LOGARITHMIC MATRIX TRANSFORMATIONS INTO G_w

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We introduced the logarithmic matrix L_t and studied it as mappings into ℓ and G in 1998 and 2000, respectively. In this paper, we study L_t as mappings into G_w .

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1. Introduction. The logarithmic power series method of summability [1], denoted by L , is the following sequence-to-function transformation: if

$$\lim_{x \rightarrow 1^-} \left\{ -\frac{1}{\log(1-x)} \sum_{k=0}^{\infty} \frac{1}{k+1} u_k x^{k+1} \right\} = A, \quad (1.1)$$

then u is L -summable to A . The matrix analogue of the L -summability method is the L_t matrix [2] given by

$$a_{nk} = -\frac{1}{\log(1-t_n)} \frac{1}{k+1} t_n^{k+1}, \quad (1.2)$$

where $0 < t_n < 1$ for all n and $\lim_n t_n = 1$. Thus, the sequence u is transformed into the sequence $L_t u$ whose n th term is given by

$$(L_t u)_n = -\frac{1}{\log(1-t_n)} \sum_{k=0}^{\infty} \frac{1}{k+1} u_k t_n^{k+1}. \quad (1.3)$$

The L_t matrix is called the logarithmic matrix. Throughout this paper, t will denote such a sequence: $0 < t_n < 1$ for all n , and $\lim_n t_n = 1$.

2. Basic notations and definitions. Let $A = (a_{nk})$ be an infinite matrix defining a sequence-to-sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad (2.1)$$

where $(Ax)_n$ denotes the n th term of the image sequence Ax . The sequence Ax is called the A -transform of the sequence x . Let y be a complex number sequence. Throughout

this paper, we will use the following basic notations and definitions:

$$\begin{aligned}
 \ell &= \left\{ \gamma : \sum_{k=0}^{\infty} |\gamma_k| \text{ is convergent} \right\}, \\
 \ell(A) &= \{ \gamma : A\gamma \in \ell \}, \\
 G &= \{ \gamma : \gamma_k = O(r^k) \text{ for some } r \in (0, 1) \}, \\
 G_w &= \{ \gamma : \gamma_k = O(r^k) \text{ for some } r \in (0, w), 0 < w < 1 \}, \\
 G_w(A) &= \{ \gamma : A\gamma \in G_w \}, \\
 c &= \{ \text{the set of all convergent sequences} \}, \\
 c(A) &= \{ \gamma : A(\gamma) \in c \}.
 \end{aligned} \tag{2.2}$$

DEFINITION 2.1. If X and Y are complex number sequences, then the matrix A is called an X - Y matrix if the image Au of u under the transformation A is in Y whenever u is in X .

DEFINITION 2.2. The summability matrix A is said to be G_w -translative for a sequence u in $G_w(A)$ provided that each of the sequences T_u and S_u is in $G_w(A)$, where $T_u = \{u_1, u_2, u_3, \dots\}$ and $S_u = \{0, u_0, u_1, \dots\}$.

DEFINITION 2.3. The matrix A is G_w -stronger than the matrix B provided that $G_w(B) \subseteq G_w(A)$.

3. Main results. Our first main result gives a necessary and sufficient condition for L_t to be G_w - G_w .

THEOREM 3.1. *The logarithmic matrix L_t is a G_w - G_w matrix if and only if $-1/\log(1-t) \in G_w$.*

PROOF. Since $0 < t_n < 1$, it follows that

$$\frac{|a_{nk}|}{\log(1-t_n)} \leq -1, \tag{3.1}$$

for all n and k . Therefore, if $-1/\log(1-t) \in G_w$, [3, Theorem 2.3] guarantees that L_t is a G_w - G_w matrix. Conversely, if $-1/\log(1-t) \notin G_w$, then the first column of L_t is not in G_w because $a_{n,0} = -t_n/\log(1-t_n) \notin G_w$. Hence, L_t is not a G_w - G_w matrix by [3, Theorem 2.3]. $\square$

COROLLARY 3.2. *If $0 < t_n < u_n < 1$ and L_t is a G_w - G_w matrix, then L_u is also a G_w - G_w matrix.*

COROLLARY 3.3. *Suppose $\alpha > -1$ and L_t is an G_w - G_w matrix, then $(1-t)^{\alpha+1} \in G_w$.*

COROLLARY 3.4. *Let $t_n = 1 - e^{-q_n}$, where $q_n = r^n$. Then L_t is a G_w - G_w matrix if and only if $r > 1/w$.*

COROLLARY 3.5. *If L_t is a G_w - G_w matrix, then it is a G - G matrix.*

The next result suggests that the logarithmic matrix L_t is G_w -stronger than the identity matrix. The result indicates that the L_t matrix is rather a strong method in the G_w - G_w setting.

THEOREM 3.6. *If L_t is a G_w - G_w matrix and the series $\sum_{k=0}^{\infty} x_k$ has bounded partial sums, then it follows that $x \in G_w(L_t)$.*

PROOF. The proof easily follows using the same techniques as in the proof of Theorem 3.10 [2]. $\square$

REMARK 3.7. Theorem 3.6 indicates that if L_t is a G_w - G_w matrix, then $G_w(L_t)$ contains the class of all conditionally convergent series. This suggests how large the size of $G_w(L_t)$ is. In fact, we can give a further indication of the size of $G_w(L_t)$ by showing that if L_t is a G_w - G_w matrix, then $G_w(L_t)$ contains also an unbounded sequence. To see this, consider the sequence x given by

$$x_k = (-1)^k (k+1)^2 (k+2)(k+3). \quad (3.2)$$

Then

$$\sum_{k=0}^{\infty} \frac{1}{k+1} x_k t_n^{k+1} = t_n \sum_{k=0}^{\infty} (-1)^k (k+1)(k+2)(k+3) t_n^k = \frac{6t_n}{(1+t_n)^4}. \quad (3.3)$$

Hence,

$$|(L_t x)_n| = \frac{6t_n}{-\log(1-t_n)(1+t_n)^4} \leq -\frac{6}{\log(1-t_n)}. \quad (3.4)$$

Thus, if L_t is an G_w - G_w matrix, then by Theorem 3.1, $-1/\log(1-t) \in G_w$, so $x \in G_w(L_t)$.

The next few results deal with the G_w -translativity of L_t . We will show that the L_t matrix is G_w -translative for some sequences in $G_w(L_t)$.

PROPOSITION 3.8. *Every G_w - G_w L_t matrix is G_w -translative for each sequence $x \in G_w$.*

THEOREM 3.9. *Suppose L_t is a G_w - G_w matrix and $\{x_k/k\}$ is a sequence such that $x_k/k = 0$ for $k = 0$, then the sequence $\{x_k/k\}$ is in $G_w(L_t)$ for each L -summable sequence x .*

PROOF. Let Y be the L_t -transform of the sequence $\{x_k/k\}$. Then we have

$$|Y_n| = -\frac{1}{\log(1-t_n)} \left| \sum_{k=0}^{\infty} \frac{1}{k(k+1)} x_k t_n^{k+1} \right| \leq C_n + D_n, \quad (3.5)$$

where

$$\begin{aligned} C_n &= -\frac{|x_1| t_n}{2 \log(1-t_n)} - \frac{|x_2| t_n}{6 \log(1-t_n)}, \\ D_n &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=3}^{\infty} \frac{1}{k(k+1)} x_k t_n^k \right|. \end{aligned} \quad (3.6)$$

By [Theorem 3.1](#), the hypothesis that L_t is G_w - G_w implies that $C \in G_w$, and hence there remains only to show $D \in G_w$ to prove the theorem. Note that

$$\begin{aligned} D_n &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=3}^{\infty} \frac{x_k}{(k+1)} \left(\int_0^{t_n} t^{k-1} dt \right) \right| \\ &= -\frac{1}{\log(1-t_n)} \left| \int_0^{t_n} dt \left(\sum_{k=3}^{\infty} \frac{1}{(k+1)} x_k t^{k-1} \right) \right|. \end{aligned} \quad (3.7)$$

The interchanging of the integral and the summation is legitimate as the radius of convergence of the power series

$$\sum_{k=3}^{\infty} \frac{1}{k+1} x_k t^{k-1} \quad (3.8)$$

is at least 1 by [\[2, Lemma 1\]](#) and hence the power series converges absolutely and uniformly for $0 \leq t \leq t_n$. Now we let

$$F(t) = \sum_{k=3}^{\infty} \frac{1}{k+1} x_k t^{k-1}. \quad (3.9)$$

Then we have

$$-\frac{F(t)}{\log(1-t)} = -\frac{1}{\log(1-t)} \sum_{k=3}^{\infty} \frac{1}{k+1} x_k t^{k-1}, \quad (3.10)$$

and the hypothesis that $x \in c(L)$ implies that

$$\lim_{t \rightarrow 1^-} \frac{F(t)}{-\log(1-t)} = A \text{ (finite)}, \quad \text{for } 0 < t < 1. \quad (3.11)$$

We also have

$$\lim_{t \rightarrow 0} \frac{F(t)}{-\log(1-t)} = 0. \quad (3.12)$$

Now [\(3.11\)](#) and [\(3.12\)](#) yield

$$\left| \frac{F(t)}{-\log(1-t)} \right| \leq M, \quad \text{for some } M > 0, \quad (3.13)$$

and hence

$$|F(t)| \leq M \log(1-t), \quad \text{for } 0 < t < 1. \quad (3.14)$$

So, we have

$$\begin{aligned}
 D_n &= -\frac{1}{\log(1-t_n)} \left| \int_0^{t_n} F(t) dt \right| \\
 &\leq -\frac{1}{\log(1-t_n)} \int_0^{t_n} |F(t)| dt \\
 &\leq -\frac{M}{\log(1-t_n)} \int_0^{t_n} -\log(1-t) dt \\
 &= -M(1-t_n) - \frac{Mt_n}{\log(1-t_n)} \\
 &\leq -\frac{M}{\log(1-t_n)}.
 \end{aligned} \tag{3.15}$$

The hypothesis that L_t is G_w - G_w implies that both $-1/\log(1-t)$ and $(1-t)$ are in G_w by [Theorem 3.1](#). Hence $D \in G_w$. $\square$

THEOREM 3.10. *Every G_w - G_w L_t matrix is G_w -translative for each L -summable sequence in $G_w(L_t)$.*

PROOF. Let $x \in c(L) \cap G_w(L_t)$. Then we will show that

- (i) $T_x \in G_w(L_t)$,
- (ii) $S_x \in G_w(L_t)$.

We first show that (i) holds. Note that

$$\begin{aligned}
 |(L_t T_x)_n| &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=0}^{\infty} \frac{1}{k+1} x_{k+1} t_n^{k+1} \right| \\
 &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=1}^{\infty} \frac{1}{k} x_k t_n^k \right| \\
 &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=1}^{\infty} \left(\frac{1}{k+1} + \frac{1}{k(k+1)} \right) x_k t_n^k \right| \\
 &\leq P_n + Q_n,
 \end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
 P_n &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=1}^{\infty} \frac{1}{k+1} x_k t_n^k \right|, \\
 Q_n &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=1}^{\infty} \frac{1}{k(k+1)} x_k t_n^k \right|.
 \end{aligned} \tag{3.17}$$

So, we have $|(L_t T_x)_n| \leq P_n + Q_n$, and if we show that both P and Q are in G_w , then (i) holds. But the condition $P \in G_w$ follows from the hypothesis that $x \in G_w(L_t)$ and $Q \in G_w$ follows from [Theorem 3.9](#).

Next we will show that (ii) holds. We have

$$\begin{aligned}
 |(L_t S_x)_n| &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=1}^{\infty} \frac{1}{k+1} x_{k-1} t_n^{k+1} \right| \\
 &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=0}^{\infty} \frac{1}{k+2} x_k t_n^{k+2} \right| \\
 &\leq W_n + U_n,
 \end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
 W_n &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=0}^{\infty} \frac{1}{k+1} x_k t_n^{k+2} \right|, \\
 U_n &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} x_k t_n^{k+2} \right|.
 \end{aligned} \tag{3.19}$$

The hypothesis that $X \in G_w(L_t)$ implies that $W \in G_w$. We can also show that $U \in G_w$ by making a slight modification in the proof of [Theorem 3.9](#), replacing the sequence $\{x_k/k\}$ with the sequence $\{x_k/(k+2)\}$. Hence, the theorem follows. $\square$

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