first m derivatives of e . (A differential ring must contain infinitely many derivatives of any of its elements.) Let $\mathbb{Q}\langle e \rangle_m(z) = \mathbb{Q}\langle e \rangle_m[z]$ denote the field generated by $\mathbb{Q}$, e , the first m derivatives of e , and the single root z . From this point on, we will write $\mathbb{Q}\langle e \rangle_m[z]$ instead of $\mathbb{Q}\langle e \rangle_m(z)$ for this field to emphasize the fact that elements in this field are polynomial in the root z . If $\mathbb{R}$ represents any of the rings or fields mentioned so far, then let $\mathbb{R}[t, \alpha]$ denote the polynomial ring in the indeterminates t and α over $\mathbb{R}$.

Let $\theta \equiv (n!) \cdot (\prod_{k=1}^n \pi_k) \cdot (\prod_{k=1}^n z_k) \cdot (\prod_{i < j} (z_i - z_j)^2)$. By our conditions on P , $\theta \neq 0$. It is also easy to show that $\theta \in \mathbb{Z}[e]$. For each $m \in \mathbb{N}$, it was proved in [4, Theorem 32, page 60] that there exists a polynomial $G_m(t, \alpha)$ in t and α satisfying the following definition.

DEFINITION 2.1. Define $G_m(t, \alpha)$ to be the polynomial in t and α such that $G_m(z, \alpha) = D^m \gamma / (\alpha \cdot \gamma)$ for each root z of P and $\theta^m \cdot G_m(t, \alpha) \in \mathbb{Z}\{e\}_m[t, \alpha]$.

A *specialization* ϕ is a ring homomorphism $\phi : \mathbb{R} \rightarrow \hat{\mathbb{R}}$ from a ring $\mathbb{R}$ into an integral domain $\hat{\mathbb{R}}$. For any polynomial $P(t) = \sum_{k=0}^N (-1)^{N-k} e_{N-k} \cdot t^k \in \mathbb{R}[t]$, $\phi(P)$ is defined to be the polynomial $(\phi P)(t) = \sum_{k=0}^N (-1)^{N-k} \phi(e_{N-k}) \cdot t^k \in \mathbb{R}[t]$. A *differential specialization* ϕ is a specialization $\phi : \mathbb{R} \rightarrow \hat{\mathbb{R}}$ from a differential ring $\mathbb{R}$ with derivation D into a differential integral domain $\hat{\mathbb{R}}$ with derivation $\hat{D}$ such that $\phi D = \hat{D} \phi$ on $\mathbb{R}$.

3. Specializing α . Let $q \in \mathbb{N}$. Let $\phi_q : \mathbb{Q}\langle e \rangle[z, \alpha] \rightarrow \mathbb{Q}\langle e \rangle[z]$ be the ring specialization such that ϕ_q is the identity on $\mathbb{Q}\langle e \rangle[z]$ and $\phi_q(\alpha) = q$. We may compute $Dz^q / (q \cdot z^q)$. Since ϕ_q is not defined to act on γ , we are not able to specialize γ to z^q in Theorem 3.1. However, ϕ_q is defined to act on $D^m \gamma / (\alpha \cdot \gamma)$ since $D^m \gamma / (\alpha \cdot \gamma) = G_m(z, \alpha) \in \theta^{-m} \cdot \mathbb{Z}\{e\}_m[z, \alpha]$. Theorem 3.1 asserts that $G_m(z, q) = D^m z^q / (q \cdot z^q)$. Theorem 3.2 asserts that $\phi_q(D^m \gamma / (\alpha \cdot \gamma)) = D^m z^q / (q \cdot z^q)$.

THEOREM 3.1. Assume all the same definitions and notations as in the introduction. Then the m th derivative of z^q can be expressed as a product of $q \cdot z^q$ and an element in $\mathbb{Q}\langle e \rangle_m[z]$. More specifically, $G_m(z, q) = D^m z^q / (q \cdot z^q)$, where $G_m(z, q) \in \mathbb{Q}\langle e \rangle_m[z]$ and $G_m(t, \alpha)$ was given in Definition 2.1.

PROOF. For brevity, write $G_m = G_m(z, \alpha)$ for the particular root z . We emphasize that G_m is $G_m(t, \alpha)$ with t specialized to the particular root z . We find that $\theta \cdot (Dz^q / (q \cdot z^q)) = \theta \cdot (Dz/z) = \theta \cdot G_1 \in \mathbb{Z}\{e\}_1[z]$. Therefore,

$$\begin{aligned} Dz^q &= q \cdot z^q \cdot G_1 \implies D^2 z^q = q \cdot (q \cdot z^{q-1} Dz \cdot G_1 + z^q \cdot DG_1) \\ &= q \cdot z^q \left(q \cdot \left(\frac{Dz}{z} \right) \cdot G_1 + DG_1 \right) = q \cdot z^q (q \cdot G_1^2 + DG_1) \\ &= q \cdot z^q \cdot \phi_q(\alpha \cdot G_1^2 + DG_1) = q \cdot z^q \cdot \phi_q(G_2). \end{aligned} \tag{3.1}$$

So, $D^m z^q = q \cdot z^q \cdot \phi_q(G_m)$ is true for $m = 1$. Now assume that it is true for $m \geq 2$. Then $D^{m+1} z^q = q \cdot (q \cdot z^{q-1} (Dz) \cdot \phi_q(G_m) + z^q \cdot D(\phi_q(G_m)))$. But ϕ_q specializes α , whose derivative is 0, to an integer whose derivative is 0. Thus, $D(\phi_q(G_m)) = \phi_q(D(G_m))$.

Hence,

$$\begin{aligned}
 D^{m+1}z^q &= q \cdot (q \cdot z^{q-1} \cdot (Dz) \cdot \phi_q(G_m) + z^q \cdot \phi_q(D(G_m))) \\
 &= q \cdot z^q \cdot \left(q \cdot \frac{Dz}{z} \cdot \phi_q(G_m) + \phi_q(D(G_m)) \right) \\
 &= q \cdot z^q \cdot (\phi_q(\alpha) \cdot G_1 \cdot \phi_q(G_m) + \phi_q(D(G_m))) \\
 &= q \cdot z^q \cdot \phi_q(\alpha \cdot G_1 \cdot G_m + D(G_m)) \\
 &= q \cdot z^q \cdot \phi_q(G_{m+1}).
 \end{aligned} \tag{3.2}$$

Therefore, $D^{m+1}z^q = q \cdot z^q \cdot G_{m+1}(z, q)$ since ϕ_q affects only α . By the principle of mathematical induction, this equation is true for all positive integers m . $\square$

Just because $Dy/(\alpha \cdot y) = Dz/z = Dz^q/(q \cdot z^q)$ implies that $Dy/(\alpha \cdot y)$ is independent of α , it does not follow that $D^m y/(\alpha \cdot y)$ is independent of α for $m \geq 2$. We can see this by observing that $D^m y/(\alpha \cdot y) \neq D^m z/z \neq D^m z^q/(q \cdot z^q) \neq D^m y/(\alpha \cdot y)$ for $m \geq 2$.

THEOREM 3.2. Assume all the same definitions and notations as in [Theorem 3.1](#) and [Section 2](#). Then, for each $m \in \mathbb{N}$, the specialization under ϕ_q of $D^m y/(\alpha \cdot y)$ is $D^m z^q/(q \cdot z^q)$. That is, $\phi_q(D^m y/(\alpha \cdot y)) = D^m z^q/(q \cdot z^q)$.

PROOF. By [Definition 2.1](#), $D^m y/(\alpha \cdot y) = G_m(z, \alpha)$. By [Theorem 3.1](#), $D^m z^q/(q \cdot z^q) = G_m(z, q)$. Putting these results together yields

$$\phi_q\left(\frac{D^m y}{\alpha \cdot y}\right) = \phi_q(G_m(z, \alpha)) = G_m(z, q) = \frac{D^m z^q}{q \cdot z^q}. \tag{3.3} \quad \square$$

4. Powersum satisfaction theorem and formula. An α -resolvent of a polynomial $P(t) \equiv \sum_{i=0}^N (-1)^{n-i} e_{N-i} t^i \in \mathbb{F}[t]$ over a differential field $\mathbb{F}$ with derivation D is a linear ordinary differential equation $\sum_{m=0}^o B_m(\alpha) \cdot D^m y = 0$ of finite order o such that each of the coefficient functions $B_m(\alpha)$ lies in the field $\mathbb{Q}\langle e \rangle(\alpha)$ (or preferably in the ring $\mathbb{Z}\{e\}[\alpha]$) such that not all $B_m(\alpha)$ are identically zero, and which is satisfied by the α -power of every root z of P . In other words, the coefficient functions of the resolvent are independent of the choice of root and are not all zero. By [\[4, Theorem 37, page 67\]](#), resolvents for any polynomial are guaranteed to exist. We state this assertion in [Theorem 4.1](#).

THEOREM 4.1. Let $P(t) \equiv \sum_{i=0}^N (-1)^{n-i} e_{N-i} t^i \in \mathbb{F}[t]$ be a polynomial of degree N in t over a d -field $\mathbb{F}$ with n distinct roots $\{z_i\}_{i=1}^n$. Then there exists an o th order differential resolvent $\sum_{m=0}^o B_m(\alpha) \cdot D^m y = 0$ with $B_m(\alpha) \in \mathbb{Z}\{e\}_n[\alpha]$, $B_0(0) = 0$, and $\deg_\alpha B_m(\alpha) \leq o(o-1)/2 - m + 1$ for some $o \in [n]$. Furthermore, o may be chosen to equal the number of $\{y_j\}_{j=1}^n$ linearly independent over constants, and all solutions of this resolvent are linear combinations over constants of these o y_j 's.

[Theorem 4.1](#) gives us an upper bound on the degree in α in an α -resolvent of P . [Theorem 4.2](#) allows us to specialize the indeterminate α to an integer q (or any number) to obtain a q -resolvent.

THEOREM 4.2 (*powersum satisfaction theorem*). Let $P \in \mathbb{F}[t]$ be a monic polynomial with n distinct roots $z = \{z_i\}_{i=1}^n$, none of which is zero and not all of which are constants. Let $q \in \mathbb{Z}$. If $R_\alpha \equiv \sum_{m=0}^o B_m(\alpha) \cdot D^m \gamma$ is an α -resolvent for P of arbitrary order o , where $B_m(\alpha) = \sum_{i \geq 0} b_{i,m} \alpha^i \in \mathbb{Z}\{e\}[\alpha]$, with $b_{i,m} \in \mathbb{Z}\{e\}$, then R_α specializes to the q -resolvent $R_q \equiv \sum_{m=0}^o B_m(q) \cdot D^m \gamma$ for $q \in \mathbb{Z}^\#$ under $\phi_q(\alpha) = q$ and $\phi_q(u) = u$ for each $u \in \mathbb{Z}\{e\}$. Furthermore, the q th powersum p_q satisfies $\sum_{m=0}^o B_m(q) \cdot D^m p_q = 0$ for each $q \in \mathbb{Z}^\#$.

PROOF. By Definition 2.1 of $G_m(t, \alpha)$, we have

$$\begin{aligned} \sum_{m=0}^o B_m(\alpha) \cdot D^m \gamma = 0 &\iff \sum_{m=0}^o B_m(\alpha) \cdot \frac{D^m \gamma}{\alpha \cdot \gamma} = 0 \\ &\iff \sum_{m=0}^o B_m(\alpha) \cdot G_m(z, \alpha) = 0. \end{aligned} \quad (4.1)$$

Now for each $q \in \mathbb{Z}^\#$, specialize this equation under ϕ_q to get $\sum_{m=0}^o B_m(q) \cdot G_m(z, q) = 0$ by Theorem 3.2, since $\phi_q|_{\mathbb{F}} = I$. For any of the roots of P , we have $G_m(z, q) = D^m z^q / (q \cdot z^q)$ by Theorem 3.1. Thus,

$$\sum_{m=0}^o B_m(q) \cdot \frac{D^m z^q}{q \cdot z^q} = 0 \iff \sum_{m=0}^o B_m(q) \cdot D^m z^q = 0 \quad (4.2)$$

for each $q \in \mathbb{Z}^\#$. Therefore, an α -resolvent specializes to a q -resolvent for each $q \in \mathbb{Z}^\#$ under ϕ_q . Now sum over the N roots of P including their multiplicities to get $\sum_{m=0}^o B_m(q) \cdot D^m p_q = 0$ for each $q \in \mathbb{Z}^\#$. $\square$

The powersum satisfaction theorem states that for any monic polynomial P , the coefficients $b_{i,m}$ of α in any α -resolvent $R_\alpha \equiv \sum_{(i,m) \in S} b_{i,m} \cdot \alpha^i D^m \gamma$ of P satisfy an infinite system of homogeneous equations

$$[q^i D^m p_q]_{q \times (i,m) \ 1 \leq q < \infty \ (i,m) \in S} \cdot [b_{i,m}]_{(i,m) \in S} = [0_q]_{1 \leq q < \infty}. \quad (4.3)$$

Here, S denotes the set of pairs (i, m) consisting of a power of α , denoted by i , and an order of a derivative, denoted by m , such that $b_{i,m} \neq 0$. Let $|S|$ denote the size of S . We will be interested in proving that the rank

$$\text{rk} [q^i D^m p_q]_{q \times (i,m) \ 1 \leq q < \infty \ (i,m) \in S} \quad (4.4)$$

of the matrix $[q^i D^m p_q]_{q \times (i,m) \ 1 \leq q < \infty \ (i,m) \in S}$ equals $|S| - 1$ under certain circumstances. Under those circumstances, one can solve this system of equations to get a nonzero solution for $b_{i,m}$. The solution is given by $b_{i,m} = F_{i,m} \equiv (-1)^{\text{sgn}(i,m)} \cdot |q^{i'} D^{m'} p_q|_{(i',m') \neq (i,m) \ q \in \Gamma}$, where $\text{sgn}(i, m)$ indicates the ordering of the term $b_{i,m}$ in the resolvent, and we take Γ to be the smallest possible set of positive integers that will guarantee a nonzero solution. In numerous examples, it has been found that $\Gamma \equiv \{k \in \mathbb{N} \ni 1 \leq k \leq |S| - 1\}$. We call this the *powersum formula* for a resolvent of P . We use the notation $F_{i,m}$ to denote the terms of the resolvent obtained by this method to suggest the word *formula*. We will denote the resolvent obtained by this formula by $\mathfrak{R}_\alpha$. So, $\mathfrak{R}_\alpha = \{F_{i,m}\}$.

If $\text{rk}[q^i D^m p_q]_{q \times (i,m) \ 1 \leq q < \infty \ (i,m) \in S} = |S|$, then the only solution would be $b_{i,m} = 0$ for all $(i,m) \in S$, contradicting the hypothesis that R_α is nonzero. Unfortunately, for a given polynomial P , one does not know a priori what the set S of nonzero $b_{i,m}$ is or how large it is. Nevertheless, we may summarize the results obtained so far in a corollary to the powersum satisfaction theorem.

COROLLARY 4.3 (the powersum formula). *Let $R_\alpha \equiv \sum_{(i,m) \in S} b_{i,m} \cdot \alpha^i D^m y$ be an α -resolvent of P , where $S \subset \mathbb{N}_0 \times \mathbb{N}_0$ is a finite set. If there exists a set of $|S| - 1$ integers $\Gamma \subset \mathbb{N}$ such that not all the $F_{i,m}$ given by the powersum formula $F_{i,m} \equiv (-1)^{\text{sgn}(i,m)} \cdot |q^{i'} D^{m'} p_q|_{(i',m') \neq (i,m) \ q \in \Gamma}$ are zero, then the linear ordinary differential equation (ODE), $\mathfrak{R}_\alpha \equiv \sum_{(i,m) \in S} F_{i,m} \cdot \alpha^i D^m y$, is an integral α -resolvent of P . If no such set of integers Γ exists, then the powersum formula yields all zeroes for $F_{i,m}$.*

The author believes that the resolvent $\mathfrak{R}_\alpha$ given by the powersum formula will be a $\mathbb{Q}\langle e \rangle$ -multiple, not just a $\mathbb{Q}\langle e \rangle(\alpha)$ -multiple of R_α , but this requires proof. For example, let $\alpha^M \cdot D^H y$ denote the highest power of α on the highest derivative of y in R_α . Even though $F_{M,L}/b_{M,L} \cdot R_\alpha$ and $\mathfrak{R}_\alpha$ are both resolvents (provided that $F_{M,L} \neq 0$) with the same coefficient function of $\alpha^M \cdot D^H y$, one must eliminate the possibility that their other terms may differ due to the possibility that P has resolvents of lower order.

5. Example. We will now apply the powersum formula to compute a particular α -resolvent of a particular trinomial. It has not yet been proved that this formula yields a nonzero differential equation for every polynomial. However, in every polynomial the author has tested, it has been possible to set up an α -resolvent, itself a polynomial in the power α , and choose the proper set of powersums such that the powersum formula yields a nonzero answer. If the powersum formula yields a nonzero answer, then it is guaranteed by [Corollary 4.3](#) that the answer is a (nonzero) resolvent of the polynomial. By a very long and difficult proof in [[4](#), Theorem 41, page 74] and [[5](#), Theorem 4.1, page 726], it has been shown that in case the distinct roots of the polynomial are differentially independent over constants (i.e., they satisfy no polynomial differential equations over $\mathbb{Q}$), then the powersum formula yields a nonzero resolvent.

The powersum formula has the advantage of giving a resolvent in an integral form. In the next example, this means the powersum formula gives a resolvent all of whose terms lie in the ring $\mathbb{Z}[x, \alpha]$.

EXAMPLE 5.1 (Sir James Cockle's resolvent of a trinomial). Cockle [[1](#)] gave a formula for a linear differential α -resolvent (although he did not call it that) for any trinomial of the form $t^n + x \cdot t^p - 1$, where $Dx \equiv 1$. Consider the particular trinomial $P(t) \equiv t^3 + x \cdot t^2 - 1$, where $n = 3$ and $p = 2$. Then, Cockle's resolvent specializes to $27 \cdot D^3 y = 4 \cdot (x \cdot D + \alpha/2)(x \cdot D + 3/2 + \alpha/2)(x \cdot D - \alpha)y$. This expands to $27 \cdot D^3 y = (4 \cdot (x \cdot D)^3 + 6 \cdot (x \cdot D)^2 - 3 \cdot \alpha \cdot (1 + \alpha) \cdot (x \cdot D) - \alpha^2 \cdot (3 + \alpha))y$. Replacing $(x \cdot D)^3$ with $x^3 \cdot D^3 + 3 \cdot x^2 \cdot D^2 + x \cdot D$ and $(x \cdot D)^2$ with $x^2 \cdot D^2 + x \cdot D$ yields $(4x^3 - 27) \cdot D^3 y + 18 \cdot x^2 \cdot D^2 y + (10 - 3 \cdot \alpha - 3 \cdot \alpha^2) \cdot x \cdot D y - \alpha^2 \cdot (3 + \alpha) \cdot y = 0$, which has the form $f_1 \cdot D^3 y + f_2 \cdot D^2 y + (f_3 + f_4 \cdot \alpha + f_5 \cdot \alpha^2) \cdot D y + (f_6 \cdot \alpha^2 + f_7 \cdot \alpha^3) \cdot y = 0$. The powersum formula requires one to know a priori the various powers of α appearing in a resolvent. Specialize α to one of the six integers $q \in \{1, 2, 3, 4, 5, 6\}$, then sum the resulting equation over each of the three

roots. Doing this for each $q \in \{1, 2, 3, 4, 5, 6\}$, one gets a system of six linear equations in the undetermined coefficient functions $\{f_k\}_{k=1}^7$ of the form $\mathfrak{N} \cdot \bar{f} = \bar{0}$, where $\mathfrak{N}$ is the 6×7 matrix defined by

$$\mathfrak{N} \equiv \begin{bmatrix} D^3 p_1 & D^2 p_1 & D p_1 & 1 \cdot D p_1 & 1^2 \cdot D p_1 & 1^2 \cdot p_1 & 1^3 \cdot p_1 \\ D^3 p_2 & D^2 p_2 & D p_2 & 2 \cdot D p_2 & 2^2 \cdot D p_2 & 2^2 \cdot p_2 & 2^3 \cdot p_2 \\ D^3 p_3 & D^2 p_3 & D p_3 & 3 \cdot D p_3 & 3^2 \cdot D p_3 & 3^2 \cdot p_3 & 3^3 \cdot p_3 \\ D^3 p_4 & D^2 p_4 & D p_4 & 4 \cdot D p_4 & 4^2 \cdot D p_4 & 4^2 \cdot p_4 & 4^3 \cdot p_4 \\ D^3 p_5 & D^2 p_5 & D p_5 & 5 \cdot D p_5 & 5^2 \cdot D p_5 & 5^2 \cdot p_5 & 5^3 \cdot p_5 \\ D^3 p_6 & D^2 p_6 & D p_6 & 6 \cdot D p_6 & 6^2 \cdot D p_6 & 6^2 \cdot p_6 & 6^3 \cdot p_6 \end{bmatrix}, \quad (5.1)$$

$\bar{f}$ is the 7×1 column vector defined by

$$\bar{f} \equiv \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}, \quad (5.2)$$

and $\bar{0}$ is the 6×1 column vector defined by

$$\bar{0} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (5.3)$$

The following program, written in Mathematica 4.0 for Students and run on a Dell Dimension XPS R400 computer using Windows 98 operating system, computes the seven terms $\{f_k\}_{k=1}^7$ by setting each f_k to the appropriate cofactor of $\mathfrak{N}$. This matrix is denoted by T in the program. The symbol $s[k]$ stands for the k th powersum p_k of the roots of P . The output is denoted by f , which is defined as the transpose of $\bar{f}$. The result is

$$\begin{bmatrix} -27 + 4x^3 & 18x^2 & 10x & -3x & -3x & -3 & -1 \end{bmatrix}, \quad (5.4)$$

which is the Cockle resolvent. The computation time is less than 5 seconds.

```
x=.; s[0]=3; s[1]=-x; s[2]=x^2;
Table[s[k+3]=Expand[-x*s[k+2]+s[k]],{k,0,3}];
T=Table[{D[s[k],{x,3}],D[s[k],{x,2}],D[s[k],x],
k*D[s[k],x],k^2*D[s[k],x],k^2*s[k],k^3*s[k]},{k,1,6}];
M=Minors[t,6];
f=Table[Simplify[m[[1,k]]*(-1)^(7-k)/(466560*x)],{k,1,7}].
```

To see the output in Mathematica for other variables, remove the semicolon after its formula. For the record, the first six powersums are (written in the form Mathematica gives) $p_1 = -x$, $p_2 = x^2$, $p_3 = 3 - x^3$, $p_4 = -4x + x^4$, $p_5 = 5x^2 - x^5$, and $p_6 = 3 - 6x^3 + x^6$. The cofactors of the matrix $\mathfrak{M}$ had to be divided by $466560 \cdot x = 2^7 \cdot 3^6 \cdot 5^1 \cdot x$ to get the resolvent in *Cohnian* form, that is, such that the only divisors in $\mathbb{Z}[x, \alpha]$ among all the terms of the resolvent are ± 1 .

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