MARCINKIEWICZ INTEGRALS ALONG SUBVARIETIES ON PRODUCT DOMAINS

AHMAD AL-SALMAN

Received 25 January 2004

We study the L^p mapping properties of a class of Marcinkiewicz integral operators on product domains with rough kernels supported by subvarieties.

2000 Mathematics Subject Classification: 42B20, 42B15, 42B25.

1. Introduction. Let $\mathbb{R}^d$ ($d = n$ or $d = m$), $d \geq 2$ be the d -dimensional Euclidean space and $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ equipped with the induced Lebesgue measure $d\sigma$. Suppose that Ω is a homogeneous function of degree zero on $\mathbb{R}^n$ which is integrable on $\mathbb{S}^{n-1}$ and $\int_{\mathbb{S}^{n-1}} \Omega(\gamma) d\sigma(\gamma) = 0$. Then the Marcinkiewicz integral operator μ_Ω which was introduced by Stein in [18] is defined by

$$\mu_\Omega(f)(x) = \left(\int_{-\infty}^{\infty} \left| \int_{|\gamma| \leq 2^t} f(x - \gamma) \Omega(\gamma) |\gamma|^{1-n} d\gamma \right|^2 2^{-2t} dt \right)^{1/2}. \quad (1.1)$$

Stein proved that if $\Omega \in \text{Lip}_\alpha(\mathbb{S}^{n-1})$, ($0 < \alpha \leq 1$), then μ_Ω is bounded on L^p for all $1 < p \leq 2$ [18]. Since then, the study of the L^p boundedness of μ_Ω under various conditions on the function Ω has attracted the attention of many authors ([1, 4, 5, 7, 10, 13], among others). In particular, Chen et al. in [8] studied the L^p boundedness of μ_Ω under the following condition on the function Ω which was introduced by Grafakos and Stefanov in their study of singular integral operators [17]:

$$\sup_{\xi \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\Omega(\gamma')| \left(\log \frac{1}{|\xi \cdot \gamma'|} \right)^{1+\alpha} d\sigma(\gamma') < \infty, \quad (1.2)$$

for some $\alpha > 0$. Chen et al. [8] showed that if Ω satisfies (1.2) for some $\alpha > 0$, then μ_Ω is bounded on L^p for $p \in ((2 + 2\alpha)/(1 + 2\alpha), 2 + 2\alpha)$. It should be pointed out here that Grafakos and Stefanov showed that for any $\alpha > 0$, the following relations hold:

$$F(\alpha, \mathbb{S}^{n-1}) \not\subset L(\log^+ L)(\mathbb{S}^{n-1}), \quad L(\log^+ L)(\mathbb{S}^{n-1}) \not\subset F(\alpha, \mathbb{S}^{n-1}), \quad (1.3)$$

where $F(\alpha, \mathbb{S}^{n-1})$ is the space of all integrable functions Ω on $\mathbb{S}^{n-1}$ which satisfy $\int_{\mathbb{S}^{n-1}} \Omega(\gamma) d\sigma(\gamma) = 0$ and (1.2). For conditions similar to (1.2), we refer the readers to consult [1, 6].

Recently, a number of authors started to study the analog of the operator μ_Ω on product domains. More precisely, let $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ be such that

$$\begin{aligned} \Omega(tx, sy) &= \Omega(x, y) \quad \text{for any } t, s > 0, \\ \int_{\mathbf{S}^{n-1}} \Omega(u, \cdot) d\sigma(u) &= \int_{\mathbf{S}^{m-1}} \Omega(\cdot, v) d\sigma(v) = 0. \end{aligned} \quad (1.4)$$

Then, the Marcinkiewicz integral operator on product domains $\mathcal{M}_{\Omega, c}$ is given by

$$\mathcal{M}_{\Omega, c} f(x, y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F_{t,s}(f)(x, y)|^2 2^{-2(t+s)} dt ds \right)^{1/2}, \quad (1.5)$$

where

$$F_{t,s}(f)(x, y) = \int_{|u| \leq 2^t} \int_{|v| \leq 2^s} f(x - u, y - v) \frac{\Omega(u, v) du dv}{|u|^{n-1} |v|^{m-1}}. \quad (1.6)$$

It has been known for quite some time that the operator $\mathcal{M}_{\Omega, c}$ is bounded on L^p for all $1 < p < \infty$ under the condition that $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ [9, 12]. Recently, the L^p boundedness of $\mathcal{M}_{\Omega, c}$ was established under the weaker condition $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$; see Choi [11] for $p = 2$ and Al-Qassem et al. [2] for all $1 < p < \infty$.

Motivated by [1, 6], the main purpose of this note is to investigate the L^p boundedness of Marcinkiewicz integral operators on product domains with kernels satisfying conditions similar to (5) in [6] (see also [1]) and supported by subvarieties determined by polynomial mappings. To be more specific, let $\mathcal{V}(d, l)$ be the set of real-valued polynomials in $\mathbb{R}^d$ which have degrees at most l . For $\mathcal{P} = (P_1, \dots, P_N) \in (\mathcal{V}(d, l))^N$ and $\mathcal{Q} = (Q_1, \dots, Q_M) \in (\mathcal{V}(d, r))^M$, consider the Marcinkiewicz integral operator

$$\mathcal{M}_{\Omega, \mathcal{P}, \mathcal{Q}} f(x, y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F_{t,s}^{\mathcal{P}, \mathcal{Q}}(f)(x, y)|^2 2^{-2(t+s)} dt ds \right)^{1/2}, \quad (1.7)$$

where

$$F_{t,s}^{\mathcal{P}, \mathcal{Q}}(f)(x, y) = \int_{|u| \leq 2^t} \int_{|v| \leq 2^s} f(x - \mathcal{P}(u), y - \mathcal{Q}(v)) \frac{\Omega(u, v) du dv}{|u|^{n-1} |v|^{m-1}}. \quad (1.8)$$

It is clear that if $N = n$, $M = m$, $\mathcal{P}(u) = u$, and $\mathcal{Q}(v) = v$, then $\mathcal{M}_{\Omega, \mathcal{P}, \mathcal{Q}} = \mathcal{M}_{\Omega, c}$. In order to formulate our main results regarding the operators (1.7), we let $\mathcal{H}(d, l)$ be the collection of all homogeneous polynomials of degree l in $\mathcal{V}(d, l)$. We will associate with $\mathcal{V}(d, l)$ the norm $\|\cdot\|$ given by $\|P\| = (\sum_{|\beta| \leq d} |a_\beta|^2)^{1/2}$, where $P(y) = \sum_{|\beta| \leq d} a_\beta y^\beta$. Let

$$\mathcal{H}^l(d, l) = \{P \in \mathcal{H}(d, l) : \|P\| = 1\}. \quad (1.9)$$

DEFINITION 1.1. For $n, m \geq 2$, $l, r \in \mathbb{N}$, and $\alpha > 0$, let $F_\alpha(\mathbf{S}^{n-1}, \mathbf{S}^{m-1}, l, r)$ be the space of all integrable functions Ω on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ that satisfy

$$\sup_{P \in \mathcal{H}^l(n, l), Q \in \mathcal{H}^l(m, r)} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{m-1}} |\Omega(u', v')| \{G_{P, Q}(u', v')\}^{1+\alpha} d\sigma(u') d\sigma(v') < \infty, \quad (1.10)$$

where $\alpha > 0$ and

$$\begin{aligned} G_{P,Q}(u', v') &= \log^+ (|P(u')|^{-1}) + \log^+ (|Q(v')|^{-1}) \\ &\quad + \log^+ (|P(u')|^{-1}) \log^+ (|Q(v')|^{-1}). \end{aligned} \quad (1.11)$$

It is clear that $L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \not\subseteq F_\alpha(\mathbf{S}^{n-1}, \mathbf{S}^{m-1}, l, r)$ for all $q > 1$. Moreover, by (1.3) it can be easily shown that

$$\begin{aligned} F_\alpha(\mathbf{S}^{n-1}, \mathbf{S}^{m-1}, l, r) &\not\subseteq L(\log^+ L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}), \\ L(\log^+ L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) &\not\subseteq F_\alpha(\mathbf{S}^{n-1}, \mathbf{S}^{m-1}, l, r) \end{aligned} \quad (1.12)$$

for any $\alpha > 0$. Therefore, by (1.12) and the results of Choi [11] and Al-Qassem et al. [2] when $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, it is natural to investigate the L^p boundedness of $\mathcal{M}_{\Omega, \mathcal{P}, \mathfrak{Q}}$ under the conditions (1.10).

Our main result is the following.

THEOREM 1.2. *Let $n, m \geq 2$, $N, M, \tilde{N}, \tilde{M} \in \mathbb{N}$, $\mathcal{P} = (P_1, \dots, P_N) \in (\mathcal{V}(n, \tilde{N}))^N$, and $\mathfrak{Q} = (Q_1, \dots, Q_M) \in (\mathcal{V}(m, \tilde{M}))^M$. If Ω satisfies (1.4) and*

$$\Omega \in \bigcap_{l=1}^{\tilde{N}} \bigcap_{r=1}^{\tilde{M}} F_\alpha(\mathbf{S}^{n-1}, \mathbf{S}^{m-1}, l, r) \quad (1.13)$$

for some $\alpha > 0$, then

$$\|\mathcal{M}_{\Omega, \mathcal{P}, \mathfrak{Q}} f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^M)} \leq C_p \|f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^M)} \quad (1.14)$$

for all $p \in ((2+2\alpha)/(1+2\alpha), 2+2\alpha)$ and $f \in L^p(\mathbb{R}^N \times \mathbb{R}^M)$. The constant C_p is independent of the coefficients of the polynomials $\{P_j, Q_k : 1 \leq j \leq N, 1 \leq k \leq M\}$.

In Section 4, we will show that

$$\bigcap_{l=1}^{\infty} \bigcap_{r=1}^{\infty} F_\alpha(\mathbf{S}^1, \mathbf{S}^1, l, r) = F_\alpha(\mathbf{S}^1, \mathbf{S}^1, 1, 1). \quad (1.15)$$

Therefore, we obtain the following.

COROLLARY 1.3. *Let $N, M, \tilde{N}, \tilde{M} \in \mathbb{N}$, $\mathcal{P} = (P_1, \dots, P_N) \in (\mathcal{V}(2, \tilde{N}))^N$, and $\mathfrak{Q} = (Q_1, \dots, Q_M) \in (\mathcal{V}(2, \tilde{M}))^M$. If Ω satisfies (1.4) and $\Omega \in F_\alpha(\mathbf{S}^1, \mathbf{S}^1, 1, 1)$ for some $\alpha > 0$, then $\|\mathcal{M}_{\Omega, \mathcal{P}, \mathfrak{Q}} f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^M)} \leq C_p \|f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^M)}$ for all $p \in ((2+2\alpha)/(1+2\alpha), 2+2\alpha)$ and $f \in L^p(\mathbb{R}^N \times \mathbb{R}^M)$. The constant C_p is independent of the coefficients of the polynomials $\{P_j, Q_k : 1 \leq j \leq N, 1 \leq k \leq M\}$.*

2. General tools. For a nonnegative C^∞ radial function Φ on $\mathbb{R}$ and a linear transformation $L : \mathbb{R}^d \rightarrow \mathbb{R}^l$, we let $\check{\Phi}_w^L$ be such that $(\check{\Phi}_w^L)^\wedge(\xi) = \Phi(|wL(\xi)|^2)$. Let $N, M \geq 2$ and let $\sigma = \{\sigma_{t,s} : t, s \in \mathbb{R}\}$ be a family of measure defined on $\mathbb{R}^N \times \mathbb{R}^M$. For linear transformations $L : \mathbb{R}^N \rightarrow \mathbb{R}^{\tilde{N}}$, $G : \mathbb{R}^M \rightarrow \mathbb{R}^{\tilde{M}}$, nonnegative C^∞ radial function Φ on $\mathbb{R}$ with

$\text{supp}(\Phi) \subset [2^{-1}, 2]$ and $0 \leq \Phi(t) \leq 1$, and positive real numbers a and b , define the family of operators $\{Z_{\sigma, \theta, \gamma}^{(a, b)} : \theta, \gamma \in \mathbb{R}\}$ by

$$Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f(x, y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\check{\Phi}_{2^a(t+\theta)}^L \otimes \check{\Phi}_{2^b(s+\gamma)}^G) * \sigma_{t, s} * f(x, y)|^2 dt ds \right)^{1/2}. \quad (2.1)$$

Also, define the operators $\mu_{\sigma, \Phi, L, G}^{(a, b, \infty, ++)}$, $\mu_{\sigma, \Phi, L, G}^{(a, b, \infty, +-)}$, $\mu_{\sigma, \Phi, L, G}^{(a, b, \infty, --)}$, $\mu_{\sigma, \Phi, L, G}^{(a, b, \infty, -+)}$, $\mu_{\sigma, \Phi, L, G}^{(a, b, 0\infty, +)}$, $\mu_{\sigma, \Phi, L, G}^{(a, b, 0\infty, -)}$, $\mu_{\sigma, \Phi, L, G}^{(a, b, \infty 0, -)}$, $\mu_{\sigma, \Phi, L, G}^{(a, b, \infty 0, +)}$, and $\mu_{\sigma, \Phi, L, G}^{(a, b, 00)}$ by

$$\mu_{\sigma, \Phi, L, G}^{(a, b, \infty, ++)} f(x, y) = \int_2^\infty \int_2^\infty Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f(x, y) d\theta d\gamma, \quad (2.2)$$

$$\mu_{\sigma, \Phi, L, G}^{(a, b, \infty, +-)} f(x, y) = \int_2^\infty \int_{-\infty}^{-2} Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f(x, y) d\theta d\gamma, \quad (2.3)$$

$$\mu_{\sigma, \Phi, L, G}^{(a, b, \infty, --)} f(x, y) = \int_{-\infty}^{-2} \int_{-\infty}^{-2} Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f(x, y) d\theta d\gamma, \quad (2.4)$$

$$\mu_{\sigma, \Phi, L, G}^{(a, b, \infty, -+)} f(x, y) = \int_{-\infty}^{-2} \int_2^\infty Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f(x, y) d\theta d\gamma, \quad (2.5)$$

$$\mu_{\sigma, \Phi, L, G}^{(a, b, 0\infty, +)} f(x, y) = \int_{-2}^2 \int_2^\infty Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f(x, y) d\theta d\gamma, \quad (2.6)$$

$$\mu_{\sigma, \Phi, L, G}^{(a, b, 0\infty, -)} f(x, y) = \int_{-2}^2 \int_{-\infty}^{-2} Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f(x, y) d\theta d\gamma, \quad (2.7)$$

$$\mu_{\sigma, \Phi, L, G}^{(a, b, \infty 0, -)} f(x, y) = \int_{-\infty}^{-2} \int_{-2}^2 Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f(x, y) d\theta d\gamma, \quad (2.8)$$

$$\mu_{\sigma, \Phi, L, G}^{(a, b, \infty 0, +)} f(x, y) = \int_2^\infty \int_{-2}^2 Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f(x, y) d\theta d\gamma, \quad (2.9)$$

$$\mu_{\sigma, \Phi, L, G}^{(a, b, 00)} f(x, y) = \int_{-2}^2 \int_{-2}^2 Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f(x, y) d\theta d\gamma. \quad (2.10)$$

We also, let M_σ denote the maximal function corresponding to σ , that is,

$$M_\sigma f(x, y) = \sup_{t, s \in \mathbb{R}} |\sigma_{t, s}| * f(x, y). \quad (2.11)$$

LEMMA 2.1. Let $N, M, \tilde{N}, \tilde{M} \geq 1$, $a, b \geq 1$, and $\alpha > 0$. Let $\sigma = \{\sigma_{t, s} : t, s \in \mathbb{R}\}$ be a family of measure defined on $\mathbb{R}^N \times \mathbb{R}^M$ and let Φ be as above. Suppose that there exist linear transformations $L : \mathbb{R}^N \rightarrow \mathbb{R}^{\tilde{N}}$ and $G : \mathbb{R}^M \rightarrow \mathbb{R}^{\tilde{M}}$ such that

- (a) $\|\sigma_{t, s}\| \leq C$,
- (b) $|(\sigma_{t, s})^\wedge(\xi, \eta)| \leq C |2^{at} L(\xi)| |2^{bs} G(\eta)|$,
- (c) $|(\sigma_{t, s})^\wedge(\xi, \eta)| \leq C |2^{at} L(\xi)| (\log^+ |2^{bs} G(\eta)|)^{-1-\alpha}$,
- (d) $|(\sigma_{t, s})^\wedge(\xi, \eta)| \leq C (\log^+ |2^{at} L(\xi)|)^{-1-\alpha} (\log^+ |2^{bs} G(\eta)|)^{-1-\alpha}$,
- (e) $|(\sigma_{t, s})^\wedge(\xi, \eta)| \leq C (\log^+ |2^{at} L(\xi)|)^{-1-\alpha} |2^{bs} G(\eta)|$,
- (f) $\|M_\sigma(f)\|_{L^p(\mathbb{R}^N \times \mathbb{R}^M)} \leq C \|f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^M)}$ for all $1 < p < \infty$.

Then the operators (2.2)–(2.10) are bounded on $L^p(\mathbb{R}^N \times \mathbb{R}^M)$ for all $p \in ((2+2\alpha)/(1+2\alpha), 2+2\alpha)$. Moreover the L^p bounds are independent of the linear transformations L and G .

PROOF. By the argument in [3, 15], we may assume that $\tilde{N} \leq N$, $\tilde{M} \leq M$, $L(\xi) = (\xi_1, \dots, \xi_{\tilde{N}})$, and $G(\eta) = (\eta_1, \dots, \eta_{\tilde{M}})$ where $\xi = (\xi_1, \dots, \xi_N)$ and $\eta = (\eta_1, \dots, \eta_M)$. Let

$\mathbf{J}_{\theta, \gamma, \Phi, L, G}^{(a, b)}$ be the operator given by

$$\mathbf{J}_{\theta, \gamma, \Phi, L, G}^{(a, b)}(f)(x, \gamma) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\check{\Phi}_{2^a(t+\theta)}^L \otimes \check{\Phi}_{2^b(s+\gamma)}^G) * f(x, \gamma)|^2 dt ds \right)^{1/2}. \quad (2.12)$$

Then by a well-known argument (see [16, 19]), we obtain

$$\|\mathbf{J}_{\theta, \gamma, \Phi, L, G}^{(a, b)}(f)\|_p \leq C_p \|f\|_p \quad (2.13)$$

for all $1 < p < \infty$.

Now for $p > 2$, by an argument similar to that used in the proof of a lemma in [14, page 544], choose a nonnegative function $\omega \in L^{(p/2)'}(\mathbb{R}^N \times \mathbb{R}^M)$ such that

$$\|Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f\|_p^2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^M} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\check{\Phi}_{2^a(t+\theta)}^L \otimes \check{\Phi}_{2^b(s+\gamma)}^G) * \sigma_{t, s} * f(x, \gamma)|^2 \omega(x, \gamma) dt ds. \quad (2.14)$$

Thus by condition (a), condition (f), and (2.13), we get

$$\|Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f\|_p \leq C \|\mathbf{J}_{\theta, \gamma, \Phi, L, G}^{(a, b)}(f)\|_p \sqrt{\|M_{\sigma}\|_{(p/2)'}} \|f\|_p \leq C \|f\|_p, \quad (2.15)$$

where C is a constant independent of the essential variables. On the other hand, by duality we get (2.15) for $1 < p < 2$. Hence, for all $1 < p < \infty$, we have

$$\|Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f\|_p \leq C \|f\|_p. \quad (2.16)$$

Next, we claim

$$\|Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)} \leq C(|\theta||\gamma|)^{-1-\alpha} \|f\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}. \quad (2.17)$$

To see (2.17), we only need to apply Plancherel's theorem along with condition (d). Namely, for $t, s, \theta, \gamma \in \mathbb{R}$, let $\mathbf{A}(t, \theta) = \{\xi \in \mathbb{R}^N : 2^{-a(t+\theta)-1} \leq |L(\xi)| \leq 2^{-a(t+\theta)+1}\}$ and let $\mathbf{B}(s, \gamma) = \{\eta \in \mathbb{R}^M : 2^{-b(s+\gamma)-1} \leq |G(\eta)| \leq 2^{-b(s+\gamma)+1}\}$. Then

$$\begin{aligned} & \|Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}^2 \\ & \leq C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbf{A}(t, \theta)} \int_{\mathbf{B}(s, \gamma)} |\hat{f}(\xi, \eta)|^2 \\ & \quad \times \{(\log^+ |2^{at} L(\xi)|)(\log^+ |2^{bs} G(\eta)|)\}^{-2-2\alpha} d\xi d\eta dt ds \quad (2.18) \\ & \leq C(|\theta||\gamma|)^{-2-2\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbf{A}(t, \theta)} \int_{\mathbf{B}(s, \gamma)} |\hat{f}(\xi, \eta)|^2 d\xi d\eta dt ds \\ & = C(|\theta||\gamma|)^{-2-2\alpha} \|f\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}^2. \end{aligned}$$

By interpolation between (2.17) and (2.16) for any $p \in ((2+2\alpha)/(1+2\alpha), 2+2\alpha)$, there exists $\delta_p > 1$ such that

$$\|Z_{\sigma, \theta, \gamma, \Phi, L, G}^{(a, b)} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C |\theta|^{-\delta_p} |\gamma|^{-\delta_p} \|f\|_{L^p}. \quad (2.19)$$

Therefore, the boundedness of the operators (2.2)–(2.9) follows by (2.19) and Minkowski's inequality. The boundedness of the operator (2.10) follows by Minkowski's inequality and (2.16). This completes the proof. $\square$

Lemma 2.1 immediately implies the following.

LEMMA 2.2. *Suppose that $N, M, \tilde{N}, \tilde{M}, a, b, \alpha, \sigma = \{\sigma_{t,s} : t, s \in \mathbb{R}\}$, L , and G are as in Lemma 2.1. If $\sigma = \{\sigma_{t,s} : t, s \in \mathbb{R}\}$ satisfies the conditions (a)–(f) of Lemma 2.1, then*

$$\mu_{\sigma}^{(a,b)} f(x, y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\sigma_{t,s} * f(x, y)|^2 dt ds \right)^{1/2} \quad (2.20)$$

is bounded on $L^p(\mathbb{R}^N \times \mathbb{R}^M)$ for all $p \in ((2 + 2\alpha)/(1 + 2\alpha), 2 + 2\alpha)$. Moreover the L^p bounds are independent of the linear transformations L and G .

PROOF. Choose a nonnegative C^∞ radial function Φ on $\mathbb{R}$ with $\text{supp}(\Phi) \subset [2^{-1}, 2]$ and $0 \leq \Phi(t) \leq 1$, and

$$\int_0^\infty t^{-1} \Phi(t) dt = 1. \quad (2.21)$$

Then, it is easy to see that

$$f(x, y) = C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\check{\Phi}_{2at}^L \otimes \check{\Phi}_{2bs}^G) * f(x, y) dt ds, \quad (2.22)$$

where C is a constant independent of L and G . Therefore, by (2.22), it follows that

$$\begin{aligned} \mu_{\sigma}^{(a,b)} f(x, y) &\leq C \{ \mu_{\sigma, \Phi, L, G}^{(a,b, \infty, ++)} f(x, y) + \mu_{\sigma, \Phi, L, G}^{(a,b, \infty, +-)} f(x, y) + \mu_{\sigma, \Phi, L, G}^{(a,b, \infty, --)} f(x, y) \\ &\quad + \mu_{\sigma, \Phi, L, G}^{(a,b, \infty, -+)} f(x, y) + \mu_{\sigma, \Phi, L, G}^{(a,b, 0\infty, +)} f(x, y) + \mu_{\sigma, \Phi, L, G}^{(a,b, 0\infty, -)} f(x, y) \\ &\quad + \mu_{\sigma, \Phi, L, G}^{(a,b, 00, -)} f(x, y) + \mu_{\sigma, \Phi, L, G}^{(a,b, 00, +)} f(x, y) + \mu_{\sigma, \Phi, L, G}^{(a,b, 00)} f(x, y) \}. \end{aligned} \quad (2.23)$$

Hence the proof is complete by Lemma 2.1. $\square$

Now, we have the following lemma which can be proved by a proper modification of the arguments in [3, 15].

LEMMA 2.3. *Let $M, N, \tilde{N}, \tilde{M} \geq 1$, and $\{\lambda_{t,s}^{(l,r)} : t, s \in \mathbb{R}, 0 \leq l \leq \tilde{N}, 0 \leq r \leq \tilde{M}\}$ be a family of Borel measures on $\mathbb{R}^N \times \mathbb{R}^M$ with $\lambda_{t,s}^{(0,r)} = 0$ and $\sigma_{k,j}^{(l,0)} = 0$ for every $t, s \in \mathbb{R}$. Let $\{N_l, M_r : 1 \leq l \leq \tilde{N}, 1 \leq r \leq \tilde{M}\} \subseteq \mathbb{N}$ and let $L_l : \mathbb{R}^N \rightarrow \mathbb{R}^{N_l}$ and $G_r : \mathbb{R}^M \rightarrow \mathbb{R}^{M_r}$ be linear transformations, $1 \leq l \leq \tilde{N}, 1 \leq r \leq \tilde{M}$. Suppose that*

- (i) $\|\lambda_{t,s}^{(l,r)}\| \leq C$;
- (ii) $|\hat{\lambda}_{t,s}^{(l,r)}(\xi, \eta)| \leq C(\log^+ |2^{lt} L_l(\xi)|)^{-1-\alpha} (\log^+ |2^{rs} G_r(\eta)|)^{-1-\alpha}$;
- (iii) $|\hat{\lambda}_{t,s}^{(l,r)}(\xi, \eta) - \hat{\lambda}_{t,s}^{(l-1,r)}(\xi, \eta)| \leq C|2^{lt} L_l(\xi)| (\log^+ |2^{rs} G_r(\eta)|)^{-1-\alpha}$;
- (iv) $|\hat{\lambda}_{t,s}^{(l,r)}(\xi, \eta) - \hat{\lambda}_{t,s}^{(l,r-1)}(\xi, \eta)| \leq C(\log^+ |2^{lt} L_l(\xi)|)^{-1-\alpha} |2^{rs} G_r(\eta)|$;
- (v) $|\hat{\lambda}_{t,s}^{(l,r)}(\xi, \eta) - \hat{\lambda}_{t,s}^{(l-1,r)}(\xi, \eta) - \hat{\lambda}_{t,s}^{(l,r-1)}(\xi, \eta) + \hat{\lambda}_{t,s}^{(l-1,r-1)}(\xi, \eta)| \leq C|2^{lt} L_l(\xi)| |2^{rs} G_r(\eta)|$;
- (vi) $|\hat{\lambda}_{t,s}^{(l,r-1)}(\xi, \eta) - \hat{\lambda}_{t,s}^{(l-1,r-1)}(\xi, \eta)| \leq C|2^{lt} L_l(\xi)|$;
- (vii) $|\hat{\lambda}_{t,s}^{(l-1,r)}(\xi, \eta) - \hat{\lambda}_{t,s}^{(l-1,r-1)}(\xi, \eta)| \leq C|2^{rs} G_r(\eta)|$;

(viii) $\|M_{\lambda^{(l,r)}} f\|_p \leq C_p \|f\|_p$ for all $1 < p < \infty$, where $M_{\lambda^{(l,r)}}$ is given by (2.11) with $\sigma_{t,s}$ replaced by $\lambda_{t,s}^{(l,r)}$.

Then there exists a family of Borel measures $\{\sigma_{t,s}^{(l,r)} : t, s \in \mathbb{R}, 1 \leq l \leq \tilde{N}, 1 \leq r \leq \tilde{M}\}$ on $\mathbb{R}^N \times \mathbb{R}^M$ such that

$$\lambda_{t,s}^{(\tilde{N}, \tilde{M})} = \sum_{l=1}^{\tilde{N}} \sum_{r=1}^{\tilde{M}} \sigma_{t,s}^{(l,r)} \quad \forall t, s \in \mathbb{R}, \quad (2.24)$$

and that for each $1 \leq l \leq \tilde{N}, 1 \leq r \leq \tilde{M}$, the family $\{\sigma_{t,s}^{(l,r)} : t, s \in \mathbb{R}\}$ satisfies the assumptions (a)–(f) of Lemma 2.1.

3. Proof of Theorem 1.2.

PROOF. Let $n, m \geq 2, N, M, \tilde{N}, \tilde{M} \geq 1$. Let $\Omega \in \bigcap_{l=1}^{\tilde{N}} \bigcap_{r=1}^{\tilde{M}} F_\alpha(\mathbf{S}^{n-1}, \mathbf{S}^{m-1}, l, r)$ for some $\alpha > 0$. Let $\mathcal{P} = (P_1, \dots, P_N) \in (\mathcal{V}(n, \tilde{N}))^N$ and $\mathcal{Q} = (Q_1, \dots, Q_M) \in (\mathcal{V}(m, \tilde{M}))^M$. Then the polynomials $P_1, \dots, P_N, Q_1, \dots, Q_M$ can be written as

$$\begin{aligned} P_j(\mathcal{Y}) &= \sum_{|\beta| \leq \tilde{N}} a_{\beta,j} \mathcal{Y}^\beta, \quad j = 1, \dots, N, \\ Q_k(\mathcal{Y}) &= \sum_{|\tilde{\beta}| \leq \tilde{M}} b_{\tilde{\beta},k} \mathcal{Y}^{\tilde{\beta}}, \quad k = 1, \dots, M, \end{aligned} \quad (3.1)$$

where $\beta \in (\mathbb{N} \cup \{0\})^N$ and $\tilde{\beta} \in (\mathbb{N} \cup \{0\})^M$.

For each $1 \leq l \leq \tilde{N}$ and $1 \leq r \leq \tilde{M}$, let

$$\begin{aligned} A(l) &= \{\beta \in (\mathbb{N} \cup \{0\})^N : |\beta| = l\}, \\ B(r) &= \{\tilde{\beta} \in (\mathbb{N} \cup \{0\})^M : |\tilde{\beta}| = r\}. \end{aligned} \quad (3.2)$$

Let $N_l = |A(l)|$ and $M_r = |B(r)|$. For each $\beta \in A(l)$ and $\tilde{\beta} \in B(r)$, let

$$\overrightarrow{a_{\beta,l}} = (a_{\beta,1}, \dots, a_{\beta,N}), \quad \overrightarrow{b_{\tilde{\beta},r}} = (b_{\tilde{\beta},1}, \dots, b_{\tilde{\beta},M}). \quad (3.3)$$

Define the linear transformations $L_l = (L_{\beta l})_{\beta \in A(l)} : \mathbb{R}^N \rightarrow \mathbb{R}^{N_l}$ and $G_r = (G_{\tilde{\beta} r})_{\tilde{\beta} \in B(r)} : \mathbb{R}^M \rightarrow \mathbb{R}^{M_r}$ by

$$\begin{aligned} L_{\beta l}(\xi) &= \xi \cdot \overrightarrow{a_{\beta,l}}, \\ G_{\tilde{\beta} r}(\eta) &= \eta \cdot \overrightarrow{b_{\tilde{\beta},r}}. \end{aligned} \quad (3.4)$$

Without loss of generality, we will assume that L_l and G_r are nonzero for all $1 \leq l \leq \tilde{N}$ and $1 \leq r \leq \tilde{M}$.

For $1 \leq l \leq \tilde{N}$ and $1 \leq r \leq \tilde{M}$, let $\mathcal{P}_l(u) = (P_{1,l}(u), \dots, P_{N,l}(u))$ and $\mathcal{Q}_r(v) = (Q_{1,r}(v), \dots, Q_{M,r}(v))$, where

$$\begin{aligned} P_{j,l}(u) &= \sum_{|\beta| \leq l} a_{\beta,j} u^\beta, \quad j = 1, \dots, N, \\ Q_{k,r}(v) &= \sum_{|\tilde{\beta}| \leq r} b_{\tilde{\beta},k} v^{\tilde{\beta}}, \quad k = 1, \dots, M. \end{aligned} \quad (3.5)$$

Also, let

$$\begin{aligned} \mathcal{H}\mathcal{P}_l(u') &= N_l^{-1/2} ((u')^\beta)_{\beta \in A(l)}, \\ \mathcal{H}\mathcal{Q}_r(v') &= M_r^{-1/2} ((v')^{\tilde{\beta}})_{\tilde{\beta} \in B(r)}. \end{aligned} \quad (3.6)$$

Then clearly

$$\mathcal{H}\mathcal{P}_l \in (\mathcal{H}^l(n, l))^{N_l}, \quad \mathcal{H}\mathcal{Q}_r \in (\mathcal{H}^l(m, r))^{M_r}. \quad (3.7)$$

Let $\sigma = \{\lambda_{t,s}^{(l,r)} : t, s \in \mathbb{R}\}$ be the family measures where $\sigma_{t,s}$ is given by

$$\iint f d\lambda_{t,s}^{(l,r)} = 2^{-t-s} \int_{|u| \leq 2^t} \int_{|v| \leq 2^s} f(\mathcal{P}_l(u), \mathcal{Q}_r(v)) \frac{\Omega(u, v) du dv}{|u|^{n-1} |v|^{m-1}}. \quad (3.8)$$

Then it is clear that

$$\mathcal{M}_{\Omega, \mathcal{P}, \mathcal{Q}} f(x, y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\lambda_{t,s}^{(\tilde{N}, \tilde{M})} * f)(x, y)|^2 dt ds \right)^{1/2}. \quad (3.9)$$

Therefore, by Lemmas 2.1, 2.2, and 2.3, it suffices to verify that the measures $\{\lambda_{t,s}^{(l,r)} : t, s \in \mathbb{R}, 0 \leq l \leq \tilde{N}, 0 \leq r \leq \tilde{M}\}$ satisfy the assumptions of Lemma 2.3.

For $t, s \in \mathbb{R}, 0 \leq l \leq \tilde{N}, 0 \leq r \leq \tilde{M}$, let $I_l(t, \xi, x)$ and $J_r(s, \eta, y)$ be the complex-valued functions defined on $\mathbb{R} \times \mathbb{R}^N \times \mathbf{S}^{n-1}$ and $\mathbb{R} \times \mathbb{R}^M \times \mathbf{S}^{m-1}$, respectively, by

$$I_l(t, \xi, u') = \int_0^1 e^{-i\xi \mathcal{P}_l(2^t u' w)} dw, \quad J_r(s, \eta, v') = \int_0^1 e^{-i\eta \cdot \mathcal{Q}_r(2^s v' w)} dw. \quad (3.10)$$

First, notice that $(d^l/dw^l)(\xi \mathcal{P}_l(2^t u' w)) = l! 2^{lt} N_l^{1/2} L_l(\xi) \cdot \mathcal{H}\mathcal{P}_l(u')$ and $(d^r/dw^r)(\eta \cdot \mathcal{Q}_r(2^s v' w)) = r! 2^{rs} M_r^{1/2} G_r(\eta) \cdot \mathcal{H}\mathcal{Q}_r(v')$. Thus, by the van der Corput lemma [19], we obtain

$$\begin{aligned} |I_l(t, \xi, u')| &\leq C |2^{lt} N_l^{1/2} L_l(\xi) \cdot \mathcal{H}\mathcal{P}_l(u')|^{-1/l}, \\ |J_r(s, \eta, v')| &\leq C |2^{rs} M_r^{1/2} G_r(\eta) \cdot \mathcal{H}\mathcal{Q}_r(v')|^{-1/r}, \end{aligned} \quad (3.11)$$

which when interpolated with the trivial estimates $|I_l(t, \xi, u')| \leq C$ and $|J_r(s, \eta, v')| \leq C$, respectively, imply that

$$\begin{aligned} |I_l(t, \xi, u')| &\leq C |2^{lt} N_l^{1/2} L_l(\xi) \cdot \mathcal{H}\mathcal{P}_l(u')|^{-1/2l}, \\ |J_r(s, \eta, v')| &\leq C |2^{rs} M_r^{1/2} G_r(\eta) \cdot \mathcal{H}\mathcal{Q}_r(v')|^{-1/2r}. \end{aligned} \quad (3.12)$$

On the other hand, it is straightforward to see that the following hold:

$$|I_l(t, \xi, x) - I_{l-1}(t, \xi, u')| \leq |2^{lt} N_l^{1/2} L_l(\xi)|, \quad (3.13)$$

$$|J_r(s, \eta, v') - J_{r-1}(s, \eta, v')| \leq |2^{rs} M_r^{1/2} G_r(\eta)|, \quad (3.14)$$

$$|I_l(t, \xi, u')| \leq 1, \quad (3.15)$$

$$|J_r(s, \eta, v')| \leq 1 \quad (3.16)$$

for $1 \leq l \leq \tilde{N}$, $1 \leq r \leq \tilde{M}$.

Now, clearly condition (i) holds trivially. Next, by combining (3.12), (3.15), and (3.16), we obtain

$$|I_l(t, \xi, u')| \leq C(\log^+ |2^{lt} L_l(\xi)|)^{-1-\alpha} \left(l + \alpha + \log^+ \left| \frac{1}{(L_l(\xi))' \cdot \mathcal{H}\mathcal{P}_l(u')} \right| \right)^{1+\alpha}, \quad (3.17)$$

$$|J_r(s, \eta, v')| \leq C(\log^+ |2^{rs} G_r(\eta)|)^{-1-\alpha} \left(r + \alpha + \log^+ \left| \frac{1}{(G_r(\eta))' \cdot \mathcal{H}\mathcal{Q}_r(v')} \right| \right)^{1+\alpha}. \quad (3.18)$$

Therefore, since $(L_l(\xi))' \cdot \mathcal{H}\mathcal{P}_l(\cdot) \in \mathcal{H}^l(n, l)$, $(G_r(\eta))' \cdot \mathcal{H}\mathcal{Q}_r(v') \in \mathcal{H}^l(m, r)$, and $\Omega \in F_\alpha(\mathbf{S}^{n-1}, \mathbf{S}^{m-1}, l, r)$ for all $1 \leq l \leq \tilde{N}$, $1 \leq r \leq \tilde{M}$, we immediately obtain the estimates (ii), (iii), (iv), (v), (vi), and (vii) by a proper use of the estimates (3.13)–(3.18). We omit the details.

Finally, we turn to the proof of (viii). It can be easily verified that

$$M_{\lambda(l,r)}(f)(x, y) \leq \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{m-1}} |\Omega(u, v)| (M_{\mathcal{P}_l, u'}^1 \circ M_{\mathcal{Q}_r, v'}^2)(f)(x, y) d\sigma(u) d\sigma(v), \quad (3.19)$$

where

$$\begin{aligned} M_{\mathcal{P}_l, u'}^1 f(x, y) &= \sup_{h>0} h^{-1} \int_{|t|<h} |f(x - \mathcal{P}_l(tu'), y)| dt, \\ M_{\mathcal{Q}_r, v'}^2 f(x, y) &= \sup_{h>0} h^{-1} \int_{|s|<h} |f(x, y - \mathcal{Q}_r(sv'))| ds, \end{aligned} \quad (3.20)$$

and $\circ$ denotes the composition of operators. Thus, the estimate (viii) follows by (3.19), Hölder's inequality, and [19, Proposition 1, page 477]. This completes the verification of the assumptions of Lemma 2.3 and hence the proof. $\square$

4. Proof of Corollary 1.3. Corollary 1.3 is a simple consequence of Theorem 1.2 and the following lemma.

LEMMA 4.1. $\bigcap_{l=1}^\infty \bigcap_{r=1}^\infty F_\alpha(\mathbf{S}^1, \mathbf{S}^1, l, r) = F_\alpha(\mathbf{S}^1, \mathbf{S}^1, 1, 1)$.

A proof of Lemma 4.1 can be obtained by adapting the one parameter argument in [6]. For readers convenience, details are presented below.

We will need the following lemma (see [6, Lemma 3.1]).

LEMMA 4.2 [6]. Let $m \in \mathbb{N}$, $a_0, a_1, \dots, a_m \in \mathcal{C}$ and $g(z) = a_0 + a_1 z + \dots + a_m z^m$ for $z \in \mathcal{C}$. If $z_1, \dots, z_l$ are the roots of $g(z)$ which lie in $\{z \in \mathcal{C} : |z| \leq 2\}$, then

$$|g(z)| \geq 6^{-m} \left(\sup_{|z|=1} |g(z)| \right) \prod_{s=1}^l |z - z_s| \quad (4.1)$$

holds for $|z| \leq 1$.

PROOF OF LEMMA 4.1. Since $\bigcap_{l=1}^{\infty} \bigcap_{r=1}^{\infty} F_{\alpha}(\mathbf{S}^1, \mathbf{S}^1, l, r) \subset F_{\alpha}(\mathbf{S}^1, \mathbf{S}^1, 1, 1)$, it suffices to show that $F_{\alpha}(\mathbf{S}^1, \mathbf{S}^1, 1, 1) \subset \bigcap_{l=1}^{\infty} \bigcap_{r=1}^{\infty} F_{\alpha}(\mathbf{S}^1, \mathbf{S}^1, l, r)$ for all $\alpha > 0$. Let $\Omega \in F_{\alpha}(\mathbf{S}^1, \mathbf{S}^1, 1, 1)$. First, as in the one parameter case in [17], we observe that the condition $\Omega \in F_{\alpha}(\mathbf{S}^1, \mathbf{S}^1, 1, 1)$ is equivalent to the following condition:

$$\sup_{\theta, \phi \in [0, 2\pi]} \iint_{[0, 2\pi]^2} |\Omega(t, s)| \{G(t, s, \theta, \phi)\}^{1+\alpha} dt ds < \infty, \quad (4.2)$$

where

$$G(t, s, \theta, \phi) = \log^+ \left(\frac{1}{|t - \theta||s - \phi|} \right) + \log^+ \left(\frac{1}{|t - \theta|} \right) \log^+ \left(\frac{1}{|s - \phi|} \right). \quad (4.3)$$

Let $l, r \in \mathbb{N}$. Then there exist $\lambda_l > 0$ and $\mu_r > 0$ such that

$$\begin{aligned} \sup_{u \in \mathbf{S}^1} |P(u)| &\geq \lambda_l \|P\|, \\ \sup_{v \in \mathbf{S}^1} |Q(v)| &\geq \mu_r \|Q\| \end{aligned} \quad (4.4)$$

hold for all $P \in \mathcal{V}(2, l)$ and $Q \in \mathcal{V}(2, r)$.

For $P \in \mathcal{V}(2, l)$ and $Q \in \mathcal{V}(2, r)$ with $\|P\| = \|Q\| = 1$, we write

$$\begin{aligned} P(u) &= P(u_1, u_2) = \sum_{j+k=l} a_{jk} u_1^j u_2^k, \\ Q(v) &= Q(v_1, v_2) = \sum_{j+k=r} b_{jk} v_1^j v_2^k. \end{aligned} \quad (4.5)$$

Then, by (4.4), and similar argument as in [6], there exist $k_1, k_2 \in \mathbb{N}$, $\{z_1, \dots, z_{k_1}\} \subset \mathbf{S}^1$, $\{w_1, \dots, w_{k_2}\} \subset \mathbf{S}^1$, and constants $C_l = C(l)$ and $C_r = C(r)$ such that

$$\begin{aligned} |P(u)| &\geq C_l \prod_{s=1}^{k_1} |u - z_s|, \\ |Q(v)| &\geq C_r \prod_{s=1}^{k_2} |v - w_s|. \end{aligned} \quad (4.6)$$

Hence, by (4.2), (4.6), we get $\Omega \in F_{\alpha}(\mathbf{S}^1, \mathbf{S}^1, l, r)$. This completes the proof. $\square$

ACKNOWLEDGMENT. The author would like to thank the referees of this note for their valuable remarks and comments.

REFERENCES

- [1] H. Al-Qassem and A. Al-Salman, *Rough Marcinkiewicz integral operators*, Int. J. Math. Math. Sci. **27** (2001), no. 8, 495–503.
- [2] H. Al-Qassem, A. Al-Salman, L. Cheng, and Y. Pan, *Marcinkiewicz integrals on product domains*, to appear in Studia Math.
- [3] H. Al-Qassem and Y. Pan, *L^p boundedness for singular integrals with rough kernels on product domains*, Hokkaido Math. J. **31** (2002), no. 3, 555–613.
- [4] A. Al-Salman and H. Al-Qassem, *Integral operators of Marcinkiewicz type*, J. Integral Equations Appl. **14** (2002), no. 4, 343–354.
- [5] A. Al-Salman, H. Al-Qassem, L. Cheng, and Y. Pan, *L^p bounds for the function of Marcinkiewicz*, Math. Res. Lett. **9** (2002), no. 5–6, 697–700.
- [6] A. Al-Salman and Y. Pan, *Singular integrals with rough kernels*, Canad. Math. Bull. **47** (2004), no. 1, 3–11.
- [7] A. Benedek, A. Calderón, and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. USA **48** (1962), 356–365.
- [8] J. Chen, D. Fan, and Y. Pan, *A note on a Marcinkiewicz integral operator*, Math. Nachr. **227** (2001), 33–42.
- [9] J. Chen, D. Fan, and Y. Ying, *Rough Marcinkiewicz integrals with $L(\log^+ L)^2$ kernels on product spaces*, Adv. Math. (China) **30** (2001), no. 2, 179–181.
- [10] ———, *The method of rotation and Marcinkiewicz integrals on product domains*, Studia Math. **153** (2002), no. 1, 41–58.
- [11] Y. Choi, *Marcinkiewicz integrals with rough homogeneous kernels of degree zero in product domains*, J. Math. Anal. Appl. **261** (2001), no. 1, 53–60.
- [12] Y. Ding, *L^2 -boundedness of Marcinkiewicz integral with rough kernel*, Hokkaido Math. J. **27** (1998), no. 1, 105–115.
- [13] Y. Ding, D. Fan, and Y. Pan, *On the L^p boundedness of Marcinkiewicz integrals*, Michigan Math. J. **50** (2002), no. 1, 17–26.
- [14] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), no. 3, 541–561.
- [15] D. Fan and Y. Pan, *Singular integral operators with rough kernels supported by subvarieties*, Amer. J. Math. **119** (1997), no. 4, 799–839.
- [16] R. Fefferman and E. M. Stein, *Singular integrals on product spaces*, Adv. in Math. **45** (1982), no. 2, 117–143.
- [17] L. Grafakos and A. Stefanov, *L^p bounds for singular integrals and maximal singular integrals with rough kernels*, Indiana Univ. Math. J. **47** (1998), no. 2, 455–469.
- [18] E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.
- [19] ———, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, New Jersey, 1993.

Ahmad Al-Salman: Department of Mathematics, Yarmouk University, Irbid, Jordan
 E-mail address: alsalman@yu.edu.jo