EULER CASE FOR A GENERAL FOURTH-ORDER DIFFERENTIAL EQUATION

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We deal with an Euler case for a general fourth-order equation and under this case, we obtain the general formula for the asymptotic form of the solutions.

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1. Introduction. In this paper, we examine the asymptotic form of a fundamental set of solutions of the fourth-order differential equation

$$(p_0 y''')'' + (p_1 y')' + \frac{1}{2} \sum_{j=0}^1 \left[\{q_{2-j} y^{(i)}\}^{(j+1)} + \{q_{2-j} y^{(j+1)}\}^{(j)} \right] + p_2 y = 0 \quad (1.1)$$

as $x \rightarrow \infty$, where x is the independent variable and the prime denotes d/dx . The functions $p_i(x)$ ($0 \leq i \leq 2$) and $q_i(x)$ ($i = 1, 2$) are defined on an interval $[a, \infty)$, are not necessarily real-valued, and are all nowhere zero in this interval. Our aims are to identify relations between q_0 , q_1 , p_0 , p_1 , and p_2 that represents an Euler case for (1.1) and to obtain the asymptotic forms of four linearly independent solutions under this case. Al-Hammadi [2] obtained an asymptotic formula of Liouville-Green type for (1.1) which extends those of Walker [9]. Also in [1], we consider (1.1) with $p_1 = q_2 = 0$ and we give a complete analysis for the case where

$$p_2^{1/3} p_0 = o(q_1^{4/3}) \quad (x \rightarrow \infty). \quad (1.2)$$

A fourth-order equation similar to (1.1) has been considered previously by Walker [9, 10]. Eastham [4] considered an Euler case for (1.1) with $p_1 = q_2 = 0$ and showed that this case represents a borderline between situations where all solutions have a certain exponential character as $x \rightarrow \infty$ and where only two solutions have this character. Al-Hammadi and Eastham [3] considered the case where the coefficients are small for large x .

The Euler case for (1.1) that has been referred to is given by

$$\begin{aligned} \frac{q_i'}{q_i} &\sim \text{const} \frac{q_1}{q_0} \quad (i = 1, 2), \\ \frac{(p_1 q_1^{-1/2})'}{p_1 q_1^{-1/2}} &\sim \text{const} \frac{q_1}{p_0} \end{aligned} \quad (1.3)$$

as $x \rightarrow \infty$.

We will use the recent asymptotic theorem of Eastham [6, Section 2] to obtain the solutions of (1.1) under the above case. The main theorem for (1.1) is given in Section 4 with some discussion in Section 5.

2. A transformation of the differential equation. We write (1.1) in the standard way [7] as a first-order system

$$Y' = AY, \quad (2.1)$$

where the first component of Y is y and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2}q_1 p_0^{-1} & p_1^{-1} & 0 \\ -\frac{1}{2}q_2 & -p_1 + \frac{1}{4}q_1^2 p_0^{-1} & -\frac{1}{2}p_0^{-1}q_1 & 1 \\ -p_2 & -\frac{1}{2}q_2 & 0 & 0 \end{bmatrix}. \quad (2.2)$$

As in [1], we express A in its diagonal form

$$T^{-1}AT = \Lambda \quad (2.3)$$

and we therefore require the eigenvalues λ_j and the eigenvectors v_j ($1 \leq j \leq 4$) of A .

The characteristic equation of A is given by

$$p_0\lambda^4 + q_1\lambda^3 + p_1\lambda^2 + q_2\lambda + p_2 = 0. \quad (2.4)$$

An eigenvector v_j of A corresponding to λ_j is

$$v_j = \left(1, \lambda_j, p_0\lambda_j^2 + \frac{1}{2}q_1\lambda_j, -\frac{1}{2}q_2 - p_2\lambda_j^{-1} \right)^t, \quad (2.5)$$

where the superscript t denotes the transpose. We assume at this stage that the λ_j are distinct, and we define the matrix T in (2.3) by

$$T = (v_1 \quad v_2 \quad v_3 \quad v_4). \quad (2.6)$$

Now from (2.2), we note that EA is symmetric, where

$$E = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.7)$$

Hence, be [5, Section 2(i)], the v_j have the orthogonality property

$$(Ev_k)^t v_j = 0 \quad (k \neq j). \quad (2.8)$$

We define the scalars m_j ($1 \leq j \leq 4$) by

$$m_j = (Ev_j)^t v_j, \quad (2.9)$$

and the row vectors

$$r_j = (Ev_j)^t. \quad (2.10)$$

Hence, by [5, Section 2],

$$T^{-1} = \begin{bmatrix} m_1^{-1} r_1 \\ m_2^{-1} r_2 \\ m_3^{-1} r_3 \\ m_4^{-1} r_4 \end{bmatrix}, \quad (2.11)$$

$$m_j = 4p_0\lambda_j^3 + 3q_1\lambda^2 + 2p_1\lambda_j + q_2. \quad (2.12)$$

Now we define the matrix U by

$$U = (v_1 \quad v_2 \quad v_3 \quad \epsilon_1 \quad v_4) = TK, \quad (2.13)$$

where

$$\epsilon_1 = \frac{p_0 p_1}{q_1^2}. \quad (2.14)$$

The matrix K is given by

$$K = dg(1, 1, 1, \epsilon_1). \quad (2.15)$$

By (2.3) and (2.13), the transformation

$$Y = UZ \quad (2.16)$$

takes (2.1) into

$$Z' = (\Lambda - U^{-1}U')Z. \quad (2.17)$$

Now by (2.13),

$$U^{-1}U' = K^{-1}T^{-1}T'K + K^{-1}K', \quad (2.18)$$

where

$$K^{-1}K' = dg(0, 0, 0, \epsilon_1^{-1}\epsilon'_1), \quad (2.19)$$

and we use (2.15).

Now if we write

$$U^{-1}U' = \phi_{ij} \quad (1 \leq i, j \leq 4), \quad (2.20)$$

$$T^{-1}T' = \psi_{ij} \quad (1 \leq i, j \leq 4), \quad (2.21)$$

then by (2.18)–(2.21), we have

$$\begin{aligned}\phi_{ij} &= \psi_{ij} \quad (1 \leq i, j \leq 3), \\ \phi_{44} &= \psi_{44} + \epsilon_1^{-1} \epsilon'_1, \\ \phi_{i4} &= \psi_{i4} \epsilon_1 \quad (1 \leq i \leq 3), \\ \phi_{4j} &= \epsilon_1^{-1} \psi_{4j} \quad (1 \leq j \leq 3).\end{aligned}\tag{2.22}$$

Now to work out ϕ_{ij} ($1 \leq i, j \leq 4$), it suffices to deal with ψ_{ij} of the matrix $T^{-1}T'$. Thus by (2.10), (2.12), (2.6), and (2.11), we obtain

$$\psi_{ii} = \frac{1}{2} \frac{m'_i}{m_i} \quad (1 \leq i \leq 4),\tag{2.23}$$

and, for $i \neq j$, $1 \leq i, j \leq 4$,

$$\psi_{ij} = m_i^{-1} \left\{ \lambda'_j \left(p_0 \lambda_i^2 + \frac{1}{2} q_1 \lambda_i \right) + \lambda_i \left(p_0 \lambda_j^2 + \frac{1}{2} q_1 \lambda_j \right)' + -\frac{1}{2} q'_2 - (p_2 \lambda_j^{-1})' \right\}.\tag{2.24}$$

Now we need to work out (2.23) and (2.24) in some detail in terms of p_0 , p_1 , p_2 , q_1 , and q_2 , then (2.22) in order to determine the form of (2.17).

3. The matrices Λ , $T^{-1}T'$, and $U^{-1}U'$. In our analysis, we impose a basic condition on the coefficients as follows.

(I) p_i ($0 \leq i \leq 2$) and q_i ($i = 1, 2$) are nowhere zero in some interval $[a, \infty)$, and

$$\begin{aligned}\frac{p_i}{q_{i+1}} &= o\left(\frac{q_{i+1}}{p_{i+1}}\right) \quad (i = 0, 1) \quad (x \rightarrow \infty), \\ \frac{q_1}{p_1} &= o\left(\frac{p_1}{q_2}\right).\end{aligned}\tag{3.1}$$

If we write

$$\epsilon_1 = \frac{p_0 p_1}{q_1^2}, \quad \epsilon_2 = \frac{q_1 q_2}{p_1^2}, \quad \epsilon_3 = \frac{p_2 p_1}{q_2^2},\tag{3.2}$$

then by (3.1) for ($1 \leq i \leq 3$),

$$\epsilon_i = o(1) \quad (x \rightarrow \infty).\tag{3.3}$$

Now as in [1], we can solve the characteristic equation (2.4) asymptotically as $x \rightarrow \infty$. Using (3.1) and (3.2), we obtain the distinct eigenvalues λ_j as

$$\lambda_1 = -\frac{p_2}{q_2} (1 + \delta_1),\tag{3.4}$$

$$\lambda_2 = -\frac{q_2}{p_1} (1 + \delta_2),\tag{3.5}$$

$$\lambda_3 = -\frac{p_1}{q_1} (1 + \delta_3),\tag{3.6}$$

$$\lambda_4 = -\frac{q_1}{p_0} (1 + \delta_4),\tag{3.7}$$

where

$$\delta_1 = 0(\epsilon_3), \quad \delta_2 = 0(\epsilon_2) + 0(\epsilon_3), \quad \delta_3 = 0(\epsilon_1) + 0(\epsilon_2), \quad \delta_4 = 0(\epsilon_1). \quad (3.8)$$

Now by (3.1), the ordering of λ_j is such that

$$\lambda_j = o(\lambda_{j+1}) \quad (x \rightarrow \infty, 1 \leq j \leq 3). \quad (3.9)$$

Now we work out m_j ($1 \leq j \leq 4$) asymptotically as $x \rightarrow \infty$; hence by (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), and (3.8), (2.12) gives, for $1 \leq j \leq 4$,

$$m_1 = q_2 \{1 + 0(\epsilon_3)\}, \quad (3.10)$$

$$m_2 = -q_2 \{1 + 0(\epsilon_2) + 0(\epsilon_3)\}, \quad (3.11)$$

$$m_3 = \frac{p_1^2}{q_1} \{1 + 0(\epsilon_1) + 0(\epsilon_2)\}, \quad (3.12)$$

$$m_4 = -\frac{q_1^3}{p_0^2} \{1 + 0(\epsilon_1)\}. \quad (3.13)$$

Also by substituting λ_j ($j = 1, 2, 3, 4$) into (2.12) and using (3.4), (3.5), (3.6), and (3.7), respectively, and differentiating, we obtain

$$\begin{aligned} m'_1 &= q'_2 \{1 + 0(\epsilon_3)\} + q_2 \{0(\epsilon'_3) + 0(\epsilon_3 \delta'_1) + 0(\epsilon'_2 \epsilon_3^2) + 0(\epsilon'_2 \epsilon_3^2 \epsilon_3^3)\}, \\ m'_2 &= -q'_2 \{1 + 0(\epsilon_2) + 0(\epsilon_3)\} + q_2 \{0(\delta'_2) + 0(\epsilon'_2) + 0(\epsilon'_1 \epsilon_2^2)\}, \\ m'_3 &= \left(\frac{p_1^2}{q_1}\right)' \{1 + 0(\epsilon_1) + 0(\epsilon_2)\} + \frac{p_1^2}{q_1} \{0(\delta'_3) + 0(\epsilon'_2) + 0(\epsilon'_1)\}, \\ m'_4 &= -\left(\frac{q_1^3}{p_0^2}\right)' \{1 + 0(\epsilon_1)\} + \frac{q_1^3}{p_0^2} \{0(\delta'_4) + 0(\epsilon'_2 \epsilon_1^2) + 0(\epsilon'_1)\}. \end{aligned} \quad (3.14)$$

At this stage we also require the following conditions.

(II)

$$\frac{p'_0}{p_0} \epsilon_i, \frac{p'_1}{p_1} \epsilon_i, \frac{q'_1}{q_1} \epsilon_i, \frac{q'_2}{q_2} \epsilon_i, \frac{p'_2}{p_2} \epsilon_2, \frac{p'_2}{p_2} \epsilon_3 \in L(a, x) \quad (1 \leq i \leq 3). \quad (3.15)$$

Further, differentiating (3.2) for ϵ_i ($1 \leq i \leq 3$), we obtain

$$\begin{aligned} \epsilon'_1 &= 0\left(\frac{p'_0}{p_0} \epsilon_1\right) + 0\left(\frac{p'_1}{p_1} \epsilon_1\right) + 0\left(\frac{q_2}{q_2} \epsilon_1\right), \\ \epsilon'_2 &= 0\left(\frac{q'_1}{q_1} \epsilon_2\right) + 0\left(\frac{q'_2}{q_2} \epsilon_2\right) + 0\left(\frac{p'_1}{p_1} \epsilon_2\right), \\ \epsilon'_3 &= 0\left(\frac{p'_2}{p_2} \epsilon_3\right) + 0\left(\frac{p'_1}{p_1} \epsilon_3\right) + 0\left(\frac{q'_2}{q_2} \epsilon_3\right). \end{aligned} \quad (3.16)$$

For reference, we note that by substituting (3.4), (3.5), (3.6), and (3.7) into (2.4) and differentiating, we obtain

$$\begin{aligned}\delta'_1 &= 0(\epsilon'_3) + 0(\epsilon'_2\epsilon_3^2) + 0(\epsilon'_2\epsilon_3^3\epsilon_2^2), \\ \delta'_2 &= 0(\epsilon'_2) + 0(\epsilon'_3) + 0(\epsilon'_1\epsilon_3^2), \\ \delta'_3 &= 0(\epsilon'_1) + 0(\epsilon'_2) + 0(\epsilon'_3\epsilon_2^2), \\ \delta'_4 &= 0(\epsilon'_1) + 0(\epsilon'_2\epsilon_1^2) + 0(\epsilon'_3\epsilon_1^3\epsilon_2^2).\end{aligned}\tag{3.17}$$

Hence by (3.16) and (3.17), and (3.15),

$$\epsilon'_j, \delta'_j \in L(a, \infty).\tag{3.18}$$

For the diagonal elements ψ_{ii} ($1 \leq j \leq 4$) in (2.23), we can now substitute the estimates (3.10), (3.11), (3.12), (3.13), and (3.14) into (2.23). We obtain

$$\begin{aligned}\psi_{11} &= \frac{1}{2} \frac{q'_2}{q_2} + 0\left(\frac{q'_2}{q_2} \epsilon_3\right) + 0(\epsilon'_3) + 0(\epsilon_3 \delta'_1) + 0(\epsilon'_2 \epsilon_3^2) + 0(\epsilon'_1 \epsilon_2^2 \epsilon_3^3), \\ \psi_{22} &= \frac{1}{2} \frac{q'_2}{q_2} + 0\left(\frac{q'_2}{q_2} \epsilon_2\right) + 0\left(\frac{q'_2}{q_2} \epsilon_3\right) + 0(\delta'_2) + 0(\epsilon'_2) + 0(\epsilon'_2 \epsilon_2^2), \\ \psi_{33} &= \frac{1}{2} \left[2 \frac{p'_1}{p_1} - \frac{q'_1}{q_1} \right] + 0\left(\frac{p'_1}{p_1} \epsilon_1\right) + 0\left(\frac{p'_1}{p_1} \epsilon_2\right) + 0\left(\frac{q'_1}{q_1} \epsilon_1\right) + 0\left(\frac{q'_1}{q_1} \epsilon_2\right) \\ &\quad + 0(\delta'_3) + 0(\epsilon'_2) + 0(\epsilon'_1), \\ \psi_{44} &= \frac{1}{2} \left[3 \frac{q'_1}{q_1} - 2 \frac{p'_0}{p_0} \right] + 0\left(\frac{q'_1}{q_1} \epsilon_1\right) + 0\left(\frac{p'_0}{p_0} \epsilon_1\right) + 0(\delta'_4) + 0(\epsilon'_2 \epsilon_1^2) + 0(\epsilon'_1).\end{aligned}\tag{3.19}$$

Now for the nondiagonal elements ψ_{ij} ($i \neq j$, $1 \leq i, j \leq 4$), we consider (2.24). Hence (2.24) gives, for $i = 1$ and $j = 2$,

$$\psi_{12} = m_1^{-1} \left\{ \lambda'_2 \left(p_0 \lambda_1^2 + \frac{1}{2} q_1 \lambda_1 \right) + \lambda_1 \left(p_0 \lambda_2^2 + \frac{1}{2} q_1 \lambda_2 \right)' + -\frac{1}{2} q'_2 - (p_2 \lambda_2^{-1})' \right\}.\tag{3.20}$$

Now by (3.4), (3.5), (3.2), and (3.10), we have

$$m_1^{-1} \lambda'_2 \left(p_0 \lambda_1^2 + \frac{1}{2} q_1 \lambda_1 \right) = \frac{1}{2} \left[2 \frac{q'_2}{q_2} - \frac{p'_1}{p_1} \right] \epsilon_2 \epsilon_3 \{1 + 0(\epsilon_3)\} + 0(\epsilon_2 \epsilon_3 \delta'_2),\tag{3.21}$$

$$\begin{aligned}m_1^{-1} \lambda_1 \left(p_0 \lambda_1^2 + \frac{1}{2} q_1 \lambda_1 \right)' &= 0(\epsilon_2 \epsilon_3 \delta'_2) + 0(\epsilon_2^2 \epsilon_1 \epsilon_3) + \left[\frac{p'_0}{p_0} + 2 \frac{q'_2}{q_2} - 2 \frac{p'_1}{p_1} \right] \\ &\quad + 0(\epsilon_2 \epsilon_3) \left[\frac{q'_1}{q_1} + \frac{q'_2}{q_2} - \frac{p'_1}{p_1} \right],\end{aligned}\tag{3.22}$$

$$-\frac{1}{2} q'_2 m_1^{-1} = -\frac{1}{2} \frac{q'_2}{q_2} + 0\left(\frac{q'_2}{q_2} \epsilon_3\right),\tag{3.23}$$

$$m_1^{-1} (p_2 \lambda_2^{-1})' = 0\left(\frac{p'_2}{p_2} \epsilon_3\right) + 0\left(\frac{p'_1}{p_1} \epsilon_3\right) + 0\left(\frac{q'_2}{q_2} \epsilon_3\right) + 0(\epsilon_3 \delta'_2).\tag{3.24}$$

Hence by (3.21), (3.22), (3.23), and (3.24), (3.20) gives

$$\begin{aligned}\psi_{12} = & -\frac{1}{2}\frac{q'_2}{q_2} + 0\left(\frac{q'_2}{q_2}\epsilon_3\right) + 0\left(\frac{p'_1}{p_1}\epsilon_3\right) + 0\left(\frac{p'_2}{p_2}\epsilon_3\right) + 0\left(\frac{p'_0}{p_0}\epsilon_1\epsilon_2^2\epsilon_3\right) \\ & + 0(\epsilon_3\delta'_2) + 0\left(\frac{q'_1}{q_1}\epsilon_2\epsilon_3\right).\end{aligned}\quad (3.25)$$

Similar work can be done for the other elements ψ_{ij} ; so we obtain

$$\begin{aligned}\psi_{13} = & -\frac{1}{2}\frac{q'_2}{q_2} + 0\left(\frac{q'_2}{q_2}\epsilon_3\right) + 0\left(\frac{p'_1}{p_1}\epsilon_3\right) + 0\left(\frac{q'_1}{q_1}\epsilon_3\right) + 0(\epsilon_3\delta'_3) + 0\left(\frac{p'_0}{p_0}\epsilon_1\epsilon_3\right) + 0\left(\frac{p'_2}{p_2}\epsilon_2\epsilon_3\right), \\ \psi_{14} = & -\frac{1}{2}\frac{q'_2}{q_2} + 0\left(\frac{q'_2}{q_2}\epsilon_3\right) + 0\left(\frac{q'_1}{q_1}\epsilon_1^{-1}\epsilon_3\right) + 0\left(\frac{p'_0}{p_0}\epsilon_1^{-1}\epsilon_3\right) + 0(\epsilon_1^{-1}\epsilon_3\delta'_4) + 0\left(\frac{p'_2}{p_2}\epsilon_1\epsilon_2\epsilon_3\right), \\ \psi_{21} = & -\frac{1}{2}\frac{q'_2}{q_2} + 0\left(\frac{q'_2}{q_2}\epsilon_2\right) + 0\left(\frac{q'_2}{q_2}\epsilon_3\right) + 0(\delta'_1) + 0\left(\epsilon_2\frac{p'_2}{p_2}\right) + 0\left(\epsilon_3\frac{p'_2}{p_2}\right) \\ & + 0\left(\frac{q'_1}{q_1}\epsilon_2\epsilon_3\right) + 0\left(\frac{p'_0}{p_0}\epsilon_1\epsilon_2^2\epsilon_3^2\right), \\ \psi_{23} = & \left[\frac{1}{2}\frac{q'_1}{q_1} - \frac{p'_1}{p_1} + \frac{1}{2}\frac{q'_2}{q_2}\right] + 0\left(\frac{q'_1}{q_1}\epsilon_1\right) + 0\left(\frac{q'_1}{q_1}\epsilon_2\right) + 0\left(\frac{q'_1}{q_1}\epsilon_3\right) + 0\left(\frac{p'_1}{p_1}\epsilon_1\right) \\ & + 0\left(\frac{p'_1}{p_1}\epsilon_2\right) + 0\left(\frac{p'_1}{p_1}\epsilon_3\right) + 0\left(\frac{q'_2}{q_2}\epsilon_2\right) + 0\left(\frac{q'_2}{q_2}\epsilon_3\right) + 0(\delta'_3) + 0\left(\frac{p'_0}{p_0}\epsilon_1\right) + 0\left(\epsilon_2\epsilon_3\frac{p'_2}{p_2}\right), \\ \psi_{24} = & \epsilon_1^{-1}\left[\frac{1}{2}\frac{q'_1}{q_1} + 0\left(\frac{q'_1}{q_1}\epsilon_1\right) + 0\left(\frac{q'_1}{q_1}\epsilon_2\right) + 0\left(\frac{q'_1}{q_1}\epsilon_3\right) + 0\left(\frac{p'_0}{p_0}\epsilon_1\right) + 0\left(\frac{p'_0}{p_0}\epsilon_2\right) \right. \\ & \left. + 0\left(\frac{p'_0}{p_0}\epsilon_3\right) + 0(\delta'_4) + 0\left(\frac{q'_2}{q_2}\epsilon_1\right) + 0\left(\frac{p'_2}{p_2}\epsilon_1^2\epsilon_2\epsilon_3\right)\right], \\ \psi_{31} = & 0\left(\frac{p'_2}{p_2}\epsilon_2\right) + 0\left(\frac{q'_2}{q_2}\epsilon_2\right) + 0(\delta'_1\epsilon_2) + 0\left(\frac{q'_1}{q_1}\epsilon_2\epsilon_3\right) + 0\left(\frac{p'_0}{p_0}\epsilon_1\epsilon_2^2\epsilon_3^2\right), \\ \psi_{32} = & 0\left(\frac{q'_2}{q_2}\epsilon_2\right) + 0\left(\frac{p'_1}{p_1}\epsilon_2\right) + 0(\epsilon_2\delta'_2) + 0\left(\epsilon_1\epsilon_2^2\frac{p'_0}{p_0}\right) + 0\left(\frac{q'_1}{q_1}\epsilon_2\right) + 0\left(\epsilon_2\epsilon_3\frac{p'_2}{p_2}\right), \\ \psi_{34} = & \epsilon_1^{-1}\left[-\frac{1}{2}\frac{q'_1}{q_1} + 0\left(\frac{q'_1}{q_1}\epsilon_1\right) + 0\left(\frac{q'_1}{q_1}\epsilon_2\right) + 0\left(\frac{p'_0}{p_0}\epsilon_1\right) + 0\left(\frac{p'_0}{p_0}\epsilon_2\right) \right. \\ & \left. + 0(\delta'_4) + 0\left(\frac{q'_1}{q_1}\epsilon_1\epsilon_2\right) + 0\left(\frac{p'_2}{p_2}\epsilon_1^2\epsilon_2^2\epsilon_3\right)\right], \\ \psi_{41} = & \epsilon_1\left[0\left(\frac{q'_1}{q_1}\epsilon_2\epsilon_3\right) + 0\left(\frac{q'_2}{q_2}\epsilon_1\epsilon_2\right) + 0\left(\frac{p'_2}{p_2}\epsilon_1\epsilon_2\right) + 0(\delta'_1\epsilon_1\epsilon_2) + 0\left(\frac{p'_0}{p_0}\epsilon_1\epsilon_2^2\epsilon_3^2\right)\right], \\ \psi_{42} = & 0\left(\frac{q'_2}{q_2}\epsilon_1\epsilon_2\right) + 0\left(\frac{p'_0}{p_0}\epsilon_1\epsilon_2\right) + 0(\delta'_2\epsilon_1\epsilon_2) + 0\left(\frac{q'_1}{q_1}\epsilon_1^2\epsilon_2^2\right) + 0\left(\frac{p'_0}{p_0}\epsilon_1^2\epsilon_2^2\right) + 0\left(\frac{p'_2}{p_2}\epsilon_1^2\epsilon_2\epsilon_3\right), \\ \psi_{43} = & \epsilon_1\left[-\frac{1}{2}\frac{q'_1}{q_1} + 0\left(\frac{p'_1}{p_1}\epsilon_1\right) + 0\left(\frac{q'_1}{q_1}\epsilon_1\right) + 0\left(\frac{q'_1}{q_1}\epsilon_2\right) + 0(\delta'_3\epsilon_1) + 0\left(\frac{p'_0}{p_0}\epsilon_1\right) \right. \\ & \left. + 0\left(\frac{p'_2}{p_2}\epsilon_1\epsilon_2^2\epsilon_3\right) + 0\left(\frac{q'_2}{q_2}\epsilon_1\epsilon_2\right)\right].\end{aligned}\quad (3.26)$$

Now we need to work out (2.22) in order to determine the form (2.17). Now by (3.19), and (3.25) and (3.26), (2.22) will give

$$\begin{aligned}
 \phi_{11} &= \frac{1}{2} \frac{q'_2}{q_2} + 0(\Delta_1), & \phi_{22} &= \frac{1}{2} \frac{q'_2}{q_2} + 0(\Delta_2), \\
 \phi_{33} &= \frac{p'_1}{p_1} - \frac{1}{2} \frac{q'_1}{q_1} + 0(\Delta_3), & \phi_{44} &= \frac{p'_1}{p_1} - \frac{1}{2} \frac{q'_1}{q_1} + 0(\Delta_4), \\
 \phi_{12} &= \frac{1}{2} \frac{q'_2}{q_2} + 0(\Delta_5), & \phi_{13} &= -\frac{1}{2} \frac{q'_2}{q_2} + 0(\Delta_6), \\
 \phi_{14} &= 0(\Delta_7), & \phi_{21} &= -\frac{1}{2} \frac{q'_2}{q_2} + 0(\Delta_8), \\
 \phi_{23} &= \frac{1}{2} \left(\frac{q'_1}{q_1} + \frac{q'_2}{q_2} \right) - \frac{p'_1}{p_1} + 0(\Delta_9), & \phi_{24} &= \frac{1}{2} \frac{q'_1}{q_1} + 0(\Delta_{10}), \\
 \phi_{31} &= 0(\Delta_{11}), & \phi_{32} &= 0(\Delta_{12}), & \phi_{34} &= -\frac{1}{2} \frac{q'_1}{q_1} + 0(\Delta_{13}), \\
 \phi_{41} &= 0(\Delta_{14}), & \phi_{42} &= 0(\Delta_{15}), & \phi_{43} &= -\frac{1}{2} \frac{q'_1}{q_1} + 0(\Delta_{16}),
 \end{aligned} \tag{3.27}$$

where

$$\Delta_i \in L(a, \infty) \quad (1 \leq i \leq 16) \tag{3.28}$$

by (3.15) and (3.18).

Now by (3.27) and (3.28), we write the system (2.17) as

$$Z' = (\Lambda + R + S)Z, \tag{3.29}$$

where

$$R = \begin{bmatrix} -\eta_1 & \eta_1 & \eta_1 & 0 \\ \eta_1 & -\eta_1 & \eta_2 - \eta_1 & -\eta_3 \\ 0 & 0 & -\eta_2 & \eta_3 \\ 0 & 0 & \eta_3 & -\eta_2 \end{bmatrix} \tag{3.30}$$

with

$$\eta_1 = \frac{1}{2} \frac{q'_2}{q_2}, \quad \eta_2 = \frac{(p_1 q_1^{-1/2})'}{p_1 q_1^{-1/2}}, \quad \eta_3 = \frac{1}{2} \frac{q'_1}{q_1}, \tag{3.31}$$

and $S \in L(a, \infty)$ by (3.28).

4. The Euler case. Now we deal with (1.3) more generally, so we write (1.3) as

$$\eta_k = \sigma_k \frac{q_1}{p_0} (1 + \varphi_k) \quad (1 \leq k \leq 3), \tag{4.1}$$

where σ_k ($1 \leq k \leq 3$) are nonzero constants, $\varphi_k(x) \rightarrow 0$ ($1 \leq k \leq 3$, $x \rightarrow \infty$), and also at this stage we let

$$\varphi'_k \in L(a, \infty) \quad (1 \leq k \leq 3). \tag{4.2}$$

We note that by (4.1), the matrix Λ no longer dominates R and so Eastham's theorem [6, Section 2] is not satisfied which means that we have to carry out a second diagonalization of the system (3.29).

First we write

$$\Lambda + R = \lambda_4 \{S_1 + S_2\}, \quad (4.3)$$

and we need to work out the matrix $S_1 = \text{const}$ with the matrix $S_2(x) = o(1)$ as $x \rightarrow \infty$ using (3.4), (3.5), (3.6), and (3.7) and the Euler case (4.1). Hence after some calculations, we obtain

$$S_1 = \begin{pmatrix} \sigma_1 & -\sigma_1 & -\sigma_1 & 0 \\ -\sigma_1 & \sigma_1 & \sigma_1 - \sigma_2 & \sigma_3 \\ 0 & 0 & \sigma_2 & -\sigma_3 \\ 0 & 0 & -\sigma_3 & \sigma_2 \end{pmatrix}, \quad (4.4)$$

$$S_2(x) = \begin{pmatrix} u_1 & u_2 & u_2 & 0 \\ u_2 & u_3 & u_4 & u_5 \\ 0 & 0 & u_6 & -u_5 \\ 0 & 0 & -u_5 & -u_7 \end{pmatrix},$$

where

$$\begin{aligned} u_1 &= \lambda_1 \lambda_4^{-1} - u_2, & u_2 &= -\sigma_1 (\varphi_1 - \delta_4) (1 + \delta_4)^{-1}, \\ u_3 &= \lambda_2 \lambda_4^{-1} - u_2, & u_4 &= -u_2 + u_7, \\ u_5 &= \sigma_3 (\varphi_3 - \delta_4) (1 + \delta_4)^{-1}, & u_6 &= \lambda_3 \lambda_4^{-1} - u_7, \\ u_7 &= -\sigma_2 (\varphi_2 - \delta_4) (1 + \delta_4)^{-1}. \end{aligned} \quad (4.5)$$

It is clear that by (3.9) and (3.8), $S_2(x) \rightarrow 0$ as $x \rightarrow \infty$. Hence we diagonalize the constant matrix S_1 . Now the eigenvalues of the matrix S_1 are given by

$$\alpha_1 = 0, \quad \alpha_2 = 2\sigma_1, \quad \alpha_3 = \sigma_2 + \sigma_3, \quad \alpha_4 = \sigma_2 - \sigma_3. \quad (4.6)$$

Let

$$\sigma_2 \neq (\pm \sigma_3, \pm \sigma_3 + 2\sigma_1). \quad (4.7)$$

Hence by (4.7), the eigenvalues α_i ($1 \leq i \leq 4$) are distinct. Thus we use the transformation

$$Z = T_1 W \quad (4.8)$$

in (3.29), where T_1 diagonalizes the constant matrix S_1 . Then (3.29) transforms to

$$W' = (\Lambda_1 + M + T_1^{-1} S T_1) W, \quad (4.9)$$

where

$$\begin{aligned}\Lambda_1 &= \lambda_4 T_1^{-1} S_1 T_1 = \text{diag}(v_1, v_2, v_3, v_4) = \lambda_4 \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \\ M &= \lambda_4 T_1^{-1} S_2 T_1, \\ T_1^{-1} S T_1 &\in L(a, \infty).\end{aligned}\tag{4.10}$$

Now we can apply the asymptotic theorem of Eastham in [6, Section 2] to (4.9) provided only that Λ_1 and M satisfy the conditions in [6, Section 2].

We first require that the v_j ($1 \leq j \leq 4$) are distinct, and this holds because the α_j ($1 \leq j \leq 4$) are distinct.

Second, we need to show that

$$\frac{M}{v_i - v_j} \rightarrow 0 \quad (x \rightarrow \infty)\tag{4.11}$$

for $i \neq j$ and $1 \leq i, j \leq 4$. Now

$$\frac{M}{v_i - v_j} = (\alpha_i - \alpha_j)^{-1} T_1^{-1} S_2 T_1 = o(1) \quad (x \rightarrow \infty).\tag{4.12}$$

Thus (4.11) holds. Third, we need to show that

$$S'_2 \in L(a, \infty).\tag{4.13}$$

Thus it suffices to show that

$$u'_i(x) \in L(a, \infty) \quad (1 \leq i \leq 8).\tag{4.14}$$

Now, by (3.4), (3.5), (3.6), (3.7), and (4.5),

$$\begin{aligned}u'_1 &= 0(\epsilon'_1 \epsilon_2 \epsilon_3) + 0(\epsilon'_2 \epsilon_1 \epsilon_3) + 0(\epsilon'_3 \epsilon_1 \epsilon_2) + 0(\varphi') + 0(\delta'_4), \\ u'_2 &= 0(\varphi') + 0(\delta'_4), \\ u'_3 &= 0(\epsilon'_1 \epsilon_2) + 0(\epsilon'_2 \epsilon_1) + 0(\delta'_2 \epsilon_1 \epsilon_2) + 0(\varphi'_1) + 0(\delta'_4), \\ u'_4 &= 0(\varphi'_1) + 0(\delta'_4) + 0(\varphi'_2), \\ u'_5 &= 0(\varphi'_3) + 0(\delta'_4), \\ u'_6 &= 0(\epsilon'_1) + 0(\epsilon_1 \delta'_3) + 0(\varphi'_2) + 0(\delta'_4), \\ u'_7 &= 0(\varphi'_2) + 0(\delta'_4).\end{aligned}\tag{4.15}$$

Thus by (4.15), (3.18), and (4.2), (4.14) holds and consequently (4.13) holds. Now we state our main theorem for (1.1).

THEOREM 4.1. *Let the coefficients q_1 , q_2 , and p_1 in (1.1) be in $C^{(2)}[a, \infty)$ and let p_0 and p_2 be $C^{(1)}[a, \infty)$. Let (3.1), (3.15), (4.1), (4.2), and (4.7) hold.*

Let

$$\begin{aligned}\text{Re } I_j(x) \quad (j = 1, 2), \\ \text{Re} [\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 - 2\eta_1 + 2\eta_2 \pm I_1 \pm I_2]\end{aligned}\tag{4.16}$$

be one sign in $[a, \infty)$, where

$$\begin{aligned} I_1 &= \left[4\eta_1^2 + (\lambda_1 - \lambda_2)^2 \right]^{1/2}, \\ I_2 &= \left[4\eta_3^2 + (\lambda_3 - \lambda_4)^2 \right]^{1/2}. \end{aligned} \quad (4.17)$$

Then (1.1) has solutions

$$\begin{aligned} y_1 &\sim q_2^{-1/2} \exp\left(\frac{1}{2} \int_a^x [\lambda_1 + \lambda_2 + I_1] dt\right), \\ y_2 &= o\left\{ q_1^{1/2} \exp\left(\frac{1}{2} \int_a^x [\lambda_1 + \lambda_2 - I_1] dt\right) \right\}, \\ y_k &= o\left\{ q_1^{-1/2} p_1^{-1} \exp\left(\frac{1}{2} \int_a^x [\lambda_3 + \lambda_4 + (-1)^{k+1} I_2] dt\right) \right\} \quad (k = 3, 4). \end{aligned} \quad (4.18)$$

PROOF. Before applying the theorem in [6, Section 2], we show that the eigenvalues μ_k of $\Lambda_1 + M$ satisfy the dichotomy condition [8]. As in [1], the dichotomy condition holds if

$$(\mu_j - \mu_k) = f + g \quad (j \neq k, 1 \leq j, k \leq 4), \quad (4.19)$$

where f has one sign in $[a, \infty)$ and g belongs to $L[a, \infty)$ [6, (1.5)]. Now since the eigenvalues of $\Lambda_1 + M$ are the same as the eigenvalues of $\Lambda + R$, hence by (2.3) and (3.23),

$$\begin{aligned} \mu_k &= \frac{1}{2}(\lambda_3 + \lambda_2 - 2\eta_1) + \frac{1}{2}(-1)^{k+1} I_1 \quad (k = 1, 2), \\ \mu_k &= \frac{1}{2}(\lambda_3 + \lambda_4 - 2\eta_2) + \frac{(-1)^{k+1}}{2} I_2 \quad (k = 3, 4). \end{aligned} \quad (4.20)$$

Thus by (4.20) and (4.16), (4.19) holds. Since (4.9) satisfies all the conditions for the asymptotic result [6, Section 2], it follows that, as $x \rightarrow \infty$, (4.9) has four linearly independent solutions

$$W_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right) \quad (4.21)$$

with e_k being the coordinate vector with k th component unity and other components being zero. Now we transform back to Y by means of (2.16) and (4.8), where T_1 in (4.8) is given by

$$T_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \quad (4.22)$$

We obtain

$$Y_k(x) = UT_1 W_k(x) \quad (1 \leq k \leq 4). \quad (4.23)$$

Now using (2.13), (2.14), (2.15), (4.20), (4.21), (4.22), and (3.31) in (4.23) and carrying out the integration of $(1/2)(q'_2/q)$ and $(q_1^{1/2} p_1^{-1})'/q_1^{1/2} p_1^{-1}$, for $1 \leq k \leq 4$, we obtain (4.18). $\square$

5. Discussion. (i) In the familiar case, the coefficients which are covered by [Theorem 4.1](#) are $p_i(x) = c_i x^{\alpha_i}$ ($i = 0, 1, 2$) and $q_i(x) = c_{i+2} x^{\alpha_{i+2}}$ ($i = 1, 2$) with real constants α_i and c_i ($0 \leq i \leq 4$). Then the Euler case ([4.2](#)) is given by

$$\alpha_0 - \alpha_3 = 1. \quad (5.1)$$

The values of σ_k ($1 \leq k \leq 3$) in ([4.1](#)) are given by

$$\sigma_1 = \frac{1}{2} \alpha_4 c_0 c_3^{-1}, \quad \sigma_2 = \left(\alpha_1 - \frac{1}{2} \alpha_3 \right) c_0 c_3^{-1}, \quad \sigma_3 = \frac{1}{2} \alpha_3 c_0 c_3^{-1}. \quad (5.2)$$

Also in this example, $\varphi_k(x) = 0$ in ([4.1](#)).

(ii) Also the theorem covered the class of the coefficients

$$\begin{aligned} p_0 &= c_0 x^{\alpha_0} e^{x^b}, & p_1 &= c_1 x^{\alpha_1} e^{(1/4)x^b}, & p_2 &= c_2 x^{\alpha_2} e^{-3x^b}, \\ q_1 &= c_3 x^{\alpha_3} e^{x^b}, & q_2 &= c_4 x^{\alpha_4} e^{-x^b} \end{aligned} \quad (5.3)$$

with real constants c_i , α_i ($0 \leq i \leq 4$) and $b(> 0)$.

The Euler case ([4.1](#)) is given by

$$\alpha_3 - \alpha_0 = b - 1. \quad (5.4)$$

The values of σ_k ($1 \leq k \leq 4$) in ([4.1](#)) are given by

$$\sigma_1 = \frac{1}{2} b c_0 c_3^{-1}, \quad \sigma_2 = \frac{1}{2} \sigma_1, \quad \sigma_3 = -\sigma_1. \quad (5.5)$$

Also

$$\varphi_1 = -\alpha_4 b^{-1} x^{-b}, \quad \varphi_2 = 4b^{-1} \left(\frac{1}{2} \alpha_3 - \alpha_1 \right) x^{-b}, \quad \varphi_3 = b^{-1} \alpha_3 x^{-b}. \quad (5.6)$$

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