ON THE LAGRANGE RESOLVENTS OF A DIHEDRAL QUINTIC POLYNOMIAL

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The cyclic quartic field generated by the fifth powers of the Lagrange resolvents of a dihedral quintic polynomial $f(x)$ is explicitly determined in terms of a generator for the quadratic subfield of the splitting field of $f(x)$.

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Let $f(x) = x^5 + px^3 + qx^2 + rx + s \in \mathbb{Q}[x]$ be an irreducible quintic polynomial with a solvable Galois group. Let $x_1, x_2, x_3, x_4, x_5 \in \mathbb{C}$ be the roots of $f(x)$. The splitting field of f is $K = \mathbb{Q}(x_1, x_2, x_3, x_4, x_5)$. Let ζ be a primitive fifth root of unity. The Lagrange resolvents of the root x_1 are

$$\begin{aligned} r_1 &= (x_1, \zeta) = x_1 + x_2\zeta + x_3\zeta^2 + x_4\zeta^3 + x_5\zeta^4 \in K(\zeta), \\ r_2 &= (x_1, \zeta^2) = x_1 + x_2\zeta^2 + x_3\zeta^4 + x_4\zeta + x_5\zeta^3 \in K(\zeta), \\ r_3 &= (x_1, \zeta^3) = x_1 + x_2\zeta^3 + x_3\zeta + x_4\zeta^4 + x_5\zeta^2 \in K(\zeta), \\ r_4 &= (x_1, \zeta^4) = x_1 + x_2\zeta^4 + x_3\zeta^3 + x_4\zeta^2 + x_5\zeta \in K(\zeta). \end{aligned} \tag{1}$$

We set

$$R_i = r_i^5, \quad i = 1, 2, 3, 4. \tag{2}$$

By [1, Theorem 2] we know that the Galois group of f is $\mathbb{Z}_5$ (cyclic group of order 5), D_5 (dihedral group of order 10), or F_{20} (Frobenius group of order 20). When $\text{Gal}(f) \simeq D_5$, the splitting field K of f contains a unique quadratic subfield, say $\mathbb{Q}(\sqrt{m})$ (m square-free integer $\neq 1$). In this note we show, for quintic polynomials f with $\text{Gal}(f) \simeq D_5$, that the fields $\mathbb{Q}(R_i)$ ($i = 1, 2, 3, 4$) are the same cyclic quartic field and we give a simple explicit generator for this field. We prove the following theorem.

THEOREM 1. *If $\text{Gal}(f) \simeq D_5$, then*

$$\mathbb{Q}(R_i) = \mathbb{Q}\left(\sqrt{-m(5 + 2\sqrt{5})}\right), \quad i = 1, 2, 3, 4, \tag{3}$$

where $\mathbb{Q}(\sqrt{m})$ is the unique quadratic subfield of the splitting field K of f .

PROOF. Expanding $(x_1, \zeta)^5 = (x_1 + x_2\zeta + x_3\zeta^2 + x_4\zeta^3 + x_5\zeta^4)^5$ we obtain

$$R_1 = l_0 + l_1\zeta + l_2\zeta^2 + l_3\zeta^3 + l_4\zeta^4, \quad (4)$$

where $l_0, l_1, l_2, l_3, l_4 \in K$ are given in [1, page 391] and satisfy

$$l_0 + l_1 + l_2 + l_3 + l_4 = (x_1 + x_2 + x_3 + x_4 + x_5)^5 = 0. \quad (5)$$

As $\text{Gal}(f) \simeq D_5$, by [1, Theorem 2, page 397] the discriminant D of f is a square in $\mathbb{Q}$. Thus, by [1, pages 392–397], l_1, l_2, l_3, l_4 are the roots of a quartic polynomial belonging to $\mathbb{Q}[x]$, which factors over $\mathbb{Q}$ into two irreducible conjugate quadratics

$$(x^2 + (T_1 + T_2\sqrt{D})x + (T_3 + T_4\sqrt{D}))(x^2 + (T_1 - T_2\sqrt{D})x + (T_3 - T_4\sqrt{D})) \quad (6)$$

with $T_1, T_2, T_3, T_4 \in \mathbb{Q}$. The roots of one of these quadratics (without loss of generality the first) are l_1 and l_4 , and the roots of the other are l_2 and l_3 . Thus

$$\begin{aligned} l_1 + l_4 &= -T_1 - T_2\sqrt{D}, & l_2 + l_3 &= -T_1 + T_2\sqrt{D}, \\ l_1 l_4 &= T_3 + T_4\sqrt{D}, & l_2 l_3 &= T_3 - T_4\sqrt{D}. \end{aligned} \quad (7)$$

Clearly $[\mathbb{Q}(l_i) : \mathbb{Q}] = 2$ ($i = 1, 2, 3, 4$). Also $l_i \in K$ ($i = 1, 2, 3, 4$) so that $\mathbb{Q}(l_i) \subseteq K$ ($i = 1, 2, 3, 4$). However K has a unique quadratic subfield $\mathbb{Q}(\sqrt{m})$. Thus $\mathbb{Q}(l_i) = \mathbb{Q}(\sqrt{m})$, $i = 1, 2, 3, 4$. Hence

$$l_1 = a + b\sqrt{m}, \quad l_4 = a - b\sqrt{m}, \quad l_2 = c + d\sqrt{m}, \quad l_3 = c - d\sqrt{m}, \quad (8)$$

where $a, b, c, d \in \mathbb{Q}$, $b \neq 0$ and $d \neq 0$. Thus

$$l_0 = -l_1 - l_2 - l_3 - l_4 = -2a - 2c. \quad (9)$$

Next we define

$$g(x) = (x - R_1)(x - R_2)(x - R_3)(x - R_4) \in K(\zeta)[x]. \quad (10)$$

Hence, as $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$, we obtain

$$\begin{aligned} R_1 &= l_0 + l_1\zeta + l_2\zeta^2 + l_3\zeta^3 + l_4\zeta^4 \\ &= (a + b\sqrt{m} + 2a + 2c)\zeta + (c + d\sqrt{m} + 2a + 2c)\zeta^2 \\ &\quad + (c - d\sqrt{m} + 2a + 2c)\zeta^3 + (a - b\sqrt{m} + 2a + 2c)\zeta^4 \in \mathbb{Q}(\sqrt{m}, \zeta). \end{aligned} \quad (11)$$

Similarly

$$\begin{aligned} R_2 &= (a + b\sqrt{m} + 2a + 2c)\zeta^2 + (c + d\sqrt{m} + 2a + 2c)\zeta^4 \\ &\quad + (c - d\sqrt{m} + 2a + 2c)\zeta + (a - b\sqrt{m} + 2a + 2c)\zeta^3 \in \mathbb{Q}(\sqrt{m}, \zeta), \\ R_3 &= (a + b\sqrt{m} + 2a + 2c)\zeta^3 + (c + d\sqrt{m} + 2a + 2c)\zeta \\ &\quad + (c - d\sqrt{m} + 2a + 2c)\zeta^4 + (a - b\sqrt{m} + 2a + 2c)\zeta^2 \in \mathbb{Q}(\sqrt{m}, \zeta), \\ R_4 &= (a + b\sqrt{m} + 2a + 2c)\zeta^4 + (c + d\sqrt{m} + 2a + 2c)\zeta^3 \\ &\quad + (c - d\sqrt{m} + 2a + 2c)\zeta^2 + (a - b\sqrt{m} + 2a + 2c)\zeta \in \mathbb{Q}(\sqrt{m}, \zeta). \end{aligned} \quad (12)$$

Using Maple we find that

$$\begin{aligned}
 g(x) = & x^4 + (10c + 10a)x^3 + (5b^2m + 5d^2m + 80ac + 35a^2 + 35c^2)x^2 \\
 & + (30cd^2m + 50c^3 + 200a^2c - 20bcdm + 30ab^2m + 20ad^2m \\
 & + 20b^2cm + 200ac^2 + 50a^3 + 20abdm)x - 10b^3dm^2 + 150a^3c \\
 & + 25a^2d^2m + 25b^2c^2m - 5b^2d^2m^2 + 275a^2c^2 + 25c^4 + 10bd^3m^2 \\
 & + 50acd^2m - 50bc^2dm + 150ac^3 + 50a^2bdm + 50c^2d^2m + 5d^4m^2 \\
 & + 25a^4 + 5b^4m^2 + 50a^2b^2m + 50ab^2cm.
 \end{aligned} \tag{13}$$

The roots of $g(x)$ are (again using Maple)

$$\begin{aligned}
 & -\frac{5}{2}a - \frac{5}{2}c + \frac{1}{2}(-a+c)\sqrt{5} + \frac{1}{2}\sqrt{-m(10(b^2+d^2) - (2b^2+8bd-2d^2)\sqrt{5})}, \\
 & -\frac{5}{2}a - \frac{5}{2}c + \frac{1}{2}(-a+c)\sqrt{5} - \frac{1}{2}\sqrt{-m(10(b^2+d^2) - (2b^2+8bd-2d^2)\sqrt{5})}, \\
 & -\frac{5}{2}a - \frac{5}{2}c - \frac{1}{2}(-a+c)\sqrt{5} + \frac{1}{2}\sqrt{-m(10(b^2+d^2) + (2b^2+8bd-2d^2)\sqrt{5})}, \\
 & -\frac{5}{2}a - \frac{5}{2}c - \frac{1}{2}(-a+c)\sqrt{5} + \frac{1}{2}\sqrt{-m(10(b^2+d^2) + (2b^2+8bd-2d^2)\sqrt{5})}.
 \end{aligned} \tag{14}$$

The quantities under the radicals are $X + Y\sqrt{5}$ and $X - Y\sqrt{5}$, where

$$X = -10m(b^2 + d^2), \quad Y = m(2b^2 + 8bd - 2d^2). \tag{15}$$

As

$$X^2 - 5Y^2 = 5m^2(4b^2 - 4bd - 4d^2)^2, \tag{16}$$

the roots of $g(x)$ belong to the cyclic quartic field $\mathbb{Q}(\sqrt{X \pm Y\sqrt{5}})$ [2, Theorem 1, page 134]. Further

$$X + Y\sqrt{5} = (-10 + 2\sqrt{5})m \left(\frac{2b - d - d\sqrt{5}}{2} \right)^2 \tag{17}$$

so that (as $b \neq 0$ and $d \neq 0$)

$$\mathbb{Q}(\sqrt{X + Y\sqrt{5}}) = \mathbb{Q}(\sqrt{(-10 + 2\sqrt{5})m}) = \mathbb{Q}(\sqrt{-m(5 + 2\sqrt{5})}), \tag{18}$$

as $(-10 + 2\sqrt{5})(-5 - 2\sqrt{5}) = (5 + \sqrt{5})^2$. $\square$

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