ON A DIFFERENCE EQUATION WITH MIN-MAX RESPONSE

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We investigate the global behavior of the (positive) solutions of the difference equation $x_{n+1} = \alpha_n + F(x_n, \dots, x_{n-k})$, $n = 0, 1, \dots$, where (α_n) is a sequence of positive reals and F is a min-max function in the sense introduced here. Our results extend several results obtained in the literature.

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1. Introduction. Let k be a positive integer and let $\mathbb{R}_+$ be the set of all positive reals. We give the following definition.

DEFINITION 1.1. A function $F: \mathbb{R}_+^{k+1} \rightarrow \mathbb{R}_+$ is called a min-max function if it satisfies the inequality

$$\frac{\bigwedge_{j=1}^{k+1} u_j}{\bigvee_{j=1}^{k+1} u_j} \leq F(u_1, u_2, \dots, u_{k+1}) \leq \frac{\bigvee_{j=1}^{k+1} u_j}{\bigwedge_{j=1}^{k+1} u_j}, \quad (1.1)$$

for all $u_j > 0$, $j = 1, \dots, k+1$, where, as usual, the symbol $\bigvee_{j=1}^n u_j$ stands for the maximum of the variables u_j , $j = 1, \dots, n$, and $\bigwedge_{j=1}^n u_j$ stands for their minimum.

In [Section 2](#), we give exact information on the form which a min-max function may have.

Simple examples of min-max functions are

$$F_1(u_1, u_2) := \frac{u_2}{u_1}, \quad F_2(u_1, u_2) := \frac{u_1}{u_2} \quad (1.2)$$

which appear as the response functions, respectively, in the difference equation

$$y_{n+1} = \alpha + \frac{y_{n-1}}{y_n} \quad (1.3)$$

studied in [\[1\]](#) and in the difference equation

$$y_{n+1} = \alpha + \frac{y_n}{y_{n-1}} \quad (1.4)$$

studied in [\[2\]](#). These two equations have completely different behavior; see [Remark 3.6](#). Also in [\[13, 14\]](#), the second author considered the closely related equation

$$x_{n+1} = \alpha_n + \frac{x_{n-1}}{x_n}, \quad (1.5)$$

where (α_n) is either a periodic sequence (with period two) or a convergent sequence of nonnegative real numbers.

Motivated by the above-mentioned works, in this paper, we study the behavior of the difference equation

$$x_{n+1} = \alpha_n + F(x_n, \dots, x_{n-k}), \quad n = 0, 1, \dots, \quad (1.6)$$

where the initial conditions $x_{-k}, \dots, x_0$ are positive real numbers, (α_n) is a sequence of positive real numbers, and F is a min-max function.

Since a min-max function takes the value 1 at the diagonal of the space $\mathbb{R}_+^{k+1}$, it follows that in case the sequence (α_n) converges to a certain α , the positive real number

$$K := \alpha + 1 \quad (1.7)$$

is the unique asymptotic equilibrium of (1.6).

Our purpose here is to discuss the boundedness and persistence of (1.6), as well as the attractivity of the asymptotic equilibrium $\alpha + 1$, where α is the limit of (α_n) whenever this exists. This follows immediately by Theorem 3.2, where we show that, if $1 < \liminf \alpha_n \leq \limsup \alpha_n < +\infty$, then any solution (x_n) satisfies the relation

$$1 \leq \frac{\limsup x_n}{\liminf x_n} \leq \frac{\limsup \alpha_n - 1}{\liminf \alpha_n - 1}. \quad (1.8)$$

Thus, if the sequence (α_n) converges to some $\alpha(> 1)$, then any solution with positive initial values converges to the asymptotic equilibrium $K = \alpha + 1$. This generalizes [1, Theorem 5.2] and part of [2, Theorem 1]. For the case $\alpha_n = 1$, for all n (in Theorem 3.3), we show that any nonoscillatory solution converges to 2, while if F satisfies the additional (sufficient) conditions

$$u_i < \vee_{j \neq i} u_j \implies F(u_1, u_2, \dots, u_{k+1}) < \frac{\vee_{j \neq i} u_j}{u_i}, \quad (1.9)$$

$$u_i > \wedge_{j \neq i} u_j \implies F(u_1, u_2, \dots, u_{k+1}) > \frac{\wedge_{j \neq i} u_j}{u_i}, \quad (1.10)$$

then it is shown in Theorem 3.4 that all solutions converge to 2. Comparing this fact with the results in [1], we see that the pair of conditions (1.9)–(1.10) seems also to be necessary. Indeed, these conditions are not satisfied in case of (1.3) and, as it is shown in [1, Theorem 4.1], it has (nontrivial) solutions which are periodic with period 2.

In Theorem 3.5, we show that if $\alpha_n = \alpha < 1$, for all n , then there is a large class of equations of the form (1.6) which have unbounded (positive) solutions. This result extends [1, Theorem 3.1]. In the Section 4, we give two examples of difference equations with min-max response to illustrate our results.

Also the so-called (2,2)-type equation defined in [6] (where about 50 types of difference equations are presented) includes the equation

$$x_{n+1} = \frac{A_1 x_n + B_1 x_{n-1}}{A_2 x_n + B_2 x_{n-1}}. \quad (1.11)$$

Under appropriate choice of the parameters, (1.11) can be written as

$$x_{n+1} = \alpha + \frac{(\beta + \gamma)x_{n-1}}{\beta x_n + \gamma x_{n-1}}, \quad (1.12)$$

which is of the type (1.6). Thus in this paper, we push further the investigation originated in [6] for such a form of (2, 2)-type difference equations.

For other closely related results, which mostly deal with difference equations and inequalities whose response is (or it can be transformed into) a min-max function, see, for instance, [7, 8, 9, 10, 11, 12, 13, 14] and the references cited therein.

2. On the min-max functions. In this section, we give a characterization of min-max functions. The result is incorporated in the following theorem.

THEOREM 2.1. *A function $F: \mathbb{R}_+^{k+1} \rightarrow \mathbb{R}_+$ is a min-max function if and only if there are nonnegative real-valued functions $a_j(u_1, u_2, \dots, u_{k+1})$, $b_j(u_1, u_2, \dots, u_{k+1})$, $j = 1, 2, \dots, k+1$, such that*

$$\begin{aligned} \sum_{j=1}^{k+1} a_j(u_1, u_2, \dots, u_{k+1}) &= \sum_{j=1}^{k+1} b_j(u_1, u_2, \dots, u_{k+1}) = 1, \\ F(u_1, u_2, \dots, u_{k+1}) &= \frac{\sum_{j=1}^{k+1} a_j(u_1, u_2, \dots, u_{k+1}) u_j}{\sum_{j=1}^{k+1} b_j(u_1, u_2, \dots, u_{k+1}) u_j}, \end{aligned} \quad (2.1)$$

for all $(u_1, u_2, \dots, u_{k+1}) \in \mathbb{R}_+^{k+1}$.

PROOF. The “if” part is easily proved by using the form of F and the conditions on the coefficients a_j , b_j .

To show the inverse, assume that $F(u_1, u_2, \dots, u_{k+1})$ is a min-max function and fix any element $(u_1, u_2, \dots, u_{k+1}) \in \mathbb{R}_+^{k+1}$. We let

$$v := \bigwedge_{j=1}^{k+1} u_j, \quad w := \bigvee_{j=1}^{k+1} u_j, \quad (2.2)$$

thus $v = u_{j_1}$ and $w = u_{j_2}$, for two indices $j_1, j_2 \in \{1, 2, \dots, k+1\}$.

From the definition of the min-max functions, we know that the value $F(u_1, u_2, \dots, u_{k+1})$ lies in the interval $[v/w, w/v]$, thus there is a number $a \in [0, 1]$ such that

$$F(u_1, u_2, \dots, u_{k+1}) = a \frac{w}{v} + (1-a) \frac{v}{w}. \quad (2.3)$$

Let

$$b := \frac{(1-a)v^2}{aw^2 + (1-a)v^2}. \quad (2.4)$$

It is clear that b belongs to the interval $[0, 1]$, and it depends on v , w (thus on $u_1, u_2, \dots, u_{k+1}$). By some simple calculations, we obtain

$$(bw + (1-b)v) \left(a \frac{w}{v} + (1-a) \frac{v}{w} \right) = aw + (1-a)v \quad (2.5)$$

and consequently we get

$$F(u_1, u_2, \dots, u_{k+1}) = a \frac{w}{v} + (1-a) \frac{v}{w} = \frac{aw + (1-a)v}{bw + (1-b)v}. \quad (2.6)$$

This proves the theorem since we can set $a_j(u_1, u_2, \dots, u_{k+1}) := 0$, if $j \neq j_1, j_2$, while $a_{j_1}(u_1, u_2, \dots, u_{k+1}) = 1-a$ and $a_{j_2}(u_1, u_2, \dots, u_{k+1}) = a$. Similar substitutions are used for the denominator. The proof is complete. $\square$

REMARK 2.2. The quotient of any two elements of the class of all $f : \mathbb{R}_+^{k+1} \rightarrow \mathbb{R}_+$ which satisfy an inequality of the form

$$\wedge_{j=1}^{k+1} u_j \leq f(u_1, u_2, \dots, u_{k+1}) \leq \vee_{j=1}^{k+1} u_j \quad (2.7)$$

produces a min-max function.

3. The main results. Our first result refers to the boundedness of the solutions.

THEOREM 3.1. Consider (1.6), where F is a min-max function and the sequence (α_n) satisfies

$$1 < C := \inf \alpha_n \leq \sup \alpha_n =: B < +\infty. \quad (3.1)$$

Then any solution (x_n) with positive initial values satisfies the condition

$$\min \left\{ \wedge_{j=1}^{k+1} x_j, \frac{LC}{L-1} \right\} \leq x_n \leq L, \quad (3.2)$$

for all $n = 1, 2, \dots$, where

$$L := \max \left\{ \vee_{j=1}^{k+1} x_j, \frac{BC}{C-1} \right\}. \quad (3.3)$$

Also, if $\alpha_n = \alpha = 1$, for all n , then it holds that

$$M \leq x_n \leq \frac{M}{M-1}, \quad (3.4)$$

for all $n \geq 1$, where

$$M := \min \left\{ \wedge_{j=1}^{k+1} x_j, \frac{\vee_{j=1}^{k+1} x_j}{\vee_{j=1}^{k+1} x_j - 1} \right\}. \quad (3.5)$$

PROOF. Let $n > k+1$. From (1.6), for all $j \geq 1$, we have

$$C < x_j \leq \vee_{i=1}^n x_i. \quad (3.6)$$

Also, for all $j = k+2, k+3, \dots, n$, we get

$$x_j \leq B + \frac{\vee_{i=j-k-1}^{j-1} x_i}{C} \leq B + \frac{\vee_{i=1}^n x_i}{C}. \quad (3.7)$$

These facts imply that

$$\vee_{j=1}^n x_j \leq \max \left\{ \vee_{i=1}^{k+1} x_i, B + \frac{\vee_{i=1}^n x_i}{C} \right\}, \quad (3.8)$$

from which we get

$$x_n \leq \vee_{i=1}^n x_i \leq \max \left\{ \vee_{i=1}^{k+1} x_i, \frac{BC}{C-1} \right\}, \quad (3.9)$$

and therefore,

$$C < x_m \leq L, \quad (3.10)$$

for all $m = 1, 2, \dots$

Next let $n > k+1$. From (3.10) and (1.6), it follows that for all $j = k+2, k+3, \dots, n$, it holds that

$$x_j \geq C + \frac{\wedge_{i=j-k-1}^{j-1} x_i}{L} \geq C + \frac{\wedge_{i=1}^n x_i}{L}. \quad (3.11)$$

Therefore, we have

$$x_j \geq \min \left\{ \wedge_{i=1}^{k+1} x_i, C + \frac{\wedge_{i=1}^n x_i}{L} \right\}, \quad (3.12)$$

for all $j = 1, 2, \dots$. This implies that

$$\wedge_{i=1}^n x_i \geq \min \left\{ \wedge_{i=1}^{k+1} x_i, C + \frac{\wedge_{i=1}^n x_i}{L} \right\} \quad (3.13)$$

and so

$$\wedge_{i=1}^n x_i \geq \min \left\{ \wedge_{i=1}^{k+1} x_i, \frac{LC}{L-1} \right\}. \quad (3.14)$$

This gives

$$x_n \geq \wedge_{i=1}^n x_i \geq \min \left\{ \wedge_{i=1}^{k+1} x_i, \frac{LC}{L-1} \right\}, \quad (3.15)$$

which, together with (3.10), proves the first result of the theorem.

Next assume that $\alpha_n = 1$, $n = 0, 1, \dots$. To show inequality (3.4), we observe that

$$M \leq x_n \leq \frac{M}{M-1}, \quad (3.16)$$

for all $n = 1, 2, \dots, k+1$. Also from (1.6), we get

$$\begin{aligned} x_{k+2} &\geq 1 + \frac{\wedge_{j=1}^{k+1} x_j}{\vee_{j=1}^{k+1} x_j} \geq 1 + \frac{M}{M/(M-1)} = M, \\ x_{k+2} &\leq 1 + \frac{\vee_{j=1}^{k+1} x_j}{\wedge_{j=1}^{k+1} x_j} \leq 1 + \frac{M/(M-1)}{M} = \frac{M}{M-1}. \end{aligned} \quad (3.17)$$

These arguments and the induction complete the proof. $\square$

THEOREM 3.2. Consider (1.6), where F is a continuous min-max function and the sequence (α_n) satisfies the condition

$$1 < \liminf \alpha_n \leq \limsup \alpha_n < +\infty. \quad (3.18)$$

Then any (positive) solution (x_n) satisfies relation (1.8). Hence, if the sequence (α_n) converges to some $\alpha (> 1)$, then (x_n) converges to (a constant, which, therefore, is equal to) $\alpha + 1 =: K$.

PROOF. Let (x_n) be a solution. From Theorem 3.1, the solution is bounded, thus there are two-sided sequences, (y_m) (upper full limiting sequence) and (z_m) (lower full limiting sequence) of (x_n) (see, e.g., [3, 4, 5]), satisfying (1.6), for all integers m , and such that

$$\liminf x_n = z_0 \leq z_m, \quad y_m \leq y_0 = \limsup x_n, \quad (3.19)$$

for all m . Let $a_0 := \liminf \alpha_n$ and $a^0 := \limsup \alpha_n$. Then from (1.6), we have

$$y_0 \leq a^0 + \frac{y_0}{z_0}, \quad z_0 \geq a_0 + \frac{z_0}{y_0}. \quad (3.20)$$

Combining these two relations, we obtain (1.8). $\square$

THEOREM 3.3. Consider (1.6), where $\alpha_n = 1$, $n = 0, 1, \dots$, and F is a min-max function. Then every nonoscillatory (positive) solution converges to the equilibrium $K = 2$.

PROOF. Assume first that $x_n \geq 2$, for all $n \geq -k$. Set $u_n := x_n - 2$. From Theorem 2.1, we know that F may take the form (2.1), where the (nonnegative) functions a_j, b_j satisfy

$$\sum_{j=1}^{k+1} a_j(x_n, \dots, x_{n-k}) = \sum_{j=1}^{k+1} b_j(x_n, \dots, x_{n-k}) = 1. \quad (3.21)$$

Then we obtain

$$\begin{aligned} u_{n+1} &= \frac{\sum_{j=1}^{k+1} a_j(x_n, \dots, x_{n-k}) u_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n, \dots, x_{n-k}) x_{n+1-j}} - \frac{\sum_{j=1}^{k+1} b_j(x_n, \dots, x_{n-k}) u_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n, \dots, x_{n-k}) x_{n+1-j}} \\ &\leq \frac{\sum_{j=1}^{k+1} a_j(x_n, \dots, x_{n-k}) u_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n, \dots, x_{n-k}) x_{n+1-j}} \leq \frac{1}{2} \vee_{n-k}^n u_j. \end{aligned} \quad (3.22)$$

Our intention is to show that $\lim u_n = 0$. To this end, we can either use [7, Lemma 1] or proceed as follows.

Let (Y_m) be an upper full limiting sequence of (u_n) with $Y_m \leq Y_0 = \limsup u_n$, for all integers m . Then, from the previous arguments, it follows that it satisfies the inequality

$$Y_0 \leq \frac{1}{2} Y_0, \quad (3.23)$$

thus we have $Y_0 = 0$. This and the fact that $u_n \geq 0$ imply that $\lim x_n = 2$.

Next, assume that $x_n \leq 2$, for all $n \geq -k$. Set $v_n := 2 - x_n$. From (1.3) and by using the form of the function F , we obtain

$$\begin{aligned} v_{n+1} &= \frac{\sum_{j=1}^{k+1} a_j(x_n, \dots, x_{n-k}) v_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n, \dots, x_{n-k}) x_{n+1-j}} - \frac{\sum_{j=1}^{k+1} b_j(x_n, \dots, x_{n-k}) v_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n, \dots, x_{n-k}) x_{n+1-j}}, \\ &\leq \frac{\sum_{j=1}^{k+1} a_j(x_n, \dots, x_{n-k}) v_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n, \dots, x_{n-k}) x_{n+1-j}} \leq \frac{1}{M} \vee_{j=n-k}^n v_j, \end{aligned} \quad (3.24)$$

where $M(> 1)$ is the number defined in Theorem 3.1. By using this fact and following the same procedure as in the first case, we derive that $\lim_{n \rightarrow \infty} v_n = 0$, which implies that $\lim x_n = 2$, as desired. $\square$

THEOREM 3.4. Consider (1.6), where $\alpha_n = 1$, $n = 0, 1, \dots$, and F is a continuous min-max function satisfying the properties (1.9) and (1.10). Then every (positive) solution converges to the equilibrium $K = 2$.

PROOF. Let (x_n) be a solution. Then by Theorem 3.1, (x_n) is bounded. Consider an upper full limiting sequence (y_m) and a lower full limiting sequence (z_m) of (x_n) , as above. From (1.6), we have

$$y_0 \leq 1 + \frac{y_0}{z_0}, \quad z_0 \geq 1 + \frac{z_0}{y_0} \quad (3.25)$$

and therefore, we get

$$y_0 z_0 = y_0 + z_0. \quad (3.26)$$

This gives

$$\frac{1}{y_0} + \frac{1}{z_0} = 1. \quad (3.27)$$

If it happens that $y_0, z_0 > 2$, or $y_0, z_0 < 2$, then we should have $1/y_0, 1/z_0 < 1/2$ and $1/y_0, 1/z_0 > 1/2$, respectively. Both these arguments contradict (3.27). Therefore, we must have

$$z_0 \leq 2 \leq y_0. \quad (3.28)$$

Assume that there is some $j \in \{-k-1, \dots, -1\}$ such that $y_j < y_0$ and let j_0 be an index such that

$$y_{j_0} = \wedge_{j=-k-1}^{-1} y_j. \quad (3.29)$$

Then from (1.9), we get

$$y_{j_0} < \vee_{j \neq j_0} y_j \leq y_0 \quad (3.30)$$

and so from (1.6) and condition (1.9), we have

$$y_0 = 1 + F(y_{-1}, \dots, y_{-k-1}) < 1 + \frac{y_0}{y_{j_0}} \leq 1 + \frac{y_0}{z_0}. \quad (3.31)$$

This gives $y_0 z_0 < y_0 + z_0$, contradicting (3.26). Thus we have $y_j = y_0$, for all $j = -k-1, \dots, -1$, and therefore,

$$y_0 = 1 + F(y_{-1}, \dots, y_{-k-1}) = 1 + F(y_0, \dots, y_0) = 2. \quad (3.32)$$

Similarly, we can use condition (1.10) to obtain $z_0 = 2$. The proof is complete. $\square$

Our final result refers to the case $\alpha \in [0, 1)$. We show that in this case, there are equations of the form (1.3) which admit unbounded solutions.

THEOREM 3.5. *Consider the equation*

$$x_{n+1} = \alpha + \frac{\sum_{i=0}^m a_i x_{n-2i-1}}{\sum_{i=0}^m b_i x_{n-2i}}, \quad (3.33)$$

where $m \in \mathbb{N}$, $\alpha \in [0, 1)$, and where the coefficients a_j and b_j , $j = 0, \dots, m$, are nonnegative constants which satisfy the conditions

$$\sum_{i=0}^m a_i = \sum_{i=0}^m b_i. \quad (3.34)$$

Then there exist unbounded solutions of (3.33).

PROOF. Obviously, without loss of the generality, we can assume that $\sum_{i=0}^m a_i = \sum_{i=0}^m b_i = 1$.

Assume that $\alpha \in (0, 1)$. We choose the initial conditions such that

$$\begin{aligned} x_{-(2m+1)}, \dots, x_{-1} &> \frac{1}{1-\alpha} > 1 + \alpha, \\ \alpha &< x_{-2m}, \dots, x_0 < 1. \end{aligned} \quad (3.35)$$

We set

$$D := \wedge_{i=0}^m x_{-(2i+1)} \quad (3.36)$$

and observe that

$$D > \frac{1}{1-\alpha}. \quad (3.37)$$

From (3.33), we have

$$\begin{aligned} x_1 &= \alpha + \frac{\sum_{i=0}^m a_i x_{-(2i+1)}}{\sum_{i=0}^m b_i x_{-2i}} > \alpha + \sum_{i=0}^m a_i x_{-(2i+1)} > \alpha + D, \\ x_2 &= \alpha + \frac{\sum_{i=0}^m a_i x_{1-(2i+1)}}{\sum_{i=0}^m b_i x_{1-2i}} < \alpha + \frac{1}{\sum_{i=0}^m b_i x_{1-2i}} \\ &= \alpha + \frac{1}{b_0 x_1 + b_1 x_{-1} + \dots + b_m x_{-2m+1}} \leq \alpha + \frac{1}{b_0(\alpha + D) + (1-b_0)(1/(1-\alpha))} \\ &\leq \alpha + \frac{1}{b_0(\alpha + 1/(1-\alpha)) + (1-b_0)(1/(1-\alpha))} = \alpha + \frac{1}{b_0 \alpha + 1/(1-\alpha)} \leq 1, \\ x_3 &= \alpha + \frac{\sum_{i=0}^m a_i x_{2-(2i+1)}}{\sum_{i=0}^m b_i x_{2-2i}} > \alpha + \sum_{i=0}^m a_i x_{2-(2i+1)} \\ &\geq \alpha + \min \{x_1, x_{-1}, \dots, x_{-2m+1}\} \geq \alpha + \min \{x_1, x_{-1}, \dots, x_{-2m-1}\} \\ &= \alpha + \min \{x_1, \min \{x_{-1}, \dots, x_{-2m-1}\}\} \geq \alpha + D. \end{aligned} \quad (3.38)$$

Following the same procedure, we get

$$x_{2j+1} > \alpha + D, \quad x_{2j+2} < 1, \quad (3.39)$$

for all $j = 0, 1, \dots, m$. By induction, we obtain

$$x_{(2m+2)j-(2s+1)} > \alpha j + D, \quad (3.40)$$

for all $j \in \mathbb{N}$ and $s = 0, 1, \dots, m$, as well as

$$\alpha < x_{2n} < 1, \quad n = -m, -(m-1), \dots, -1, \dots \quad (3.41)$$

Inequality (3.40) implies the desired result in case $\alpha > 0$.

Assume that $\alpha = 0$. Choose $\varepsilon \in (0, 1)$ and the initial conditions such that

$$\begin{aligned} x_{-(2m+1)}, \dots, x_{-1} &> \frac{1}{1-\varepsilon}, \\ 0 &< x_{-2m}, \dots, x_0 < 1-\varepsilon. \end{aligned} \quad (3.42)$$

From (3.33), we have

$$\begin{aligned} x_1 &= \frac{\sum_{i=0}^m a_i x_{-(2i+1)}}{\sum_{i=0}^m b_i x_{-2i}} > \frac{1/(1-\varepsilon)}{1-\varepsilon} = \frac{1}{(1-\varepsilon)^2} > \frac{1}{1-\varepsilon}, \\ x_2 &= \frac{\sum_{i=0}^m a_i x_{1-(2i+1)}}{\sum_{i=0}^m b_i x_{1-2i}} < \frac{1-\varepsilon}{b_0 x_1 + (1-b_0)(1/(1-\varepsilon))} \\ &\leq \frac{1-\varepsilon}{b_0(1/(1-\varepsilon)^2) + (1-b_0)(1/(1-\varepsilon))} < 1-\varepsilon. \end{aligned} \quad (3.43)$$

Following the same procedure, we get

$$\begin{aligned} x_{2j+1} &> \frac{1}{(1-\varepsilon)^2} > \frac{1}{1-\varepsilon}, \\ x_{2j+2} &< 1-\varepsilon, \end{aligned} \quad (3.44)$$

for all $j = 0, 1, \dots, m$. By induction, we obtain

$$x_{(2m+2)j-(2s+1)} > \frac{1}{(1-\varepsilon)^{j+1}}, \quad (3.45)$$

for all $j \in \mathbb{N}$ and $s = 0, 1, \dots, m$, as well as

$$0 < x_{2n} < 1-\varepsilon, \quad n = 1, 2, \dots \quad (3.46)$$

From (3.45), the result follows. $\square$

REMARK 3.6. Equation (3.33) includes the special case (1.3). Thus for $\alpha \in (0, 1)$, Theorem 3.5 applies and therefore, (1.3) has unbounded solutions with positive initial values. On the other hand, (3.33) does not include the case (1.4) and as proved in [2], for the same values of α , (1.4) has a global attractor.

REMARK 3.7. By some modifications of the proof of Theorem 3.5, we can prove the following result.

THEOREM 3.8. *Consider the equation*

$$x_{n+1} = \alpha_n + \frac{\sum_{i=0}^m a_i x_{n-2i-1}}{\sum_{i=0}^m b_i x_{n-2i}}, \quad (3.47)$$

where $m \in \mathbb{N}$, (α_n) is a sequence of positive real numbers such that $\lim_{n \rightarrow \infty} \alpha_n =: A \in [0, 1)$, and where the coefficients a_j and b_j , $j = 0, \dots, m$, are nonnegative constants which satisfy the conditions

$$\sum_{i=0}^m a_i = \sum_{i=0}^m b_i. \quad (3.48)$$

Then there exist unbounded solutions of (3.47).

4. Some illustrative examples

EXAMPLE 4.1. Consider the difference equation

$$x_{n+1} = \alpha + \frac{\beta x_n + \gamma x_n^2 + \delta x_{n-1}^2}{\beta x_n + \gamma x_n x_{n-1} + \delta x_{n-1}^2}, \quad (4.1)$$

where all the coefficients are positive real numbers. The rational function on the right-hand side is a min-max function, since it can be written in the form

$$\frac{((\beta + \gamma x_n)/(\beta + \gamma x_n + \delta x_{n-1}))x_n + (\delta x_{n-1}/(\beta + \gamma x_n + \delta x_{n-1}))x_{n-1}}{(\beta/(\beta + \gamma x_n + \delta x_{n-1}))x_n + ((\gamma x_n + \delta x_{n-1})/(\beta + \gamma x_n + \delta x_{n-1}))x_{n-1}}. \quad (4.2)$$

Thus, from Theorems 3.2 and 3.4, we conclude that, for every fixed $\alpha \geq 1$, any solution of (4.1) converges to the equilibrium $\alpha + 1$. Notice that conditions (1.9) and (1.10) are also satisfied.

EXAMPLE 4.2. Consider the difference equation

$$x_{n+1} = \alpha + \frac{\sum_{j_i \in \{n, n-1, n-2\}} x_{j_1} x_{j_2} x_{j_3}}{x_n^3 + x_{n-1}^3 + x_{n-2}^3 + 6x_n x_{n-1} x_{n-2}}, \quad (4.3)$$

where $\alpha \geq 0$. This is a third-order difference equation whose response on the right-hand side is a min-max function. Indeed, this can be written in the form

$$\frac{\sum_{j_i \in \{n, n-1, n-2\}, j_1 \neq j_2 \neq j_3 \neq j_1} ((x_{j_1}^2 + x_{j_1} x_{j_2} + x_{j_1} x_{j_3}) / (x_n + x_{n-1} + x_{n-2})^2) x_{j_1}}{\sum_{j_i \in \{n, n-1, n-2\}, j_1 \neq j_2 \neq j_3 \neq j_1} ((x_{j_1}^2 + 2x_{j_2} x_{j_3}) / (x_n + x_{n-1} + x_{n-2})^2) x_{j_1}}. \quad (4.4)$$

Here, again, Theorems 3.2 and 3.4 apply and we conclude that in case $\alpha \geq 1$, any solution of (4.3) converges to $\alpha + 1$.

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