THE BESSEL-STRUVE INTERTWINING OPERATOR ON $\mathbb{C}$ AND MEAN-PERIODIC FUNCTIONS

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We give a description of all transmutation operators from the Bessel-Struve operator to the second-derivative operator. Next we define and characterize the mean-periodic functions on the space $\mathcal{H}$ of entire functions and we characterize the continuous linear mappings from $\mathcal{H}$ into itself which commute with Bessel-Struve operator.

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1. Introduction. Let A and B be two differential operators on a linear space X . We say that χ is a transmutation operator of A into B if χ is an isomorphism from X into itself such that $A\chi = \chi B$. This notion was introduced by Delsarte in [2] and some generalization and applications were given in [1, 3, 7, 10].

In the case where A and B are two differential operators having the same order and without any singularity on the complex plan, acting on the space of entire functions on $\mathbb{C}$ denoted here by $\mathcal{H}$, Delsarte showed in [3] the existence of a transmutation operator between A and B and gave some applications on the theory of mean-periodic functions on $\mathbb{C}$.

In this paper, we consider the operator ℓ_α , $\alpha > -1/2$, on $\mathbb{C}$, given by

$$\ell_\alpha f(z) = \frac{d^2 f}{dz^2}(z) + \frac{2\alpha+1}{z} \left[\frac{df}{dz}(z) - \frac{df}{dz}(0) \right], \quad (1.1)$$

where f is an entire function on $\mathbb{C}$. We call this operator Bessel-Struve operator on $\mathbb{C}$.

The Bessel-Struve kernel $S_\alpha(\lambda \cdot)$, $\lambda \in \mathbb{C}$, which is the unique solution of the initial value problem $\ell_\alpha u(z) = \lambda^2 u(z)$ with the initial conditions $u(0) = 1$ and $u'(0) = \lambda \Gamma(\alpha+1)/\sqrt{\pi} \Gamma(\alpha+3/2)$, is given by

$$S_\alpha(\lambda z) = j_\alpha(i\lambda z) - ih_\alpha(i\lambda z) \quad \forall z \in \mathbb{C}, \quad (1.2)$$

where j_α and h_α are the normalized Bessel and Struve functions (see [4]).

Moreover, the Bessel-Struve kernel is a holomorphic function on $\mathbb{C} \times \mathbb{C}$ and it can be expanded in a power series in the form

$$S_\alpha(\lambda z) = \sum_{n=0}^{+\infty} \frac{(\lambda z)^n}{c_n(\alpha)}, \quad c_n(\alpha) = \frac{\sqrt{\pi} n! \Gamma(n/2 + \alpha + 1)}{\Gamma(\alpha + 1) \Gamma((n+1)/2)}. \quad (1.3)$$

The Bessel-Struve intertwining operator χ_α is defined from the space $\mathcal{H}$ into itself by

$$\chi_\alpha f(z) = \sum_{n=0}^{+\infty} \frac{d^n f}{dz^n}(0) \frac{z^n}{c_n(\alpha)} \quad \forall f \in \mathcal{H}, z \in \mathbb{C}. \quad (1.4)$$

The dual intertwining operator ${}^t\chi_\alpha$ of χ_α is defined on $\mathcal{H}'$ (the dual space of $\mathcal{H}$) by

$$\langle {}^t\chi_\alpha T, g \rangle = \langle T, \chi_\alpha g \rangle \quad \forall g \in \mathcal{H}, T \in \mathcal{H}'. \quad (1.5)$$

The Bessel-Struve transform $\mathcal{F}_\alpha$ is defined on $\mathcal{H}'$ by

$$\mathcal{F}_\alpha(T)(\lambda) = \langle T, S_\alpha(-i\lambda \cdot) \rangle \quad \forall \lambda \in \mathbb{C}. \quad (1.6)$$

We use the transmutation operator χ_α to define the Bessel-Struve translation operators $\tau_z, z \in \mathbb{C}$, associated with ℓ_α , and the Bessel-Struve convolution on $\mathcal{H}$ and $\mathcal{H}'$. A function f in $\mathcal{H}$ is said to be mean periodic if the closed subspace $\Omega(f)$ generated by $\tau_z f, z \in \mathbb{C}$, satisfies $\Omega(f) \neq \mathcal{H}$.

The objective of this paper is to characterize every transmutation operator of ℓ_α into the second derivative operator from $\mathcal{H}$ into itself. Next, we study the mean-periodic functions associated with the Bessel-Struve operator and we characterize the continuous linear mappings from $\mathcal{H}$ into itself which commute with ℓ_α .

We point out that the harmonic analysis associated with differential and differential-difference operators allows many applications as the study of integral representations (see [9]), Plancherel, and reconstruction formulas and other applications as the use of wavelets packets in the inversion of transmutation operators for the J. L. Lions operator and the Dunkl operator (see [5, 6]).

The content of this paper is as follows.

In [Section 2](#), we prove that the Bessel-Struve intertwining operator χ_α is a topological isomorphism from $\mathcal{H}$ into itself satisfying

$$\begin{aligned} \forall f \in \mathcal{H}, \quad \ell_\alpha \chi_\alpha f &= \chi_\alpha \frac{d^2}{dz^2} f, \\ \chi_\alpha f(0) &= f(0), \quad (\chi_\alpha f)'(0) = \frac{f'(0)}{c_1(\alpha)}. \end{aligned} \quad (1.7)$$

Using this operator and its dual, we study the harmonic analysis associated with the operator ℓ_α (Bessel-Struve transform, Bessel-Struve translation operators, and Bessel-Struve convolution). Next, we determine all transmutation operators W from the Bessel-Struve operator ℓ_α to the second derivative operator d^2/dz^2 .

In [Section 3](#), we study the mean-periodic functions associated with ℓ_α . Next, we give the central result of the paper, which characterizes the continuous linear mappings from $\mathcal{H}$ into itself which commute with ℓ_α .

2. Bessel-Struve transmutation operators. In this section, we consider the normalized Bessel and Struve functions which allow to define the Bessel-Struve kernel. Next, we define the Bessel-Struve intertwining operator χ_α and its dual ${}^t\chi_\alpha$; after that, we study the harmonic analysis associated with the operator ℓ_α . The aim of this section is to characterize every transmutation operator of ℓ_α into d^2/dz^2 from $\mathcal{H}$ into itself.

Let $\alpha > -1/2$. The normalized Bessel function j_α is the kernel defined on $\mathbb{C}$ by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{+\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n+\alpha+1)}, \quad (2.1)$$

where J_α is the Bessel function of order α (see [4, 12]).

The normalized Struve function h_α is the kernel defined on $\mathbb{C}$ by

$$h_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{\mathbf{H}_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{+\infty} \frac{(-1)^n (z/2)^{2n+1}}{\Gamma(n+3/2) \Gamma(n+\alpha+3/2)}, \quad (2.2)$$

where $\mathbf{H}_\alpha$ is the Struve function of order α (see [4, 12]).

This function has the following Poisson integral representation:

$$h_\alpha(z) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^1 (1-t^2)^{\alpha-1/2} \sin(zt) dt. \quad (2.3)$$

The function $z \rightarrow h_\alpha(i\lambda z)$, $\lambda, z \in \mathbb{C}$, is the unique solution of the differential equation

$$\begin{aligned} \ell_\alpha u(z) &= \lambda^2 u(z), \\ u(0) &= 0, \quad u'(0) = \frac{\lambda \Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+3/2)}. \end{aligned} \quad (2.4)$$

The functions h_α and j_α are related by the formula

$$h_\alpha(z) = \frac{\Gamma(\alpha+1)z}{\sqrt{\pi}\Gamma(\alpha+3/2)} \int_0^{\pi/2} j_{\alpha+1/2}(z \sin \varphi) \sin \varphi d\varphi. \quad (2.5)$$

The Bessel-Struve kernel is the function S_α defined on $\mathbb{C}$ by

$$S_\alpha(z) = j_\alpha(iz) - ih_\alpha(iz). \quad (2.6)$$

This kernel can be expanded in a power series in the form

$$S_\alpha(z) = \sum_{n=0}^{+\infty} \frac{z^n}{c_n(\alpha)}, \quad c_n(\alpha) = \frac{\sqrt{\pi} n! \Gamma(n/2 + \alpha + 1)}{\Gamma(\alpha+1) \Gamma((n+1)/2)}, \quad (2.7)$$

and has the following integral representation:

$$S_\alpha(z) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^1 (1-t^2)^{\alpha-1/2} \exp(zt) dt. \quad (2.8)$$

The function $z \rightarrow S_\alpha(\lambda z)$, $\lambda \in \mathbb{C}$, is the unique solution of the differential equation

$$\begin{aligned} \ell_\alpha u(z) &= \lambda^2 u(z), \\ u(0) &= 1, \quad u'(0) = \frac{\lambda \Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 3/2)}. \end{aligned} \quad (2.9)$$

NOTATIONS.

- (i) We denote by $\mathcal{H}$, the space of entire functions on $\mathbb{C}$, with the topology of the uniform convergence on compact subsets of $\mathbb{C}$. Thus $\mathcal{H}$ is a Fréchet space.
- (ii) We denote by $\mathcal{H}'$, the dual space of $\mathcal{H}$.

PROPOSITION 2.1. *The operator χ_α defined by*

$$\chi_\alpha f(z) = \sum_{n=0}^{+\infty} \frac{d^n f}{dz^n}(0) \frac{z^n}{c_n(\alpha)}, \quad \forall f \in \mathcal{H}, z \in \mathbb{C}, \quad (2.10)$$

is an isomorphism from $\mathcal{H}$ into itself satisfying the transmutation relation

$$\begin{aligned} \forall f \in \mathcal{H}, \quad \ell_\alpha \chi_\alpha f &= \chi_\alpha \frac{d^2}{dz^2} f, \\ \chi_\alpha f(0) &= f(0), \quad (\chi_\alpha f)'(0) = \frac{f'(0)}{c_1(\alpha)}. \end{aligned} \quad (2.11)$$

The inverse of χ_α is given by

$$\chi_\alpha^{-1}(f)(z) = \sum_{n=0}^{+\infty} \ell_\alpha^n(f)(0) \frac{z^{2n}}{(2n)!} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_\alpha^n f)}{dz}(0) \frac{z^{2n+1}}{(2n+1)!} \quad \forall f \in \mathcal{H}, z \in \mathbb{C}. \quad (2.12)$$

PROOF. First we prove that the image of the function f in $\mathcal{H}$ by χ_α is an entire function, and that χ_α is a continuous linear operator.

Since f is an entire function, from the Cauchy integral formula, we have

$$\forall n \in \mathbb{N}, \quad \frac{d^n f}{dz^n}(0) = \frac{n!}{2i\pi} \int_{C_R} \frac{f(w)}{w^{n+1}} dw, \quad (2.13)$$

where C_R is a circle with center 0 and radius $R > 0$. Hence there exists a positive constant M such that

$$\forall n \in \mathbb{N}, \quad \left| \frac{d^n f}{dz^n}(0) \frac{1}{c_n(\alpha)} \right| \leq MR^{-n} \|f\|_R, \quad (2.14)$$

where

$$\|f\|_R = \max_{|z| \leq R} |f(z)|. \quad (2.15)$$

As R is arbitrary, the radius of convergence of the power series in (2.10) is infinite. Thus $\chi_\alpha(f)$ is an entire function.

Using (2.14), we obtain

$$\forall f \in \mathcal{H}, \quad \|\chi_\alpha(f)\|_R \leq 2M\|f\|_{2R}. \quad (2.16)$$

Thus χ_α defines a continuous linear mapping from $\mathcal{H}$ into itself. Furthermore, using the fact that

$$\forall n \geq 2, \quad \ell_\alpha(z^n) = \frac{c_n(\alpha)}{c_{n-2}(\alpha)} z^{n-2}, \quad (2.17)$$

we get

$$\forall z \in \mathbb{C}, \quad \ell_\alpha \chi_\alpha f(z) = \sum_{n=2}^{+\infty} \frac{d^n f}{dz^n}(0) \frac{z^{n-2}}{c_{n-2}(\alpha)} = \sum_{n=0}^{+\infty} \frac{d^{n+2} f}{dz^{n+2}}(0) \frac{z^n}{c_n(\alpha)} = \chi_\alpha \frac{d^2}{dz^2} f(z). \quad (2.18)$$

It is clear that

$$\chi_\alpha f(0) = f(0), \quad (\chi_\alpha f)'(0) = \frac{f'(0)}{c_1(\alpha)}. \quad (2.19)$$

Suppose now that $\chi_\alpha f = 0$ for a certain $f \in \mathcal{H}$. Then, according to (2.10), $(d^n f/dz^n)(0) = 0$, $n \in \mathbb{N}$. Hence $f = 0$, thus we prove that χ_α is a one-to-one mapping from $\mathcal{H}$ into itself.

Now we consider the operator ψ on $\mathcal{H}$ defined by

$$\psi f(z) = \sum_{n=0}^{+\infty} \ell_\alpha^n f(0) \frac{z^{2n}}{(2n)!} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_\alpha^n f)}{dz}(0) \frac{z^{2n+1}}{(2n+1)!} \quad \forall z \in \mathbb{C}. \quad (2.20)$$

In the same way as for χ_α and by a simple calculation, we prove that ψ is a continuous linear mapping from $\mathcal{H}$ into itself and

$$\forall f \in \mathcal{H}, \quad \chi_\alpha \psi f = \psi \chi_\alpha f = f. \quad (2.21)$$

Then χ_α is a topological isomorphism from $\mathcal{H}$ into itself. $\square$

REMARKS 2.2. (i) The operator χ_α which is a transmutation operator from ℓ_α into d^2/dz^2 on $\mathcal{H}$ will be called the Bessel-Struve intertwining operator on $\mathcal{C}$.

(ii) Formula (2.10) means that the Taylor coefficients of the image of an entire function by χ_α are multiplied by the Taylor coefficients of the Bessel-Struve kernel.

COROLLARY 2.3. (i) For $\lambda, z \in \mathbb{C}$,

$$S_\alpha(\lambda z) = \chi_\alpha(e^{\lambda \cdot})(z). \quad (2.22)$$

(ii) Every function f in $\mathcal{H}$ can be expanded in a power series:

$$\forall z \in \mathbb{C}, \quad f(z) = \sum_{n=0}^{+\infty} \ell_\alpha^n f(0) \frac{z^{2n}}{c_{2n}(\alpha)} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_\alpha^n f)}{dz}(0) \frac{z^{2n+1}}{c_{2n+1}(\alpha)}. \quad (2.23)$$

DEFINITION 2.4. The dual intertwining operator ${}^t\chi_\alpha$ of χ_α is defined on $\mathcal{H}'$ by

$$\langle {}^t\chi_\alpha(T), g \rangle = \langle T, \chi_\alpha(g) \rangle \quad \forall g \in \mathcal{H}. \quad (2.24)$$

REMARK 2.5. From the properties of the operator χ_α , we deduce that the operator ${}^t\chi_\alpha$ is an isomorphism from $\mathcal{H}'$ into itself; the inverse operator $({}^t\chi_\alpha)^{-1}$ is given by

$$\langle ({}^t\chi_\alpha)^{-1}(T), g \rangle = \langle T, \chi_\alpha^{-1}(g) \rangle \quad \forall g \in \mathcal{H}. \quad (2.25)$$

NOTATIONS.

- (i) We denote by $\text{Exp}_a(\mathbb{C})$, $a > 0$, the space of functions of exponential type a . It is the space of functions $f \in \mathcal{H}$ such that

$$N_a(f) = \sup_{z \in \mathbb{C}} |f(z)| e^{-a|z|} < +\infty. \quad (2.26)$$

- (ii) We denote by $\text{Exp}(\mathbb{C})$, the space of functions with exponential type. It is given by

$$\text{Exp}(\mathbb{C}) = \bigcup_{a>0} \text{Exp}_a(\mathbb{C}). \quad (2.27)$$

The space $\text{Exp}(\mathbb{C})$ is endowed with the inductive limit topology.

- (iii) We denote by $\mathcal{F}$, the classical Fourier transform defined on $\mathcal{H}'$ by

$$\mathcal{F}(T)(\lambda) = \langle T, e^{-i\lambda \cdot} \rangle \quad \forall \lambda \in \mathbb{C}. \quad (2.28)$$

- (iv) We denote by $*_o$, the classical convolution product given by

$$T *_o f(z) = \langle T_w, f(w+z) \rangle \quad \forall T \in \mathcal{H}', f \in \mathcal{H}, z \in \mathbb{C}. \quad (2.29)$$

DEFINITION 2.6. The Bessel-Struve transform $\mathcal{F}_\alpha$ of $T \in \mathcal{H}'$ is given by

$$\mathcal{F}_\alpha(T)(\lambda) = \langle T, S_\alpha(-i\lambda \cdot) \rangle \quad \forall \lambda \in \mathbb{C}. \quad (2.30)$$

REMARK 2.7. From [Corollary 2.3\(i\)](#) and [Definition 2.4](#), we obtain

$$\forall T \in \mathcal{H}', \quad \mathcal{F}_\alpha(T)(\lambda) = \mathcal{F}_\alpha({}^t\chi_\alpha(T))(\lambda). \quad (2.31)$$

PROPOSITION 2.8. The Bessel-Struve transform $\mathcal{F}_\alpha$ is a topological isomorphism from $\mathcal{H}'$ into $\text{Exp}(\mathbb{C})$.

PROOF. According to [\[8\]](#), the classical Fourier transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{H}'$ into $\text{Exp}(\mathbb{C})$. Then the result follows from [\(2.25\)](#) and [\(2.31\)](#). $\square$

LEMMA 2.9. *Let $f \in \mathcal{H}$. The Cauchy problem*

$$\begin{aligned} \ell_{\alpha,z} u(z, w) &= \ell_{\alpha,w} u(z, w), \\ u(0, w) &= f(w), \quad \frac{\partial}{\partial z} u(0, w) = f'(w) \end{aligned} \quad (2.32)$$

has a unique solution that is an entire function on $\mathbb{C} \times \mathbb{C}$ given by

$$u(z, w) = \chi_{\alpha,z} \chi_{\alpha,w} [\chi_{\alpha}^{-1}(f)(z + w)] \quad \forall z, w \in \mathbb{C}. \quad (2.33)$$

PROOF. From Proposition 2.1, (2.32) is equivalent to the Cauchy problem

$$\begin{aligned} \frac{\partial^2}{\partial z^2} v(z, w) &= \frac{\partial^2}{\partial w^2} v(z, w), \\ v(0, w) &= \chi_{\alpha}^{-1}(f)(w), \quad \frac{\partial}{\partial z} v(0, w) = \frac{d(\chi_{\alpha}^{-1}f)}{dz}(w), \end{aligned} \quad (2.34)$$

where

$$v(z, w) = \chi_{\alpha,z}^{-1} \chi_{\alpha,w}^{-1} u(z, w). \quad (2.35)$$

But the solution of (2.34) is given by

$$v(z, w) = \chi_{\alpha}^{-1}(f)(z + w) \quad \forall z, w \in \mathbb{C}. \quad (2.36)$$

□

DEFINITION 2.10. The Bessel-Struve translation operators τ_z , $z \in \mathbb{C}$, associated with the operator ℓ_{α} , is defined on $\mathcal{H}$ by

$$\tau_z f(w) = \chi_{\alpha,z} \chi_{\alpha,w} [\chi_{\alpha}^{-1}(f)(z + w)] \quad \forall w \in \mathbb{C}. \quad (2.37)$$

The operator τ_z , $z \in \mathbb{C}$, satisfies the following properties.

- (i) For all $z \in \mathbb{C}$, the operator τ_z is linear continuous from $\mathcal{H}$ into itself.
- (ii) For all $f \in \mathcal{H}$ and $z, w \in \mathbb{C}$,

$$\begin{aligned} \tau_z f(w) &= \tau_w f(z), & \tau_0 f(w) &= f(w), \\ \tau_z(\tau_w f) &= \tau_w(\tau_z f), & \ell_{\alpha} \tau_z f &= \tau_z \ell_{\alpha} f. \end{aligned} \quad (2.38)$$

- (iii) The following product formula holds:

$$\forall z, w \in \mathbb{C}, \quad \tau_z(S_{\alpha}(\lambda \cdot))(w) = S_{\alpha}(\lambda w) S_{\alpha}(\lambda z). \quad (2.39)$$

COROLLARY 2.11. *Let $f \in \mathcal{H}$ and $z \in \mathbb{C}$. Then the function $w \rightarrow \tau_z f(w)$ can be expanded in the Taylor series:*

$$\forall w \in \mathbb{C}, \quad \tau_z f(w) = \sum_{n=0}^{+\infty} \ell_{\alpha}^n f(z) \frac{w^{2n}}{c_{2n}(\alpha)} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_{\alpha}^n f)}{dz}(z) \frac{w^{2n+1}}{c_{2n+1}(\alpha)}. \quad (2.40)$$

PROOF. For $z, w \in \mathbb{C}$, we have

$$\tau_z f(w) = \chi_{\alpha, z} \chi_{\alpha, w} [\chi_{\alpha}^{-1}(f)(z + w)]. \quad (2.41)$$

Applying [Corollary 2.3\(ii\)](#) to the function $w \rightarrow \tau_z f(w)$, we obtain

$$\begin{aligned} \tau_z f(w) &= \sum_{n=0}^{+\infty} \ell_{\alpha}^n[\tau_z f](0) \frac{w^{2n}}{c_{2n}(\alpha)} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_{\alpha}^n[\tau_z f])}{dz}(0) \frac{w^{2n+1}}{c_{2n+1}(\alpha)} \\ &= \sum_{n=0}^{+\infty} \tau_z[\ell_{\alpha}^n f](0) \frac{w^{2n}}{c_{2n}(\alpha)} + c_1(\alpha) \sum_{n=0}^{+\infty} \tau_z \left[\frac{d(\ell_{\alpha}^n f)}{dz} \right](0) \frac{w^{2n+1}}{c_{2n+1}(\alpha)}, \end{aligned} \quad (2.42)$$

which proves the result. $\square$

DEFINITION 2.12. (i) The convolution product of two elements T and K in $\mathcal{H}'$ is defined by

$$\langle T * K, f \rangle = \langle T_z, \langle K_w, \tau_z f(w) \rangle \rangle \quad \forall f \in \mathcal{H}. \quad (2.43)$$

(ii) Let $T \in \mathcal{H}'$ and $f \in \mathcal{H}$. The convolution product of T and f is the function in $\mathcal{H}$ defined by

$$T * f(z) = \langle T_w, \tau_z f(w) \rangle \quad \forall z \in \mathbb{C}. \quad (2.44)$$

The convolution $*$ satisfies the following properties.

(i) Let $T, K \in \mathcal{H}'$ and let $f \in \mathcal{H}$. Then

$$T * (K * f) = (T * K) * f. \quad (2.45)$$

(ii) Let $T, K \in \mathcal{H}'$. Then

$$\mathcal{F}_{\alpha}(T * K) = \mathcal{F}_{\alpha}(T) \mathcal{F}_{\alpha}(K). \quad (2.46)$$

PROPOSITION 2.13. Let $T \in \mathcal{H}'$ and let $f \in \mathcal{H}$. Then

$$\begin{aligned} ({}^t\chi_{\alpha})^{-1}(T) * \chi_{\alpha}(f) &= \chi_{\alpha}(T *_o f), \\ {}^t\chi_{\alpha}(T) *_o \chi_{\alpha}^{-1}(f) &= \chi_{\alpha}^{-1}(T * f), \end{aligned} \quad (2.47)$$

where $*_o$ is the classical convolution product given by [\(2.29\)](#).

PROOF. From [Definition 2.12](#), we have

$$\begin{aligned} \forall z \in \mathbb{C}, \quad ({}^t\chi_{\alpha})^{-1}(T) * \chi_{\alpha}(f)(z) \\ = \left\langle ({}^t\chi_{\alpha})^{-1}(T)_{\xi}, \tau_z(\chi_{\alpha}(f))(\xi) \right\rangle = \left\langle T_{\xi}, \chi_{\alpha, \xi}^{-1} \tau_z(\chi_{\alpha}(f))(\xi) \right\rangle. \end{aligned} \quad (2.48)$$

But from [Definition 2.10](#), we obtain

$$\forall \xi \in \mathbb{C}, \quad \chi_{\alpha, \xi}^{-1} \tau_z(\chi_{\alpha}(f))(\xi) = \chi_{\alpha, z}(f)(\xi - z). \quad (2.49)$$

Thus

$$\begin{aligned} &({}^t\chi_\alpha)^{-1}(T) * \chi_\alpha(f)(z) \\ &= \langle T_\xi, \chi_{\alpha,z}(f)(\xi - z) \rangle = \chi_{\alpha,z}(\langle T_\xi, f(\xi - z) \rangle) = \chi_\alpha(T *_o f)(z), \end{aligned} \quad (2.50)$$

which proves the first relation.

For the second relation, we have

$$\begin{aligned} \forall z \in \mathbb{C}, \quad {}^t\chi_\alpha(T) *_o ({}^t\chi_\alpha)^{-1}(f)(z) \\ = \langle {}^t\chi_\alpha(T)_\xi, \chi_\alpha^{-1}(f)(\xi - z) \rangle = \langle T_\xi, \chi_{\alpha,\xi} \chi_\alpha^{-1}(f)(\xi - z) \rangle. \end{aligned} \quad (2.51)$$

But

$$\forall z, \xi \in \mathbb{C}, \quad \chi_{\alpha,\xi} \chi_\alpha^{-1}(f)(\xi - z) = \chi_{\alpha,z}^{-1}(\tau_z f)(\xi). \quad (2.52)$$

So

$$\forall z \in \mathbb{C}, \quad {}^t\chi_\alpha(T) * (\chi_\alpha)^{-1}(f)(z) = \chi_{\alpha,z}^{-1} \langle T_\xi, \tau_z f(\xi) \rangle = \chi_\alpha^{-1}(T * f)(z), \quad (2.53)$$

which finishes the proof. $\square$

Now we are in position to derive the main result of this section.

NOTATIONS.

- (i) We denote $D = d/dz$.
- (ii) We denote by $\mathcal{G}_{D^2}$, the group of isomorphisms Y from $\mathcal{H}$ into itself such that

$$YD^2 = D^2Y. \quad (2.54)$$

THEOREM 2.14. *Every transmutation operator W of ℓ_α into D^2 from $\mathcal{H}$ into itself is of the form*

$$Wf(z) = ({}^t\chi_\alpha)^{-1}T_0 * \chi_\alpha(f)(z) + ({}^t\chi_\alpha)^{-1}T_1 * \chi_\alpha(f)(-z) \quad \forall z \in \mathbb{C}, \quad (2.55)$$

where $T_0, T_1 \in \mathcal{H}'$.

PROOF. It is clear that every transmutation operator W of ℓ_α into D^2 from $\mathcal{H}$ into itself is of the form $W = \chi_\alpha Y$, where $Y \in \mathcal{G}_{D^2}$. Then according to [3], every element Y of $\mathcal{G}_{D^2}$ has the form

$$Yf(z) = T_0 *_o f(z) + T_1 *_o f(-z), \quad (2.56)$$

where $T_0, T_1 \in \mathcal{H}'$. Thus, we can write

$$\forall z \in \mathbb{C}, \quad Wf(z) = \chi_\alpha(T_0 *_o f)(z) + \chi_\alpha(T_1 *_o f)(-z). \quad (2.57)$$

Hence the result follows from [Proposition 2.13](#). $\square$

3. Mean-periodic functions and commutators of ℓ_α

3.1. Mean-periodic functions

DEFINITION 3.1. A function f in $\mathcal{H}$ is said to be mean periodic if the closed subspace $\Omega(f)$ generated by $\tau_z f$, $z \in \mathbb{C}$, satisfies

$$\Omega(f) \neq \mathcal{H}. \quad (3.1)$$

From Hahn-Banach theorem, this definition is equivalent to the following.

DEFINITION 3.2. A function f in $\mathcal{H}$ is said to be mean periodic if there exists $T \in \mathcal{H}' \setminus \{0\}$ such that

$$\forall z \in \mathbb{C}, \quad T * f(z) = 0. \quad (3.2)$$

DEFINITION 3.3. Let $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$. The function $S_{\alpha, \ell}(\lambda, \cdot)$ is defined by

$$S_{\alpha, \ell}(\lambda, z) = \frac{d^\ell}{d\mu^\ell} S_\alpha(\mu z) \Big|_{\mu=-i\lambda} \quad \forall z \in \mathbb{C}. \quad (3.3)$$

LEMMA 3.4. Let $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$. Then the function $S_{\alpha, \ell}(\lambda, \cdot)$ is mean periodic and

$$\forall z \in \mathbb{C}, \quad S_{\alpha, \ell}(\lambda, z) = \chi_\alpha(\xi^\ell \exp(-i\lambda\xi))(z). \quad (3.4)$$

PROOF. Let $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$. According to [Proposition 2.8](#), there exists $T \in \mathcal{H}' \setminus \{0\}$ such that

$$\forall j = 0, \dots, \ell, \quad \frac{d^j}{d\mu^j} (\mathcal{F}_\alpha(T))(\mu) \Big|_{\mu=\lambda} = 0. \quad (3.5)$$

Then from the properties of the Bessel-Struve translation for every $z \in \mathbb{C}$, we can write

$$\begin{aligned} (T * S_{\alpha, \ell}(\lambda \cdot))(z) &= \left\langle T(w), \frac{d^\ell}{d\mu^\ell} (\tau_w(S_\alpha(\mu \cdot))(z)) \Big|_{\mu=-i\lambda} \right\rangle \\ &= \left\langle T(w), \frac{d^\ell}{d\mu^\ell} (S_\alpha(\mu z) S_\alpha(\mu w)) \Big|_{\mu=-i\lambda} \right\rangle \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{d^{\ell-j}}{d\mu^{\ell-j}} (S_\alpha(\mu z)) \Big|_{\mu=-i\lambda} \frac{d^j}{d\mu^j} \mathcal{F}_\alpha(T)(\mu) \Big|_{\mu=\lambda} \\ &= 0. \end{aligned} \quad (3.6)$$

Thus we prove that $S_{\alpha, \ell}(\lambda, \cdot)$ is a mean-periodic function. The result follows from [\(1.3\)](#) and [\(2.10\)](#). $\square$

Let $f \in \mathcal{H}$. The following proposition characterizes the functions which belong to $\Omega(f)$.

PROPOSITION 3.5. Let $f \in \mathcal{H}$, $\ell \in \mathbb{N}$, and $\lambda \in \mathbb{C}$. The function $S_{\alpha,j}(\lambda, \cdot)$, $0 \leq j \leq \ell$, belongs to $\Omega(f)$ if and only if for all T in $\mathcal{H}'$ satisfying

$$\forall z \in \mathbb{C}, \quad T * f(z) = 0, \quad (3.7)$$

then

$$\frac{d^j}{d\mu^j} (\mathcal{F}_\alpha(T))(\mu) \Big|_{\mu=\lambda} = 0, \quad 0 \leq j \leq \ell. \quad (3.8)$$

PROOF. If $S_{\alpha,j}(\lambda, \cdot)$, $0 \leq j \leq \ell$, belongs to $\Omega(f)$, then for all $T \in \mathcal{H}'$ satisfying (3.7) we have

$$\langle T, S_{\alpha,j}(\lambda, \cdot) \rangle = 0. \quad (3.9)$$

Then

$$\begin{aligned} \langle T, S_{\alpha,j}(\lambda, \cdot) \rangle &= \frac{d^j}{d\mu^j} \left\langle T, S_\alpha(\mu \cdot) \right|_{\mu=-i\lambda} \rangle \\ &= \frac{d^j}{d\mu^j} \mathcal{F}_\alpha(T)(\mu) \Big|_{\mu=\lambda} = 0. \end{aligned} \quad (3.10)$$

The converse follows from the Hahn-Banach theorem. $\square$

DEFINITION 3.6. Let $f \in \mathcal{H}$ be a mean-periodic function. The spectrum $\text{Sp}(f)$ of f is the set

$$\text{Sp}(f) = \{(\lambda, \ell), \lambda \in \mathbb{C}, \ell \in \mathbb{N}, S_{\alpha,j}(\lambda \cdot) \in \Omega(f), 0 \leq j \leq \ell\}. \quad (3.11)$$

REMARKS 3.7. (i) From Proposition 3.5, we have

$$\text{Sp}(f) = \left\{ (\lambda, \ell), \lambda \in \mathbb{C}, \ell \in \mathbb{N}, \frac{d^j}{d\mu^j} \mathcal{F}_\alpha(T)(\mu) \Big|_{\mu=\lambda} = 0, j = 0, 1, \dots, \ell, T \in (\Omega(f))^\perp \right\}. \quad (3.12)$$

(ii) If $\text{Sp}(f) \neq \emptyset$, we say that $\Omega(f)$ admits a spectral analysis associated with ℓ_α .

PROPOSITION 3.8. Let $f \in \mathcal{H}$. Denote by $\$(f)$ the closed subspace of $\mathcal{H}$ generated by $\{D^k \ell_\alpha^n f\}_{n \in \mathbb{N}; k=0,1}$. Then $\Omega(f) = \$(f)$.

PROOF. According to Corollary 2.11, we have, for every $g \in \mathcal{H}$,

$$Dg = \lim_{w \rightarrow 0} \frac{1}{w} [\tau_w g - g], \quad (3.13)$$

$$\ell_\alpha g = \lim_{w \rightarrow 0} \frac{c_2(\alpha)}{w^2} [\tau_w g - g - w Dg], \quad (3.14)$$

$$D\ell_\alpha g = \lim_{w \rightarrow 0} \frac{c_3(\alpha)}{c_1(\alpha)w^2} \left[\tau_w g - g - wg - \frac{w^2}{c_2(\alpha)} \ell_\alpha g \right] \quad (3.15)$$

in the sense of the convergence in $\mathcal{H}$.

Suppose that $g \in \Omega(f)$. Then, for every $w \in \mathbb{C}$, $\tau_w g \in \Omega(f)$. Hence we conclude that for $k = 0, 1$, $D^k \ell_\alpha g \in \Omega(f)$. By induction, we can prove that, for every $n \in \mathbb{N}$ and $k = 0, 1$, $D^k \ell_\alpha^n g \in \Omega(f)$. In particular, for every $n \in \mathbb{N}$ and $k = 0, 1$, $D^k \ell_\alpha^n f \in \Omega(f)$. Thus we conclude that $\$(f) \subset \Omega(f)$.

Let now $g \in \$(f)$. Using once more [Corollary 2.11](#), we prove that, for every $w \in \mathbb{C}$, $\tau_w g \in \$(f)$. In particular, for every $w \in \mathbb{C}$, $\tau_w f \in \$(f)$. Hence, $\Omega(f) = \$(f)$. $\square$

COROLLARY 3.9. *Let $f \in \mathcal{H}$. Then f is a mean periodic if and only if $\$(f) \neq \mathcal{H}$.*

COROLLARY 3.10. *Let $f \in \mathcal{H}$. Then f is a mean-periodic function if and only if $\chi_\alpha^{-1}(f)$ is a classical mean-periodic function.*

THEOREM 3.11. *Let $f \in \mathcal{H}$. Then f is a mean-periodic function if and only if f is a limit of finite linear combination of the functions $S_{\alpha,j}(\lambda, \cdot)$, $0 \leq j \leq \ell$, such that $(\lambda, \ell) \in \text{Sp}(f)$.*

PROOF. To see this property, we can use [Lemma 3.4](#) and a celebrated result about classical mean-periodic functions established in [[11](#), page 926]. $\square$

COROLLARY 3.12. *Every mean-periodic function such that $\text{Sp}(f) = \emptyset$ is zero.*

3.2. The commutator of ℓ_α

NOTATIONS.

(i) We denote by $\mathcal{G}_\alpha$, the group of isomorphisms Y of $\mathcal{H}$ into itself such that

$$Y \ell_\alpha = \ell_\alpha Y; \quad (3.16)$$

(ii) We denote by $\mathfrak{G}_\alpha(f)$ (resp., $\mathfrak{G}_{D^2}(f)$), the closed subspaces of $\mathcal{H}$ generated by Yf , $Y \in \mathcal{G}_\alpha$, (resp., $\mathcal{G}_{D^2}$).

PROPOSITION 3.13. (i) *The group $\mathcal{G}_\alpha$ is isomorphic to $\mathcal{G}_{D^2}$.*

(ii)

$$\forall f \in \mathcal{H}, \quad \mathfrak{G}_\alpha(f) = \chi_\alpha \mathfrak{G}_{D^2}(\chi_\alpha^{-1}(f)). \quad (3.17)$$

PROPOSITION 3.14. *The set of functions f in $\mathcal{H}$ satisfying*

$$\mathfrak{G}_\alpha(f) \neq \mathcal{H} \quad (3.18)$$

with the set of mean-periodic functions is identified.

PROOF. From [Proposition 3.13](#), $f \in \mathcal{H}$ satisfies (3.18) if and only if $\chi_\alpha^{-1}(f)$ satisfies

$$\mathfrak{G}_{D^2} \chi_\alpha^{-1}(f) \neq \mathcal{H}. \quad (3.19)$$

But these functions are classical mean-periodic functions. The result follows from [Proposition 3.13](#). $\square$

Now we are able to state the main result of this paper.

THEOREM 3.15. *Let L be a continuous linear mapping from $\mathcal{H}$ into itself. The following statements are equivalent.*

- (i) L commutes with Bessel-Struve translation operators τ_z , $z \in \mathbb{C}$, on $\mathcal{H}$, that is, $\tau_z L = L \tau_z$, $z \in \mathbb{C}$, on $\mathcal{H}$.
- (ii) L commutes with the Bessel-Struve operator ℓ_α on $\mathcal{H}$, that is, $\ell_\alpha L = L \ell_\alpha$ on $\mathcal{H}$.
- (iii) There exists a unique element T in $\mathcal{H}'$ such that $Lf = T * f$, $f \in \mathcal{H}$.
- (iv) There exists a complex Borel regular measure γ having compact support on $\mathbb{C}$, for which for all $f \in \mathcal{H}$,

$$L(f)(z) = \int_{\mathbb{C}} (\tau_z f)(w) d\gamma(w) \quad \forall z \in \mathbb{C}. \quad (3.20)$$

- (v) There exists $\Psi, \Phi \in \text{Exp}(\mathbb{C})$ such that for all $f \in \mathcal{H}$, $Lf = \Psi(\ell_\alpha)f + D\Phi(\ell_\alpha)f$, where $\Psi(\ell_\alpha)f$ and $D\Phi(\ell_\alpha)f$ are given by

$$\begin{aligned} [\Psi(\ell_\alpha)f](z) &= \sum_{n=0}^{+\infty} a_{2n} \ell_\alpha^n f(z), \quad \forall z \in \mathbb{C}, \\ [D\Phi(\ell_\alpha)f](z) &= c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} \frac{d(\ell_\alpha^n f)}{dz}(z), \quad \forall z \in \mathbb{C}, \end{aligned} \quad (3.21)$$

where $\Psi(z) = \sum_{n=0}^{+\infty} a_{2n} z^n$ and $\Phi(z) = c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} z^n$.

PROOF. (i) $\Rightarrow$ (ii). From (3.13) and (3.14), we have

$$\begin{aligned} D(Lg) &= \lim_{w \rightarrow 0} \frac{1}{w} [\tau_w Lg - Lg - wDLg] = L \left(\lim_{w \rightarrow 0} \frac{1}{w} [\tau_w g - g] \right) = L(Dg), \\ \ell_\alpha(Lg) &= \lim_{w \rightarrow 0} \frac{c_2(\alpha)}{w^2} [\tau_w Lg - g - wDLg] = L \left(\lim_{w \rightarrow 0} \frac{c_2(\alpha)}{w^2} [\tau_w g - g - wDg] \right) = L(\ell_\alpha g). \end{aligned} \quad (3.22)$$

Hence (i) implies (ii).

(ii) $\Rightarrow$ (i). We decide the results from Corollary 2.11.

(i) $\Rightarrow$ (iii). Assume that (i) holds. We define the functional T on $\mathcal{H}$ as follows:

$$\langle T, f \rangle = L(f)(0), \quad f \in \mathcal{H}. \quad (3.23)$$

It is clear that T is in $\mathcal{H}'$ and $Lf = T * f$, $f \in \mathcal{H}$.

(iii) $\Rightarrow$ (iv). It follows immediately from Hahn-Banach and Riesz representation theorems.

(iv) $\Rightarrow$ (v). Suppose that for all $f \in \mathcal{H}$, we have

$$\forall z \in \mathbb{C}, \quad L(f)(z) = \int_{\mathbb{C}} (\tau_z f)(w) d\gamma(w), \quad (3.24)$$

where γ is a complex Borel regular measure with compact support.

According to [Corollary 2.11](#), we obtain for all $z \in \mathbb{C}$,

$$L(f)(z) = \sum_{n=0}^{+\infty} \ell_{\alpha}^n f(z) \int_{\mathbb{C}} \frac{w^{2n}}{c_{2n}(\alpha)} d\gamma(w) + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_{\alpha}^n f)}{dz}(z) \int_{\mathbb{C}} \frac{w^{2n+1}}{c_{2n+1}(\alpha)} d\gamma(w). \quad (3.25)$$

Hence

$$Lf = \Psi(\ell_{\alpha})f + D\Phi(\ell_{\alpha})f, \quad (3.26)$$

where

$$\Psi(z) = \sum_{n=0}^{+\infty} a_{2n} z^n, \quad \Phi(z) = c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} z^n, \quad (3.27)$$

with, for every $n \in \mathbb{N}$,

$$a_n = \int_{\mathbb{C}} \frac{w^n}{c_n(\alpha)} d\gamma(w). \quad (3.28)$$

Since γ has compact support on $\mathbb{C}$, for certain a and C , we have

$$\forall n \in \mathbb{N}, \quad |a_n| \leq C \frac{a^n}{c_n(\alpha)}. \quad (3.29)$$

Then we have

$$\forall z \in \mathbb{C}, \quad |\Psi(z)| \leq C \sum_{n=0}^{+\infty} \frac{(|z|a)^n}{c_n(\alpha)} = CS_{\alpha}(|z|a) \leq Ce^{|z|a}. \quad (3.30)$$

Similarly we have

$$\forall z \in \mathbb{C}, \quad |\Phi(z)| \leq c_1(\alpha) Ce^{|z|a}. \quad (3.31)$$

Thus we have proved that (v) is true.

(v) $\Rightarrow$ (i). Suppose now that, for every $f \in \mathcal{H}$ and $z \in \mathbb{C}$,

$$(Lf)(z) = \sum_{n=0}^{+\infty} a_{2n} (\ell_{\alpha}^n f)(z) + c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} \frac{d(\ell_{\alpha}^n f)}{dz}(z), \quad (3.32)$$

for a certain $a_k \in \mathbb{C}$, $k \in \mathbb{N}$, where the series converges in $\mathcal{H}$.

Hence, if $f \in \mathcal{H}$, since $\tau_z \ell_\alpha f = \ell_\alpha \tau_z f$, $z \in \mathbb{C}$, using (2.38) and the fact that τ_z is a continuous linear mapping from $\mathcal{H}$ into itself, we obtain for every $z, w \in \mathbb{C}$,

$$\begin{aligned} \tau_w(Lf)(z) &= \sum_{n=0}^{+\infty} a_{2n} \tau_w(\ell_\alpha^n f)(z) + c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} \tau_w\left(\frac{d(\ell_\alpha^n f)}{dz}\right)(z) \\ &= \sum_{n=0}^{+\infty} a_{2n} \ell_\alpha^n(\tau_w f)(z) + c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} \frac{d(\ell_\alpha^n(\tau_w f))}{dz}(z) \\ &= L(\tau_w f)(z). \end{aligned} \quad (3.33)$$

Hence (v) implies (i). $\square$

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