ON THE EXTENDIBILITY OF THE DIOPHANTINE TRIPLE $\{1, 5, c\}$

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We study the problem of extendibility of the triples of the form $\{1, 5, c\}$. We prove that if $c_k = s_k^2 + 1$, where (s_k) is a binary recursive sequence, k is a positive integer, and the statement that all solutions of a system of simultaneous Pellian equations $z^2 - c_k x^2 = c_k - 1$, $5z^2 - c_k y^2 = c_k - 5$ are given by $(x, y, z) = (0, \pm 2, \pm s_k)$, is valid for $2 \leq k \leq 31$, then it is valid for all positive integer k .

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1. Introduction. Let n be an integer. A set of positive integers $\{a_1, a_2, \dots, a_m\}$ is said to have the property $D(n)$ if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$; such a set is called a Diophantine m -tuple or a P_n set of size m . The problem of construction of such sets was studied by Diophantus (see [4]). A famous conjecture related to this problem is as follows.

CONJECTURE 1.1. *There does not exist a Diophantine quadruple with the property $D(-1)$.*

For certain triples $\{a, b, c\}$ with $1 \notin \{a, b, c\}$, the validity of this conjecture can be verified by simple use of congruences (see [5]). The case $a = 1$ is more involved and the first important result concerning this conjecture was proved in 1985 by Mohanty and Ramasamy [8]; they proved that the triple $\{1, 5, 10\}$ cannot be extended. Also, Brown [5] proved the conjecture for the triples $\{n^2 + 1, (n + 1)^2 + 1, (2n + 1)^2 + 1\}$, where $n \not\equiv 0 \pmod{4}$, for the triples $\{2, 2n^2 + 2n + 1, 2n^2 + 6n + 5\}$, where $n \equiv 1 \pmod{4}$, and proved nonextendibility of triples $\{17, 26, 68\}$ and $\{1, 2, 5\}$. In 1998, Kedlaya [7] verified it for the triples $\{1, 2, 145\}$, $\{1, 2, 4901\}$, $\{1, 5, 65\}$, $\{1, 5, 20737\}$, $\{1, 10, 17\}$, and $\{1, 26, 37\}$. Since Dujella [6] has proved the conjecture for all triples of the form $\{1, 2, c\}$, the consideration of triples of the form $\{1, 5, c\}$ seems to be the natural next step.

In the present paper, we will study the extendibility of all triples of the form $\{1, 5, c\}$. In our proof, we will follow the strategy of [6].

2. Preliminaries. Since the triple $\{1, 5, c\}$ satisfies the property $D(-1)$, therefore there exist integers s, t such that $c - 1 = s^2$ and $5c - 1 = t^2$ which imply

$$t^2 - 5s^2 = 4. \quad (2.1)$$

If this triple can be extended to a Diophantine quadruple, then there are integers d, x, y, z such that

$$d - 1 = x^2, \quad 5d - 1 = y^2, \quad cd - 1 = z^2. \quad (2.2)$$

Eliminating d , we get

$$z^2 - cx^2 = c - 1, \quad z^2 - cy^2 = c - 5; \quad (2.3)$$

it is obvious that if all the solutions of this system are given by $(x, y, z) = (0, \pm 2, \pm\sqrt{c-1})$, then, from (2.2), we get $d = 1$, so the triple $\{1, 5, c\}$ cannot be extended.

The Pell equation (2.1) has three classes of solutions and all the solutions are given by

$$\begin{aligned} t'_k + s'_k &= (3 + \sqrt{5})(9 + 4\sqrt{5})^k, \\ t''_k + s''_k &= (-3 + \sqrt{5})(9 + 4\sqrt{5})^k, \\ t'''_k + s'''_k &= 2(9 + 4\sqrt{5})^k. \end{aligned} \quad (2.4)$$

Hence, if the triple $\{1, 5, c\}$ is a Diophantine triple with the property $D(-1)$, then there exists a positive integer k such that the integer c has the following three formulas (see [3]):

$$c = c'_k = \frac{1}{10}[(7 + 3\sqrt{5})^2(161 + 72\sqrt{5})^k + (7 - 3\sqrt{5})^2(161 - 72\sqrt{5})^k + 6], \quad (2.5)$$

$$c = c''_k = \frac{1}{10}[(7 - 3\sqrt{5})^2(161 + 72\sqrt{5})^k + (7 + 3\sqrt{5})^2(161 - 72\sqrt{5})^k + 6], \quad (2.6)$$

$$c = c'''_k = \frac{1}{5}[(161 + 72\sqrt{5})^k + (161 - 72\sqrt{5})^k + 3]. \quad (2.7)$$

The main result of this paper is in the following theorem, where c_k denotes one of the formulas in (2.5), (2.6), and (2.7).

THEOREM 2.1. *Let k be a positive integer and let $c_k = s_k^2 + 1$, where (s_k) is a binary recursive sequence. If the statement that all solutions of a system of simultaneous Pellian equations*

$$z^2 - c_k x^2 = c_k - 1, \quad 5z^2 - c_k y^2 = c_k - 5 \quad (2.8)$$

are given by $(x, y, z) = (0, \pm 2, \pm s_k)$ is valid for $k \leq 31$, then it is valid for all positive integer k .

REMARK 2.2. The theorem is true when $k = 0$ [5] and $k = 1$ (see [1, 7, 8]). So we will suppose that $k \geq 2$. For simplicity, we will omit the index k and we will divide the proof of the theorem into many lemmas.

3. A system of Pellian equations. There are finite sets

$$\begin{aligned} \{z_0^{(i)} + x_0^{(i)}\sqrt{c} : i = 1, 2, \dots, i_0\}, \\ \{z_1^{(j)}\sqrt{5} + y_1^{(j)}\sqrt{c} : j = 1, 2, \dots, j_0\}, \end{aligned} \quad (3.1)$$

of elements of $Z[\sqrt{c}]$ and $Z[\sqrt{5c}]$, respectively, such that all solutions of (2.8) are given by

$$z + x\sqrt{c} = (z_0^{(i)} + x_0^{(i)}\sqrt{c})(2c - 1 + 2s\sqrt{c})^m, \quad i = 1, \dots, m \geq 0, \quad (3.2)$$

$$z\sqrt{5} + y\sqrt{c} = (z_1^{(j)}\sqrt{5} + y_1^{(j)}\sqrt{c})(10c_k - 1 + 2t\sqrt{5c})^n, \quad i = 1, \dots, n \geq 0, \quad (3.3)$$

respectively (see [6]).

From (3.2), we conclude that $z = v_m^{(i)}$ for some index i and integer m , where

$$v_0^{(i)} = z_0^{(i)}, \quad v_1^{(i)} = (2c - 1)z_0^{(i)} + 2scx_0^{(i)}, \quad v_{m+2}^{(i)} = (4c - 2)v_{m+1}^{(i)} - v_m^{(i)}, \quad (3.4)$$

and from (3.3), we conclude that $z = w_n^{(j)}$ for some index j and integer n , where

$$w_0^{(j)} = z_1^{(j)}, \quad w_1^{(j)} = (10c - 1)z_1^{(j)} + 2tcy_1^{(j)}, \quad w_{n+2}^{(j)} = (20c - 2)w_{n+1}^{(j)} - w_n^{(j)}. \quad (3.5)$$

Thus we reformulated system (2.8) to finitely many Diophantine equations of the form

$$v_m^{(i)} = w_n^{(j)}. \quad (3.6)$$

If we choose representatives $z_0^{(i)} + x_0^{(i)}\sqrt{c}$ and $z_1^{(j)}\sqrt{5} + y_1^{(j)}\sqrt{c}$ such that $|z_0^{(i)}|$ and $|z_1^{(j)}|$ are minimal, then, by [9, Theorem 108a], we have the following estimates:

$$\begin{aligned} 0 < |z_0^{(i)}| &\leq \sqrt{\frac{1}{2}2c(c-1)} < c, \\ 0 < |z_1^{(j)}| &\leq \sqrt{c \cdot (c-5)} < c. \end{aligned} \quad (3.7)$$

4. Application of congruence relations. In the following lemma, we prove that if (2.2) has a nontrivial solution, then the initial terms of sequences $v_m^{(i)}$ and $w_n^{(j)}$ are restricted.

LEMMA 4.1. *Let $k \geq 2$ be the least positive integer (if it exists) for which the statement of Theorem 2.1 is not valid. Let $1 \leq i \leq i_0$, $1 \leq j \leq j_0$, and let $v_m^{(i)}$ and $w_n^{(j)}$ be the sequences defined in (3.4) and (3.5). If the equation $v_m^{(i)} = w_n^{(j)}$ has a solution, then $|z_0^{(i)}| = |z_1^{(j)}| = s$.*

PROOF. From (3.4) and (3.5), it follows easily by induction that

$$\begin{aligned} v_{2m}^{(i)} &\equiv z_0^{(i)} \pmod{2c}, \\ w_{2n}^{(j)} &\equiv z_1^{(j)} \pmod{2c}, \\ v_{2m+1}^{(i)} &\equiv -z_0^{(i)} \pmod{2c}, \\ w_{2n+1}^{(j)} &\equiv -z_1^{(j)} \pmod{2c}. \end{aligned} \quad (4.1)$$

Therefore, if the equation $v_m^{(i)} = w_n^{(j)}$ has a solution in integers m and n , then we must have $|z_0^{(i)}| = |z_1^{(j)}|$. Now, let $d_0 = ((z_0^{(i)})^2 + 1)/c$; then we have

$$\begin{aligned} d_0 - 1 &= (x_0^{(i)})^2, & 5d_0 - 1 &= (y_1^{(j)})^2, & cd_0 - 1 &= (z_0^{(i)})^2, \\ d_0 &\leq \frac{c^2 - c + 1}{c} < c. \end{aligned} \quad (4.2)$$

Assume that $d_0 > 1$. It follows from (4.2) that there exists a positive integer $l < k$ such that $d_0 = c_l$. But now the system

$$z^2 - c_l x^2 = c_l - 1, \quad 5z^2 - c_l y^2 = c_l - 5 \quad (4.3)$$

has a nontrivial solution $(x, y, z) = (s_k, t_k, z_0^{(i)})$, contradicting the minimality of k . So, $d_0 = 1$ and $|z_0^{(i)}| = s$. $\square$

The following lemma can be proved easily by induction (we will omit the superscripts (i) and (j)).

LEMMA 4.2. *Let $\{v_m\}$ and $\{w_n\}$ be the sequences which have the initial terms in Lemma 4.1; then*

$$\begin{aligned} v_m &\equiv (-1)^m (z_0 - 2cm^2 z_0 - 2csmx_0) \pmod{8c^2}, \\ w_n &\equiv (-1)^n (z_1 - 10cn^2 z_1 - 2ctny_1) \pmod{8c^2}. \end{aligned} \quad (4.4)$$

REMARK 4.3. Since we may restrict ourselves to positive solutions of system (2.8), we may assume that $z_0 = z_1 = s$. Notice that $x_0 = 0$ and $y_1 = \pm 2$.

LEMMA 4.4. *If $v_m = w_n$, then m and n are both even or odd.*

PROOF. Suppose m is odd and n is even and let $m = 2r$ and $n = 2l + 1$. Lemma 4.2 and the relation $z_0 = z_1 = s$ imply

$$s \equiv cs(2l+1)^2 + 20cr^2s \pm 4ctr \pmod{4c^2} \quad (4.5)$$

and we have a contradiction to the fact that c does not divide s .

The same proof holds for the case where m is even and n odd. $\square$

LEMMA 4.5. *If $v_m = w_n$, then $n \leq m \leq n\sqrt{5}$.*

PROOF. From relations (3.4) and (3.5), $w_1 > v_1$. Let $w_l > v_l$, where $l > 0$; then

$$w_{l+2} < (20c - 2)w_{l+1} - v_l = (20c - 2)w_{l+1} - [(4c - 2)v_{l+1} - v_{l+2}], \quad (4.6)$$

hence

$$w_{l+2} - v_{l+2} < (20c - 2)w_{l+1} - (4c - 2)v_{l+1}. \quad (4.7)$$

But $(20c - 2)w_{l+1} - (4c - 2)v_{l+1} > 0$, which implies $w_{l+2} < v_{l+2}$. So, if the equation $v_m = w_n$ has a solution and $n \neq 0$, then $v_n < v_m = w_n$. But the sequence v_m is increasing, so $m > n$.

Now, from (3.4), we have

$$v_m = \frac{s}{2}[(2c-1+2s\sqrt{c})^m + (2c-1-2s\sqrt{c})^m] > \frac{1}{2}(2c-1+2s\sqrt{c})^m, \quad (4.8)$$

and from (3.5), we have

$$\begin{aligned} w_n &= \frac{1}{2\sqrt{5}}(s\sqrt{5} \pm 2\sqrt{c})[(10c-1+2t\sqrt{5c})^n + (10c-1-2t\sqrt{5c})^n] \\ &< \frac{s\sqrt{5}+2\sqrt{c}+1}{2\sqrt{5}}(10c-1+2t\sqrt{5c})^n < \frac{1}{2}(10c-1+2t\sqrt{5c})^{n+1/2}. \end{aligned} \quad (4.9)$$

Since $k \geq 2$, therefore from (2.5), (2.6), and (2.7), we have $c \geq 3026$. Thus $v_m = w_n$ implies

$$\frac{m}{n+1/2} < \frac{\ln(10c-1+2t\sqrt{5c})}{\ln(2c-1+2s\sqrt{c})} < 1.1712. \quad (4.10)$$

If $n = 0$, then $m = 0$, and if $n \geq 1$, then (4.10) implies

$$m < 1.1712n + 0.5856 < n\sqrt{5}. \quad (4.11)$$

□

LEMMA 4.6. *If $v_m = w_n$ and $n \neq 0$, then $n > (1/2)^{4/\sqrt{c}}$.*

PROOF. (1) The case where m and n are both even.

We assume that $n < (1/2)^{4/\sqrt{c}}$. Using Lemma 4.2 and from $v_m = w_n$, we get

$$2c(2m)^2s + 2cs(2m)x_0 \equiv 10c(2n)^2s - 2ct(2n)y_1 \pmod{8c^2}. \quad (4.12)$$

But $x_0 = 0$ and $y_1 = \pm 2$, so

$$8cm^2s \equiv 40cn^2s \pm 8ctn \pmod{8c^2}, \quad (4.13)$$

which implies

$$s(5n^2 - m^2) \equiv \pm tn \pmod{c}. \quad (4.14)$$

On the other hand, we have, from Lemma 4.5,

$$|s(5n^2 - m^2)| \leq \sqrt{c}4n^2 < 4\sqrt{c}\left(\frac{1}{2}\sqrt[4]{c}\right)^2 = c. \quad (4.15)$$

Also, since $c > \sqrt{5/4}\sqrt[4]{c^3}$, then

$$tn < \sqrt{5cn} < \sqrt{5}\sqrt{c}\frac{1}{2}\sqrt[4]{c} = \sqrt{\frac{5}{4}}\sqrt[4]{c^3} < c. \quad (4.16)$$

So, from (4.14), (4.15), and (4.16), we get

$$s(5n^2 - m^2) = \pm tn. \quad (4.17)$$

Also, from (4.14), we have

$$s^2(5n^2 - m^2)^2 \equiv t^2 n^2 \pmod{c}. \quad (4.18)$$

But $s^2 \equiv t^2 \pmod{c}$, so (4.18) becomes

$$(m^2 - 5n^2)^2 \equiv n^2 \pmod{c}. \quad (4.19)$$

Now, since

$$\begin{aligned} (5n^2 - m^2)^2 &\leq (4n^2)^2 = 16n^4 \leq 16 \cdot \left(\frac{1}{2} \sqrt[4]{c}\right)^4 = c, \\ n &< \frac{1}{2} \sqrt[4]{c} \Rightarrow n^2 < \frac{1}{4} \sqrt{c} < c, \end{aligned} \quad (4.20)$$

so, from (4.19), (4.20), we get

$$(5n^2 - m^2)^2 = n^2. \quad (4.21)$$

Finally, from (4.17) and (4.21), we get $t^2 = s^2$, which is impossible.

(2) The case where m and n are both odd.

We assume that $n < (1/2) \sqrt[4]{c}$. Using Lemma 4.2 and from $v_m = w_n$, where $x_0 = 0$ and $y_1 = \pm 2$, we get

$$s(5n^2 - m^2) \equiv \pm 2tn \pmod{c}. \quad (4.22)$$

As above,

$$|s(5n^2 - m^2)| < c, \quad (4.23)$$

and since

$$2tn < 2\sqrt{5cn} < \sqrt{5c} \sqrt[4]{c} < c, \quad (4.24)$$

therefore (4.22), (4.23), and (4.24) imply

$$s(5n^2 - m^2) = \pm 2tn. \quad (4.25)$$

Also, from (4.22), we have

$$s^2(5n^2 - m^2)^2 \equiv 4t^2 n^2 \pmod{c}, \quad (4.26)$$

which implies $(m^2 - 5n^2)^2 \equiv 4n^2 \pmod{c}$. But $(5n^2 - m^2)^2 < c$ and $4n^2 < c$, so

$$(5n^2 - m^2)^2 = 4n^2. \quad (4.27)$$

Finally, from (4.25) and (4.27), we get $t^2 = s^2$, which is impossible. $\square$

5. Linear forms in logarithms

LEMMA 5.1. *If $v_m = w_n$, then*

$$0 < n \log(10c - 1 + 2t\sqrt{5c}) - m \log(10c - 1 + 2t\sqrt{5c}) + \log \frac{s\sqrt{5} \pm 2\sqrt{c}}{\sqrt{5c}} < (4c)^{1-n}. \quad (5.1)$$

PROOF. We suppose that

$$p = s(2c - 1 + 2s\sqrt{c})^m, \quad q = \frac{1}{\sqrt{5}}(s\sqrt{5} \pm 2\sqrt{c})(10c - 1 + 2t\sqrt{5c})^n. \quad (5.2)$$

If $v_m = w_n$, then, from (4.8) and (4.9), we get

$$p + s^2 p^{-1} = q + \frac{c-5}{5} q^{-1}. \quad (5.3)$$

It is clear that $p > 1$ and $q > 1$; also

$$p - q = \frac{c-5}{5} q^{-1} - s^2 p^{-1} < (c-1)q^{-1} - (c-1)p^{-1} = (c-1)(p-q)p^{-1}q^{-1}. \quad (5.4)$$

If $p > q$, then from (5.4), we get $pq < c-1$, which is impossible since $q > 1$ and $p > (4s\sqrt{c})s = 4s^2\sqrt{c} = 4(c-1)\sqrt{c} > c > c-1$. Hence $q > p$, and we may assume that $m \geq 1$. Furthermore

$$0 < \log\left(\frac{p}{q}\right)^{-1} = -\log\left(\frac{p}{q}\right) = -\log\left(1 - \frac{q-p}{q}\right). \quad (5.5)$$

Since $-\log(1-x) < x + x^2$, therefore, from (5.5), we get

$$0 < \log\left(\frac{q}{p}\right) < \frac{q-p}{q} + \left(\frac{q-p}{q}\right)^2. \quad (5.6)$$

But from (5.3), we deduce that $p > q - (c-1)p^{-1} > q - (c-1)$, so

$$p^{-1} < (q - (c-1))^{-1}; \quad (5.7)$$

hence, from (5.3) and (5.7), we get

$$q - p < (c-1)(q - (c-1))^{-1} - \frac{c-5}{5} q^{-1} < \frac{4cq + c^2 + 5}{q(5q - 5c + 5)} < \frac{4cq + c^2 + 5}{q}. \quad (5.8)$$

But $q > (s\sqrt{5} - 2\sqrt{c})(4c)$ implies

$$\frac{c^2}{q} < \frac{c^2}{(s\sqrt{5} - 2\sqrt{c})(4c)} < \frac{c^2}{4c} = \frac{c}{4}, \quad (5.9)$$

so (5.8) becomes

$$\frac{q-p}{q} < \left[4c + \frac{c^2}{q} + \frac{5}{q}\right]q^{-1} < \left[4c + \frac{c}{4} + 5\right]q^{-1} = \left(\frac{17}{4}c + 5\right)q^{-1}. \quad (5.10)$$

From (5.6) and (5.10), we get

$$0 < \log \frac{q}{p} < \left(\frac{17}{4}c + 5 \right) q^{-1} + \left(\frac{17}{4}c + 5 \right)^2 q^{-2}. \quad (5.11)$$

Now, we will estimate $((17/4)c + 5)q^{-1}$. From (2.5), (2.6), and (2.7), we have $c > 20$, so

$$\left(\frac{17}{4}c + 5 \right) q^{-1} < \left(\frac{17}{4}c + 5 \right) c^{-1} = \frac{17}{4} + \frac{5}{c} < \frac{17}{4} + \frac{1}{4} = \frac{9}{2}. \quad (5.12)$$

Thus (5.11) becomes

$$\begin{aligned} 0 &< \log \frac{q}{p} < \left(\frac{17}{4}c + 5 \right) q^{-1} + \left(\frac{17}{4}c + 5 \right)^2 q^{-2} \\ &= \left(\frac{17}{4}c + 5 \right) q^{-1} \left[1 + \left(\frac{17}{4}c + 5 \right) q^{-1} \right] < \frac{11}{2} \left(\frac{17}{4}c + 5 \right) q^{-1} \\ &= \frac{11}{2} \left(\frac{17}{4}c + 5 \right) \frac{\sqrt{5}}{s\sqrt{5} \pm 2\sqrt{c}} (10c - 1 + 2t\sqrt{5c})^{-n} \\ &< 11\sqrt{5} \left(\frac{17}{8}c + \frac{5}{2} \right) (10c - 1 + 2t\sqrt{5c})^{-n} \\ &< 11(\sqrt{5}) \left(3c + \frac{5}{2} \right) (4\sqrt{5c} - 1\sqrt{5c})^{-n} \\ &< 11(\sqrt{5}) \left(3c + \frac{5}{2} \right) (4c)^{-n} \\ &< 4c(4c)^{-n}. \end{aligned} \quad (5.13)$$

But

$$\log \frac{q}{p} = n \log (10c - 1 + 2t\sqrt{5c}) - m \log (10c - 1 + 2t\sqrt{5c}) + \log \frac{s\sqrt{5} \pm 2\sqrt{c}}{\sqrt{5c}}. \quad (5.14)$$

So, (5.13) and (5.14) complete the proof of the lemma. $\square$

Now, to prove the theorem, we apply the following theorem.

THEOREM 5.2 [2]. For a linear form $\Omega \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \dots, \alpha_l$ with rational coefficients $b_1, \dots, b_l$,

$$\log |\Omega| \geq -18(l+1)!l^{l+1}(32d)^{l+2}h'(\alpha_1) \cdots h'(\alpha_l) \log(2ld) \log B, \quad (5.15)$$

where $B = \max(|b_1|, \dots, |b_l|)$ and where d is the degree of the number field generated by $\alpha_1, \dots, \alpha_l$.

Here

$$h'(\alpha) = \frac{1}{r} \max(h(\alpha), |\log \alpha|, 1) \quad (5.16)$$

and $h(\alpha)$ denotes the standard logarithmic Weil height of α .

6. Proof of Theorem 2.1. (1) The case where m and n are both even.

We consider the equation $v_{2m} = w_{2n}$ with $n \neq 0$. We apply the above theorem and we have $l = 3$, $d = 4$, $B = 2m$, where

$$\begin{aligned}\alpha_1 &= 10c - 1 + 2t\sqrt{5c}, \\ \alpha_2 &= 2c - 1 + 2s\sqrt{c}, \\ \alpha_3 &= \frac{s\sqrt{5} + 2\sqrt{c}}{\sqrt{5}s}.\end{aligned}\tag{6.1}$$

The equations satisfied by $\alpha_1, \alpha_2, \alpha_3$ are

$$\begin{aligned}\alpha_1^2 - (20c - 2)\alpha_1 + 1 &= 0, \\ \alpha_2^2 - (4c - 2)\alpha_2 + 1 &= 0, \\ (5c - 5)\alpha_3^2 - (10c - 10)\alpha_3 + c - 5 &= 0 \iff \alpha_3^2 - 2\alpha_3 + \frac{c-5}{5c-5} = 0.\end{aligned}\tag{6.2}$$

Hence

$$\begin{aligned}h'(\alpha_1) &= \frac{1}{2} \log \alpha_1 < \frac{1}{2} \log 20c, \\ h'(\alpha_2) &= \frac{1}{2} \log \alpha_2 < \frac{1}{2} \log 4c, \\ h'(\alpha_3) &= \frac{1}{2} \log \frac{s\sqrt{5} + 2\sqrt{c}}{\sqrt{5}s} < \frac{1}{2} \log(1 + 2c).\end{aligned}\tag{6.3}$$

From Lemma 5.1, where n is even, we have

$$\log \Omega < (4c)^{1-2n} = -(2n-1) \log 4c.\tag{6.4}$$

So, from Theorem 5.2, we get

$$(2n-1) \log 4c \leq 18 \times 4! \times 3^4 (32 \times 4)^5 \times \frac{1}{2} \log(20c) \times \frac{1}{2} \log(4c) \times \frac{1}{2} \log(2c+1) \log 24 \times \log 2m.\tag{6.5}$$

Now, using Lemmas 4.5 and 4.6, we get

$$(2n-1) \leq 2.07431 \times 10^{14} \times \log 8000n^4 \times \log(800n^4 + 1) \times (\log 2\sqrt{5n}),\tag{6.6}$$

which implies that

$$n < 2 \times 10^{19},\tag{6.7}$$

and finally,

$$c < 256(10^{76}).\tag{6.8}$$

To find k in the first class, substitute in (2.5); hence

$$k \log(161 + 72\sqrt{5}) < \log 256 + 77 \log 10 - \log(7 - 3\sqrt{5}), \quad (6.9)$$

which implies $k \leq 31$. Similarly, we find that in the other two classes, $k \leq 31$.

(2) The case where m and n are both odd.

In this case, using Lemma 5.1, where n is odd, relation (6.4) becomes

$$\log \Omega < (4c)^{-2n} = -(2n-1) \log 4c. \quad (6.10)$$

Hence (6.6) becomes

$$2n \leq 2.07431 \times 10^{14} \times \log 8000n^4 \times \log(800n^4 + 1) \times (\log 2\sqrt{5n}), \quad (6.11)$$

which implies that $n < 2 \times 10^{19}$, and finally $c < 256(10^{76})$, hence $k \leq 31$.

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