INTERPOLATION METHODS TO ESTIMATE EIGENVALUE DISTRIBUTION OF SOME INTEGRAL OPERATORS

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We study the asymptotic distribution of eigenvalues of integral operators T_k defined by kernels k which belong to Triebel-Lizorkin function space $F_{pu}^\sigma(F_{qv}^\tau)$ by using the factorization theorem and the Weyl numbers x_n . We use the relation between Triebel-Lizorkin space $F_{pu}^\sigma(\Omega)$ and Besov space $B_{pq}^\tau(\Omega)$ and the interpolation methods to get an estimation for the distribution of eigenvalues in Lizorkin spaces $F_{pu}^\sigma(F_{qv}^\tau)$.

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1. Lizorkin kernels. We will use the following notation: $l_{p,q}$, $S_{pq}^{(x)}$, B_{pq}^s , and F_{pq}^s to denote Lorentz sequence space, Schatten class, Besov function space, and Triebel-Lizorkin function space, respectively. By π_p , s_n , and x_n we denote p -summing norms, s -number function, and Weyl numbers, respectively, see [2, 4, 5].

THEOREM 1.1 (see [1]). *Let $s \in \mathbb{R}$, $p \in (0, \infty)$ and let $q, u, v \in (0, \infty]$. Then $B_{pu}^s \subset F_{pq}^s \subset B_{pv}^s$ if and only if*

$$0 < u \leq \min(p, q), \quad \max(p, q) \leq v \leq \infty. \quad (1.1)$$

That is, if and only if $0 < u \leq q \leq v$.

PROPOSITION 1.2 (see [4]). *Let $\Phi \in [B_{pu}^\sigma(0, 1), X]$ and $r = \max(p, u)$. Then*

$$\Phi_{\text{op}} : a \longrightarrow (\Phi(\cdot), a) \quad (1.2)$$

(where Φ_{op} is an approximable operator from X' into $B_{pu}^\sigma(0, 1)$) defines an absolutely r -summing operator from X' into $B_{pu}^\sigma(0, 1)$. Moreover,

$$\|\Phi_{\text{op}} \mid \pi_r\| \leq \|\Phi \mid [B_{pu}^\sigma, X]\|. \quad (1.3)$$

We restate the previous proposition in the following form in the case of Triebel-Lizorkin space $F_{pu}^\sigma(\Omega)$.

PROPOSITION 1.3. *Let X be a Banach space, $\Omega \subset \mathbb{R}^N$ a bounded domain, $\sigma > 0$, and $1 \leq p < \infty$. Let $\Phi \in F_{pu}^\sigma(\Omega; X)$ and $r = \max(p, u)$. Then*

$$\Phi_{\text{op}} : x \longrightarrow (\Phi(\cdot), x) \quad (1.4)$$

defines an absolutely r -summing operator from X' into $F_{pu}^\sigma(X)$. Moreover,

$$\pi_r(\Phi_{\text{op}}) \leq \|\Phi\|_{p,u,\sigma;\Omega,X}. \quad (1.5)$$

PROOF. Given $x_1, \dots, x_n \in X'$, Jessen's inequality [4] yields

$$\left(\int_{\Omega} \left[\sum_{i=1}^n |(\Phi(\xi), x_i)|^r \right]^{p/r} d\xi \right)^{1/p} \leq \left(\sum_{i=1}^n \left[\int_{\Omega} |(\Phi(\xi), x_i)|^p d\xi \right]^{r/p} \right)^{1/r}. \quad (1.6)$$

Therefore,

$$\left\| \left(\sum_{i=1}^n |(\Phi(\cdot), x_i)|^r \right)^{1/r} \right\|_{L_p} \leq \left(\sum_{i=1}^n \|(\Phi(\cdot), x_i)\|_{L_p}^r \right)^{1/r} \leq \|\Phi\|_{L_p} \|(x_i)\|_{\pi_r}. \quad (1.7)$$

Applying this result to $\Delta_\tau^m \Phi$, we obtain

$$\left\| \left(\sum_{i=1}^n |(\Delta_\tau^m \Phi(\cdot), x_i)|^r \right)^{1/r} \right\|_{L_p} \leq \|\Delta_\tau^m \Phi\|_{L_p} \|(x_i)\|_{\pi_r}. \quad (1.8)$$

Hence,

$$\begin{aligned} & \left[\int_{\Omega} \left(\sum_{i=1}^n \left[\tau^{-\sigma} \|(\Delta_\tau^m \Phi(\cdot), x_i)\|_{L_p} \right]^r \right)^{u/r} \frac{d\tau}{\tau} \right]^{1/u} \\ & \leq \left[\sum_{i=1}^n \left(\int_{\Omega} \left[\tau^{-\sigma} \|(\Delta_\tau^m \Phi(\cdot), x_i)\|_{L_p} \right]^u \right)^{r/u} \right]^{1/r} \\ & \leq \left\| \left(\left[\int_{\Omega} \tau^{-\sigma} |(\Delta_\tau^m \Phi)|^u \frac{d\tau}{\tau} \right] \right)^{1/u} \right\|_{L_p} \|(x_i)\|_{\pi_r}. \end{aligned} \quad (1.9)$$

Finally, we conclude from the preceding inequalities that

$$\begin{aligned} & \left\| \left(\sum_{i=1}^n |(\Phi(\cdot), x_i)|^r \right)^{1/r} \right\|_{F_{pu}^\sigma} \\ & \leq \left\| \left(\sum_{i=1}^n |(\Phi(\cdot), x_i)|^r \right)^{1/r} \right\|_{L_p} + \left[\int_{\Omega} \left(\sum_{i=1}^n \left[\tau^{-\sigma} \|(\Delta_\tau^m \Phi(\cdot), x_i)\|_{L_p} \right]^r \right)^{u/r} \frac{d\tau}{\tau} \right]^{1/u} \\ & \leq \left(\|\Phi\|_{L_p} + \left\| \left(\left[\int_{\Omega} \tau^{-\sigma} |(\Delta_\tau^m \Phi)|^u \frac{d\tau}{\tau} \right] \right)^{1/u} \right\|_{L_p} \right) = \|\Phi\|_{F_{pu}^\sigma} \|(x_i)\|_{\pi_r}. \end{aligned} \quad (1.10)$$

This shows that Φ_{op} is absolutely r -summing. $\square$

COROLLARY 1.4 (see [2]). *Let X and Y be Banach spaces, $2 \leq p < \infty$, and $T \in \pi_{p,2}(X, Y)$. Then $T \in S_{p,\infty}^x(X, Y)$, and for any $n \in \mathbb{N}$,*

$$x_n(T) \leq n^{-1/p} \pi_{p,2}(T). \quad (1.11)$$

We are interested in the following theorem.

THEOREM 1.5 (see [3]). *Let $1 \leq p \leq \max(2, q) \leq \infty$. Then*

$$x_n(I_{p,q}^m : l_p^m \rightarrow l_q^m) \asymp \begin{cases} n^{1/q-1/p} & \text{for } 1 \leq p \leq q \leq 2, \\ n^{1/2-1/p} & \text{for } 1 \leq p \leq 2 \leq q < \infty, \\ 1 & \text{for } 2 \leq p \leq q < \infty. \end{cases} \quad (1.12)$$

THEOREM 1.6 ((multiplication theorem) [4]). *If $1/p + 1/q = 1/r$ and $1/u + 1/v = 1/w$, then*

$$S_{pu}^{(x)} \circ S_{qv}^{(x)} \subseteq S_{rw}^{(x)}. \quad (1.13)$$

THEOREM 1.7 ((eigenvalue theorem) [2]). *Let $0 < p < \infty$, $0 < q \leq \infty$, and let X be a Banach space. Then any operator $T \in L(X)$ which has Weyl numbers $(x_n(T)) \in l_{p,q}$, $T \in S_{p,q}^{(x)}(X)$ is a Riesz operator, the eigenvalue sequence of which is in $l_{p,q}$, and the following inequality holds*

$$\|(\lambda_n(T))\|_{p,q} \leq c \|(x_n(T))\|_{p,q}. \quad (1.14)$$

2. Eigenvalue theorem for Lizorkin kernels. The following theorem contains the main result of this note.

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $1 \leq p, q, u, v < \infty$, and $\sigma + \tau > N(1/p + 1/q - 1)$. Define r by $1/r = (\sigma + \tau)/N + 1/q^+$, where $q^+ = \max(q', 2)$. Then the eigenvalues of any kernel $k \in F_{pu}^\sigma(\Omega; F_{qv}^\tau(\Omega))$ belong to the Lorentz sequence space $l_{r,p}$ with*

$$\|(\lambda_n(k))_{n \in \mathbb{N}}\|_{l_{r,p}} \leq c \|k\|_{F_{pu}^\sigma(F_{qv}^\tau)}. \quad (2.1)$$

The constant c depends only on the indices and Ω .

PROOF. First, we assume that $p \leq q'$.

We will show that there exists an imbedding map $\text{id} : F_{pu}^\sigma(\Omega) \hookrightarrow F_{qv}^\tau(\Omega)'$ and then estimate its Weyl numbers $x_n(\text{id})$. We factories an imbedding map $\text{id} : F_{pu}^\sigma(\Omega) \hookrightarrow F_{qv}^\tau(\Omega)'$ with the help of maps A and B such that

$$\begin{array}{ccc} & \text{id} & \\ F_{pu}^\sigma(\Omega) & \xrightarrow{\quad} & F_{qv}^\tau(\Omega)' \\ A \downarrow & & \uparrow B \\ l_p^m(\Omega) & \xrightarrow{\text{id}^l} & l_{q'}^m(\Omega). \end{array} \quad (2.2)$$

This means that

$$x_n(\text{id}) \leq \|A\| x_n(\text{id}^l) \|B\| \quad (2.3)$$

if we are able to estimate $\|A\|$ and $\|B\|$ suitably; from [6], operators A and B are defined exactly as they are in [4], and assume that Ω contains the unit cube in $\mathbb{R}^N$ and divide the unit cube in the usual way into 2^{jN} congruent cubes with side length 2^{-j} .

From [1], we have

$$\|A\| \leq c_1 2^{-j(\sigma-N/p)}, \quad \|B\| \leq c_2 2^{j(-\tau-N/q')}. \quad (2.4)$$

Substituting (2.4) in (2.3), we get

$$x_n(\text{id}) \leq c 2^{-j(\sigma+\tau)+jN(1/p-1/q')} x_n(\text{id}^l). \quad (2.5)$$

By Theorem 1.5, we have

$$x_n(\text{id} : F_{pu}^\sigma(\Omega) \hookrightarrow F_{qv}^\tau(\Omega)') < n^{-\rho}, \quad (2.6)$$

where

$$\rho = \frac{\sigma + \tau}{N} + \begin{cases} 0, & \text{if } 1 \leq p \leq q' \leq 2, \\ \frac{1}{2} - \frac{1}{q}, & \text{if } 1 \leq p \leq 2 \leq q' < \infty, \\ 1 - \frac{1}{p} - \frac{1}{q}, & \text{if } 2 \leq p \leq q' < \infty, \end{cases} \quad (2.7)$$

and $n = 2^{Nj}$.

Hence,

$$\text{id} \in S_{1/\rho, \infty}^{(x)}(F_{pu}^\sigma(\Omega) \hookrightarrow F_{qv}^\tau(\Omega)'). \quad (2.8)$$

To estimate the Weyl number of T_k in $F_{qv}^\tau(\Omega)'$, we use the factorization

$$F_{qv}^\tau(\Omega)' \xrightarrow{T_k} F_{pu}^\sigma(\Omega) \xrightarrow{\text{id}} F_{qv}^\tau(\Omega)'. \quad (2.9)$$

By Proposition 1.3, $k \in F_{pu}^\sigma(\Omega; X)$ implies that $T_k : X' \hookrightarrow F_{pu}^\sigma(\Omega)$ is p -summing. By Corollary 1.4, we have

$$x_n(T_k : F_{qv}^\tau(\Omega)' \rightarrow F_{pu}^\sigma(\Omega)) \leq \pi_p(T_k) n^{-1/\max(p,2)} \leq c_1 \|k\|_{F_{pu}^\sigma(X)} n^{-1/\max(p,2)}, \quad (2.10)$$

that is,

$$T_k \in S_{s, \infty}^{(x)}(F_{pu}^\sigma(\Omega), F_{qv}^\tau(\Omega)'), \quad s = \max(p, 2). \quad (2.11)$$

We conclude from the multiplication theorem that $\text{id} \circ T_k \in S_{r, \infty}^{(x)}(F_{qv}^\tau(\Omega)'),$ where $1/r = \rho + 1/s$.

In the case when $p > q'$, then we have

$$k \in F_{pu}^\sigma(\Omega; F_{qv}^\tau(\Omega)) \implies k \in F_{qu}^\sigma(\Omega; F_{qv}^\tau(\Omega)). \quad (2.12)$$

In this way the second case is reduced to the first one.

So, we have shown that the map $k \rightarrow \text{id} \circ T_k$, which assigns to every kernel the corresponding operator, acts as follows:

$$\text{op} : F_{pu}^\sigma(F_{qv}^\tau) \longrightarrow S_{r,\infty}^{(x)}(F_{qv}^\tau(\Omega)'). \quad (2.13)$$

This result can be improved by interpolation. To this end, choose p_0, p_1 , and θ such that $1/p = 1 - \theta/p_0 + \theta/p_1$. We now apply the formula

$$(F_{p_0u}^\sigma(E), F_{p_1u}^\sigma(E))_{\theta,p} = F_{pu}^\sigma(E), \quad E = F_{qv}^\tau. \quad (2.14)$$

Then, using interpolation as in [2], where $1/r = 1 - \theta/r_0 + \theta/r_1$, hence

$$(S_{r_0,\infty}^{(x)}, S_{r_1,\infty}^{(x)})_{\theta,p} \subseteq S_{r,p}^{(x)}. \quad (2.15)$$

Hence the interpolation property yields

$$\text{op} : F_{pu}^\sigma(F_{qv}^\tau) \longrightarrow S_{r,p}^{(x)}(F_{qv}^\tau(\Omega)'). \quad (2.16)$$

By the eigenvalue theorem (Theorem 1.7), we therefore obtain $(\lambda_n(k)) \in l_{r,p}$. $\square$

THEOREM 2.2 (eigenvalue theorem for Sobolev kernels). *Let $1 \leq p < \infty$, $1 < q < \infty$, $1/r = m + n + 1/q^+$, and $w = \min(q, 2)$.*

Then

$$k \in [W_p^m(0, 1), W_q^n(0, 1)] \implies (\lambda_n(k)) \in l_{r,w}. \quad (2.17)$$

PROOF. See [4]. $\square$

The following example proves that our result improves the previous theorem of [4].

THEOREM 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $1 \leq p, q, v < \infty$, and $\tau > 0$ with $\tau > N(1/p + 1/q - 1)$, $p \leq v$, and $1/r := \tau/N + 1/\max(2, q')$. Then the eigenvalues of any kernel $k \in L_p(F_{qv}^\tau)$ belong to the Lorentz sequence space $l_{r,v}$ with*

$$\|(\lambda_n(k))_{n \in \mathbb{N}}\|_{l_{r,v}} \leq c \|k\|_{L_p(F_{qv}^\tau)}. \quad (2.18)$$

PROOF. We may assume that $p \leq q'$. Then, reasoning similarly as in the proof of Theorem 2.1, it follows that the map $k \rightarrow T_k$, which assigns to every kernel the corresponding operator, acts as follows:

$$\text{op} : L_p(F_{qv}^\tau) \longrightarrow S_{r,\infty}^{(x)}(L_p(\Omega)). \quad (2.19)$$

This result can be improved by interpolation. To this end, we apply the imbedding

$$(L_p, (E_0, E_1)_{\theta, m}) \subseteq ((L_p, E_0), (L_p, E_1))_{\theta, m}, \quad p < m, \quad (2.20)$$

to the interpolation couple $(F_{q, v_0}^{\tau_0}, F_{q, v_1}^{\tau_1})$. The interpolation property now implies that

$$\text{op} : L_p(F_{qv}^{\tau}) \rightarrow S_{r, v}^{(x)}(L_p(\Omega)). \quad (2.21)$$

By the eigenvalue theorem (Theorem 1.7), we therefore obtain $(\lambda_n(k)) \in l_{r, v}$. $\square$

EXAMPLE 2.4. (1) In this example, we will indicate a special case of the Lizorkin space $F_{pu}^{\sigma}(\mathbb{R}^N)$. When $1 < p < \infty$ and $s \in \mathbb{N}_0$, then

$$F_{p, 2}^s(\mathbb{R}^N) = W_p^s(\mathbb{R}^N) \quad (2.22)$$

are the classical Sobolev spaces.

We compare this case with Theorem 2.2. We find that

$$k \in W_p^{\sigma}(W_q^{\tau}) \Rightarrow (\lambda_n(k)) \in l_{r, w}, \quad (2.23)$$

where $w = \min(q, 2)$, and

$$k \in F_{pu}^{\sigma}(F_{qv}^{\tau}) \Rightarrow (\lambda_n(k)) \in l_{r, p}. \quad (2.24)$$

We conclude that if $p < w = \min(q, 2)$, $2 \leq q < \infty$, $1 < p < 2$, then

$$l_{r, w} \subset l_{r, p}, \quad (2.25)$$

that is,

$$\|(\lambda_n(k))_{n \in \mathbb{N}}\|_{r, p} \leq \|(\lambda_n(k))_{n \in \mathbb{N}}\|_{r, w}. \quad (2.26)$$

(2) We compare

$$k \in W_p^{\sigma}(W_q^{\tau}) \Rightarrow (\lambda_n(k)) \in l_{r, w}, \quad (2.27)$$

where $w = \min(q, 2)$, with

$$k \in L_p(F_{qv}^{\tau}) \Rightarrow (\lambda_n(k)) \in l_{r, v}, \quad (2.28)$$

where $p \leq v$. We conclude that if $v < w = \min(q, 2)$, $2 \leq q < \infty$, $1 < p < 2$, then

$$l_{r, w} \subset l_{r, v}, \quad (2.29)$$

that is,

$$\|(\lambda_n(k))_{n \in \mathbb{N}}\|_{r, v} \leq \|(\lambda_n(k))_{n \in \mathbb{N}}\|_{r, w}. \quad (2.30)$$

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