STABILITY OF MULTIPLIERS ON BANACH ALGEBRAS

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Suppose $A$ is a Banach algebra without order. We show that an approximate multiplier $T : A \rightarrow A$ is an exact multiplier. We also consider an approximate multiplier $T$ on a Banach algebra which need not be without order. If, in addition, $T$ is approximately additive, then we prove the Hyers-Ulam-Rassias stability of $T$.

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1. Introduction and statement of results. It seems that the stability problem of functional equations had been first raised by Ulam (cf. [5, Chapter VI] and [6]): for what metric groups $G$ is it true that a $\varepsilon$-automorphism of $G$ is necessarily near to a strict automorphism?

An answer to the above problem has been given as follows. Suppose $E_1$ and $E_2$ are two real Banach spaces and $f : E_1 \rightarrow E_2$ is a mapping. If there exist $\delta \geq 0$ and $p \geq 0, p \neq 1$, such that

$$||f(x + y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$, then there is a unique additive mapping $T : E_1 \rightarrow E_2$ such that $||f(x) - T(x)|| \leq 2\varepsilon||x||^p / |2 - 2^p|$ for every $x \in E_1$. This result is called the Hyers-Ulam-Rassias stability of the additive Cauchy equation $g(x + y) = g(x) + g(y)$. Indeed, Hyers [2] obtained the result for $p = 0$. Then Rassias [3] generalized the above result of Hyers to the case where $0 \leq p < 1$. Gajda [1] solved the problem for $1 < p$, which was raised by Rassias. In the same paper, Gajda also gave an example that a similar result does not hold for $p = 1$. We can also find another example in [4]. If $p < 0$, then $||x||^p$ is meaningless for $x = 0$. In this case, if we assume that $||0||^p$ means $\infty$, then the proof given in [3] shows the existence of a mapping $T : E_1 \setminus \{0\} \rightarrow E_2$ such that $||f(x) - T(x)|| \leq 2\varepsilon||x||^p / |2 - 2^p|$ for every $x \in E_1 \setminus \{0\}$. Moreover, if we define $T(0) = 0$, then we see that the extended mapping, denoted by the same letter $T$, is additive. The last inequality is valid for $x = 0$ since we assume $||0||^p = \infty$. Thus, the Hyers-Ulam-Rassias stability holds for $p \in \mathbb{R} \setminus \{1\}$, where $\mathbb{R}$ denotes the real number field.

Suppose $A$ is a Banach algebra. We say that a mapping $T : A \rightarrow A$ is a multiplier if $a(Tb) = (Ta)b$ for all $a, b \in A$. Recall that a Banach algebra $A$ is not without order if there exist $x_0, y_0 \in A \setminus \{0\}$ such that $x_0A = Ay_0 = \{0\}$. Therefore, $A$ is without order if and only if for all $x \in A, xA = \{0\}$ implies $x = 0$, or, for all $x \in A, Ax = \{0\}$ implies $x = 0$. We first prove the superstability of multipliers on a Banach algebra without order; that is, each approximate multiplier is an exact multiplier.
We first show that $T$ is homogeneous, that is, $T(\lambda a) = \lambda T a$ for all $\lambda \in \mathbb{C}$ and $a \in A$. To do this, pick $\lambda \in \mathbb{C}$, $a \in A$ and fix $x \in A$ arbitrarily. Put $s = (1 - p)/|1 - p|$. For each $n \in \mathbb{N}$, it follows from (1.2) that

\begin{align*}
||n^s x [T(\lambda a) - \lambda T a]| &\leq ||n^s x[T(\lambda a) - T(n^s x)](\lambda a)|| \\
&+ ||[T(n^s x)](\lambda a) - n^s x(\lambda T a)|| \\
&\leq \varepsilon ||n^s x||^p ||\lambda a||^p + |\lambda| \varepsilon ||n^s x||^p ||a||^p \\
&\leq n^s p \varepsilon (|\lambda|^p + |\lambda|) ||x||^p ||a||^p,
\end{align*}

for some $\varepsilon > 0$, $p \geq 0$, $p \neq 1$, and $a \in A$. Therefore, $T$ is a multiplier.
and hence
\[ \| x [ T(\lambda a) - \lambda Ta ] \| \leq n^{s(p-1)} \varepsilon (|\lambda|^p + |\lambda|) \| x \|^p \| a \|^p \] (2.2)
for all \( n \in \mathbb{N} \). Since \( s(p-1) < 0 \), we obtain by letting \( n \to \infty \) in (2.2) that \( x [ T(\lambda a) - \lambda Ta ] = 0 \). Similarly to the argument above, we can also get \( [ T(\lambda a) - \lambda Ta ] x = 0 \). Since \( A \) is without order, we conclude that \( T(\lambda a) = \lambda Ta \), which implies the homogeneity of \( T \).

Now we are ready to prove that \( T \) is a multiplier. Since \( T \) is homogeneous, \( T(a) = n^{-s} T(n^s a) \) for all \( n \in \mathbb{N} \). Recall that, by definition, \( s(p-1) < 0 \). We thus obtain for all \( a,b \in A \),
\[ \| a(Tb) - (Ta)b \| = n^{-s} \| n^s a(Tb) - T(n^s a)b \| \leq n^{-s} \varepsilon \| n^s a \|^p \| b \|^p = n^{s(p-1)} \varepsilon \| a \|^p \| b \|^p \] (2.3)
\[ \to 0 \quad \text{as} \quad n \to \infty. \]
Hence \( a(Tb) = (Ta)b \), proving \( T \) is a multiplier. \( \square \)

**Proof of Theorem 1.2.** Since \( T(0) = 0 \), it suffices to show that \( a(Tb) = (Ta)b \) for all \( a,b \in A \setminus \{0\} \). So, fix \( a,b \in A \setminus \{0\} \) arbitrarily. In this case, inequalities (2.1) and (2.2) are also valid for \( p < 0 \). Recall that we assume \( \|0\|^p = \infty \), and hence
\[ x [ T(\lambda a) - \lambda Ta ] = 0, \quad [ T(\lambda a) - \lambda Ta ] x = 0, \] (2.4)
for \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( x \in A \setminus \{0\} \). Note that (2.4) is also true for \( x = 0 \). Since \( A \) is without order, we thus obtain \( T(\lambda a) = \lambda Ta \) for all \( \lambda \in \mathbb{C} \setminus \{0\} \). An argument similar to (2.3) shows \( a(Tb) = (Ta)b \), and the proof is complete. \( \square \)

**Remark 2.1.** A result similar to Theorem 1.1 need not be true for \( p = 1 \), that is, there exists an approximate multiplier which is not an exact multiplier. More explicitly, to each \( \varepsilon > 0 \) there corresponds a function \( f : \mathbb{C} \to \mathbb{C} \) which is not a multiplier such that
\[ |z_1 f(z_2) - f(z_1)z_2| \leq \varepsilon |z_1| |z_2| \] (2.5)
for all \( z_1,z_2 \in \mathbb{C} \). Fix \( \varepsilon > 0 \) arbitrarily. By the continuity of the function \( t \mapsto e^{it} \), there corresponds a \( \delta \) with \( 0 < \delta < 1 \) such that \( |t| < 2\pi(1 - \delta) \) implies \( |e^{it} - 1| < \varepsilon \). With this \( \delta \), we define the mapping \( f : \mathbb{C} \to \mathbb{C} \) by
\[ f(z) = \begin{cases} 0 & \text{if } z = 0, \\ |z| e^{i\delta \theta} & \text{if } z \in \mathbb{C} \setminus \{0\}, \end{cases} \] (2.6)
where \( \theta \in [0,2\pi) \) denotes the argument of \( z \). Then we see that \( f \) satisfies inequality (2.5) for all \( z_1,z_2 \in \mathbb{C} \). Since the case where \( z_1 = 0 \) or \( z_2 = 0 \) is trivial, we only consider \( z_1,z_2 \in \mathbb{C} \setminus \{0\} \). If \( z_j = |z_j| e^{i\theta_j} \) for \( j = 1,2 \), then we get
\[ |z_1 f(z_2) - f(z_1)z_2| = |z_1| |z_2| |e^{i(1-\delta)(\theta_1 - \theta_2)} - 1|. \] (2.7)
Note that \( |\theta_1 - \theta_2| < 2\pi \). By the definition of \( \delta \), we obtain (2.5), which implies that \( f \) is an approximate multiplier. Moreover, \( f \) is not an exact multiplier, and hence Theorem 1.1 does not hold for \( p = 1 \) in general.
Remark 2.2. Suppose $A$ is a unital commutative Banach algebra. If $f : A \to A$ is a mapping such that
\[
\|af(b) - f(a)b\| \leq \varepsilon \|a\|\|b\| \quad (a, b \in A)
\] (2.8)
for some $\varepsilon \geq 0$, then there is an exact multiplier $T : A \to A$ such that
\[
\|f(a) - Ta\| \leq \varepsilon \|a\| \quad (a \in A).
\] (2.9)
Indeed, let $e \in A$ be a unit element. Taking $b = e$ in (2.8), we obtain
\[
\|af(e) - f(a)\| \leq \varepsilon \|a\| \quad (a \in A).
\] (2.10)
If we consider the mapping $T : A \to A$ defined by
\[
Ta = af(e) \quad (a \in A),
\] (2.11)
then $T$ is a multiplier such that $\|f(a) - Ta\| \leq \varepsilon \|a\|$ for all $a \in A$.

Proof of Theorem 1.3. Suppose $p \neq 1$. By (1.3), it follows from a theorem of Rassias [3] and Gajda [1] that there exists a unique additive mapping $T : A \to A$ such that (1.5) holds. So, we need to show that $a(Tb) = (Ta)b$ for all $a, b \in A$. Since $T$ is additive, $T(0) = 0$, and hence it is enough to consider $a, b \in A \setminus \{0\}$. Put $s = (1 - p)/|1 - p|$ and fix $a, b \in A \setminus \{0\}$ arbitrarily. Since $T$ is additive, we see that $Ta = n^{-s}T(n^s a)$ for each $n \in \mathbb{N}$. Now it follows from (1.5) that
\[
\|n^{-s}f(n^s b) - Tb\| \leq n^{-s} \frac{2\varepsilon}{|2 - 2^p|} \|n^s b\|^p = n^{s(p-1)} \frac{2\varepsilon}{|2 - 2^p|} \|b\|^p
\] (2.12)
for all $n \in \mathbb{N}$, and hence
\[
\|n^{-s}f(n^s b) - Tb\| \to 0 \quad \text{as } n \to \infty.
\] (2.13)
Since $f$ is an approximate multiplier, we get
\[
\|n^{-s}af(n^s b) - f(a)b\| = n^{-s} |af(n^s b) - f(a)n^s b| \\
\leq n^{-s} \varepsilon \|a\|^p \|n^s b\|^p \\
= n^{s(p-1)} \varepsilon \|a\|^p \|b\|^p
\] (2.14)
for all $n \in \mathbb{N}$. Hence,
\[
\|n^{-s}af(n^s b) - f(a)b\| \to 0 \quad \text{as } n \to \infty.
\] (2.15)
Now it follows from (2.13) and (2.15) that
\[
\|a(Tb) - (Ta)b\| \\
\leq \|a\|\|Tb - n^{-s}f(n^s b)\| + \|n^{-s}af(n^s b) - f(a)b\| + \|f(a)b - (Ta)b\| \\
\to \|f(a)b - (Ta)b\| \quad \text{as } n \to \infty.
\] (2.16)
By (1.5), we obtain
\[ \|a(Tb) - (Ta)b\| \leq \frac{2\varepsilon}{|2 - 2^p|} \|a\|_p \|b\|. \quad (2.17) \]

An argument similar to (2.3) implies \( \|a(Tb) - (Ta)b\| = 0 \), proving \( T \) is a multiplier.

\[ \square \]

REFERENCES


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