THE STRUCTURE OF A SUBCLASS OF AMENABLE BANACH ALGEBRAS

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We give sufficient conditions that allow contractible (resp., reflexive amenable) Banach algebras to be finite-dimensional and semisimple algebras. Moreover, we show that any contractible (resp., reflexive amenable) Banach algebra in which every maximal left ideal has a Banach space complement is indeed a direct sum of finitely many full matrix algebras. Finally, we characterize Hermitian $^*$-algebras that are contractible.

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1. Introduction. The purpose of this note is to establish the structure of some class of amenable Banach algebras. Let $\mathcal{A}$ be a Banach algebra over the complex field $\mathbb{C}$. We define a Banach left $\mathcal{A}$-module $\mathcal{X}$ to be a Banach space which is also a unital left $\mathcal{A}$-module such that the linear map $\mathcal{A} \times \mathcal{X} \to \mathcal{X}$, $(a,x) \to ax$, is continuous. Right modules are defined analogously. A Banach $\mathcal{A}$-bimodule is a Banach space with a structural $\mathcal{A}$-bimodule such that the linear map $\mathcal{A} \times \mathcal{X} \times \mathcal{A} \to \mathcal{X}$, $(a \times x \times b) \to axb$, is jointly continuous, where $\mathcal{A} \times \mathcal{X} \times \mathcal{A}$ carries the Cartesian product topology. A submodule $\mathcal{Y}$ of a Banach $\langle$ left, right, bi-$\rangle$ $\mathcal{A}$-module $\mathcal{X}$ is a closed subspace of $\mathcal{X}$ with the structural Banach $\langle$ left, right, or bi-$\rangle$ $\mathcal{A}$-module. A Banach left $\mathcal{A}$-module morphism $\theta: \mathcal{X} \to \mathcal{Y}$ is a continuous linear map between two left Banach $\mathcal{A}$-modules such that $\theta(ax) = a\theta(x)$ for all $a \in \mathcal{A}$ and all $x \in \mathcal{X}$. A Banach right $\mathcal{A}$-module morphism and a Banach $\mathcal{A}$-bimodule morphism are defined analogously. For each Banach $\langle$ left, bi-$\rangle$ module $\mathcal{X}$ on $\mathcal{A}$, the dual $\mathcal{X}^*$ is naturally a Banach $\langle$ left, bi-$\rangle$ $\mathcal{A}$-bimodule with the module actions defined by $\langle aT(x) = T(xa) \rangle$, $aT(x) = T(xa)$, and $Ta(x) = T(ax)$, for all $a \in \mathcal{A}$, $T \in \mathcal{X}^*$, and $x \in \mathcal{X}$, where $T(x)$ denotes the evaluation of $T$ at $x$. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are Banach $\langle$ left, or bi-$\rangle$ $\mathcal{A}$-modules and $\theta: \mathcal{X} \rightarrow \mathcal{Y}$, $\beta: \mathcal{Y} \rightarrow \mathcal{Z}$ are $\langle$ left, bi-$\rangle$ module morphisms, then the sequence

$$
\Sigma: 0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0 \tag{1.1}
$$

is exact if $\theta$ is one-to-one, $\mathcal{Y} \beta = \mathcal{Z}$, and $\mathcal{Y} \theta = \text{ker } \beta$. The exact sequence $\Sigma$ is admissible if $\beta$ has a continuous right inverse, equivalently, $\text{ker } \beta$ has a Banach space complement in $\mathcal{Y}$. The admissible exact sequence splits if the right inverse of $\beta$ is Banach $\langle$ left, bi-$\rangle$ module, equivalently, $\text{ker } \beta$ is a Banach space complement in $\mathcal{Y}$ which is an $\mathcal{A}$-submodule.

A derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is a linear operator $D: \mathcal{A} \rightarrow \mathcal{X}$ which satisfies $D(ab) = D(a)b + aD(b)$, for all $a, b \in \mathcal{A}$. Recall that for any $x \in \mathcal{X}$, the mapping $\delta_x: \mathcal{A} \rightarrow \mathcal{X}$ defined by $\delta_x(a) = ax - xa$, $a \in \mathcal{A}$, is a continuous derivation,
called an inner derivation. A Banach algebra $\mathcal{A}$ is said to be contractible if for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$, each continuous derivation from $\mathcal{A}$ into $\mathcal{X}$ is inner. We say that $\mathcal{A}$ is amenable whenever every continuous derivation from $\mathcal{A}$ into $\mathcal{X}^*$ is inner for each Banach $\mathcal{A}$-bimodule $\mathcal{X}$. Obviously, every contractible Banach algebra is an amenable Banach algebra and the converse is true in the finite-dimension case. It is well known that a finite-dimensional algebra is semisimple if and only if it is isomorphic to a finite Cartesian product of a family of full matrix algebras. Using Theorem 2.1, it is easy to check that a finite Cartesian product of a family of full matrix algebras is contractible.

The purpose of this note is to contribute to the study of the following questions, raised, respectively, in [2], [3, page 817], and [5, page 212].

**Question 1.1.** Is every contractible Banach algebra semisimple?

**Question 1.2.** Is every reflexive amenable Banach algebra finite-dimensional and semisimple?

**Question 1.3.** Is every contractible Banach algebra finite-dimensional?

Recall that a Banach algebra is called a reflexive Banach algebra if it is reflexive as a Banach space. In this note, we will present two situations in which a contractible Banach algebra is finite-dimensional. First, we will give a partial answer to the above questions, where we assume that each maximal left ideal is complemented as a Banach space. This result improves [5, Proposition IV.4.3] for contractible Banach algebras and [3, Corollary 2.3] for reflexive amenable Banach algebras, where the authors suppose only that all of their primitive ideals have finite codimensions. Second, we will show that a Hermitian Banach $^*$-algebra is contractible if and only if it is a finite-dimensional semisimple algebra.

2. Preliminaries. In this section, we recall some facts about the structure of contractible and amenable Banach algebras. Let $\mathcal{A}$ be a Banach algebra over the complex field $\mathbb{C}$ and let $\mathcal{A}^{**}$ be the bidual of $\mathcal{A}$ with the usual multiplication defined by $\psi \cdot \phi(f) = \psi(f)\phi(f)$ for all $\psi, \phi \in \mathcal{A}^{**}$ and $f \in \mathcal{A}$. Consider on $\mathcal{A}^{**}$ the Banach $\mathcal{A}$-bimodule structure defined by $aT = \eta(a)T, Ta = T\eta(a)$ with $\eta : \mathcal{A} \to \mathcal{A}^{**}$ the canonical map. Notice that if a Banach algebra $\mathcal{A}$ has a bounded approximate identity, then its bidual $\mathcal{A}^{**}$ has an identity. It is a fact that a contractible Banach algebra has an identity and an amenable Banach algebra admits bounded right, left, bilateral approximate identities. Of course, a reflexive amenable Banach algebra must be unital. We denote the identity element of $\mathcal{A}$ by 1 and we write $\mathcal{A} \hat{\otimes} \mathcal{A}$ for the completed projective tensorial product (see [4]). The Banach space $\mathcal{A} \hat{\otimes} \mathcal{A}$ is a Banach $\mathcal{A}$-bimodule if we define

$$a(b \otimes c) = ab \otimes c, \quad (b \otimes c)a = b \otimes ca, \quad a, b, c \in \mathcal{A}. \quad (2.1)$$

For a unital Banach algebra $\mathcal{A}$, a diagonal of $\mathcal{A}$ is an element $d \in \mathcal{A} \hat{\otimes} \mathcal{A}$ such that $ad = da$, for all $a \in \mathcal{A}$, and $\pi(d) = 1$, where $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}$ is the canonical Banach $\mathcal{A}$-bimodule morphism. For such a Banach algebra $\mathcal{A}$, a virtual diagonal of $\mathcal{A}$ is an element
$d \in (A \otimes A)^{**}$ such that
\[ ad = da, \quad \forall a \in A, \quad \pi^{**}(d) = 1, \] (2.2)
where $\pi^{**} : (A \otimes A)^{**} \to A^{**}$ is the bidual Banach $A$-module morphism of $\pi$. In the following theorems, we present characterizations of contractible (resp., amenable) Banach algebras. We recall, respectively, [1, Theorem 6.1] and [6, Theorem 1.3].

**Theorem 2.1.** Let $A$ be a Banach algebra. The following are equivalent:
1. $A$ is contractible;
2. $A$ has a diagonal.

**Theorem 2.2.** Let $A$ be a Banach algebra. The following are equivalent:
1. $A$ is amenable;
2. $A$ has a virtual diagonal.

We choose as a basis of the algebra $M_n(C)$ of all $n \times n$ complex matrices the set of elementary matrices $e_{ij}$. Consider $d = \sum_{i,j} \delta_{ij} e_{ij} \otimes e_{ji} \in M_n(C) \otimes M_n(C)$. Then $Md = dM$, for all $M \in M_n(C)$, and $\pi (d) = 1$, where $\pi : M_n(C) \otimes M_n(C) \to M_n(C)$ is the canonical morphism. It follows that $M_n(C)$ is contractible.

Next, the following propositions hold.

**Proposition 2.3.** Let $A$ be a (contractible, amenable) Banach algebra. Then, if $\theta : A \to B$ is a continuous homomorphism from $A$ into another Banach algebra $B$ with dense range, then $B$ is (contractible, amenable). In particular, if $A$ is a closed two-sided ideal of a (contractible, amenable) Banach algebra $A$, then $A/\mathfrak{I}$ is (contractible, amenable) too.

**Proof.** Assume that $A$ is contractible. Let $X$ be a Banach $B$-bimodule. Consider on $X$ the structure of $A$-bimodule defined by $a \cdot x = \theta(a)x$ and $x \cdot a = x\theta(a)$. Since $\theta$ is continuous, $X$ is a Banach $A$-bimodule. Now, let $D : B \to X$ be a continuous derivation. It is easy to see that $D \circ \theta$ is a continuous derivation from $A$ to the Banach $A$-bimodule $X$, and thus it is inner. Therefore, there exists $x \in X$ such that $D(\theta(a)) = a \cdot x - x \cdot a = \theta(a)\Delta - x\theta(a)$ for all $a \in A$. Since $\Delta(A)$ is dense in $B$, we have $D(b) = bx - xb$ for all $b \in B$. It follows that $D$ is inner and $B$ is contractible. If $A$ is amenable, we will consider a continuous derivation $D : B \to \mathcal{X}^*$ from $B$ to the dual of the bimodule $X$ and we use the same way to prove that $B$ is amenable.

**Proposition 2.4** [1, Theorems 2.3 and 2.5]. Let $A$ be an amenable Banach algebra and let
\[ \Sigma : 0 \to \mathcal{X}^* \to \mathfrak{Y} \to \mathfrak{X} \to 0 \] (2.3)
be an admissible short exact sequence of Banach (left, right, or bi-) modules with $\mathcal{X}^*$ a dual of $\mathcal{X}$. Then $\Sigma$ splits.

**Proposition 2.5** [1, Theorem 6.1]. Let $A$ be a contractible Banach algebra and let
\[ \Sigma : 0 \to \mathcal{X} \to \mathfrak{Y} \to \mathfrak{X} \to 0 \] (2.4)
be an admissible short exact sequence of Banach (left, right, or bi-) modules. Then $\Sigma$ splits.
Remark 2.6. Notice that for each closed two-sided ideal \( \mathcal{I} \) of a reflexive Banach algebra, \( \mathcal{I} \) and the quotient \( \mathcal{A}/\mathcal{I} \) are reflexive Banach algebras too.

Proposition 2.7. Let \( \mathcal{A} \) be a contractible or reflexive amenable Banach algebra and assume that \( \mathcal{I} \) is a closed (left, two-sided) ideal of \( \mathcal{A} \) which has a Banach space complement. Then there exists a closed (left, two-sided) ideal \( \mathcal{J} \) of \( \mathcal{A} \) such that

\[
\mathcal{A} = \mathcal{I} + \mathcal{J}.
\] (2.5)

Proof. Let \( \mathcal{A} \) be an amenable Banach algebra and let \( \mathcal{I} \) be a closed (left, two-sided) ideal of \( \mathcal{A} \) which has a Banach space complement. Then the short exact sequence \( \Sigma : 0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{A}/\mathcal{I} \to 0 \) is admissible. If \( \mathcal{A} \) is reflexive, then the space \( \mathcal{I} \) will be the same, and so it will be the dual of the Banach (left, bi-) \( \mathcal{A} \)-module \( \mathcal{I}^* \). By Proposition 2.4, \( \Sigma \) splits and \( \mathcal{I} \) has a Banach space complement which is a (left, two-sided) ideal. When \( \mathcal{A} \) is contractible, by Proposition 2.5, we have the result. \( \square \)

3. Main results

Theorem 3.1. Let \( \mathcal{A} \) be a contractible or reflexive amenable Banach algebra. Assume that each maximal left ideal of \( \mathcal{A} \) is complemented as a Banach space in \( \mathcal{A} \). Then there are \( n_1, n_2, \ldots, n_k \in \mathbb{N} \) such that

\[
\mathcal{A} \cong \mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_2}(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}_{n_k}(\mathbb{C}).
\] (3.1)

Proof. By Section 2, the algebra \( \mathcal{A} \) has an identity \( 1_{\mathcal{A}} \). Let \( (\mathcal{M}_i)_{i \in I} \) be the family of all maximal left ideals. Since \( \mathcal{M}_i \) is complemented as a Banach space for each \( i \), there exists a left ideal \( \mathcal{J}_i \) such that \( \mathcal{A} = \mathcal{M}_i \oplus \mathcal{J}_i \). Notice that

\[
\text{Rad}(\mathcal{A}) = \bigcap_i \mathcal{M}_i
\] (3.2)

is the Jacobson radical of \( \mathcal{A} \) and

\[
\bigoplus_i \mathcal{J}_i \subseteq \text{Soc}(\mathcal{A}),
\] (3.3)

where \( \text{Soc}(\mathcal{A}) \) is the socle of the algebra \( \mathcal{A} \), that is, it is the sum of all minimal left ideals of \( \mathcal{A} \) and it coincides with the sum of all minimal right ideals of \( \mathcal{A} \). Recall that every minimal left ideal of \( \mathcal{A} \) is of the form \( e\mathcal{A}e \), where \( e \) is a minimal idempotent, that is, \( e^2 = e \neq 0 \) and \( e\mathcal{A}e = \mathbb{C}e \). On the other hand, for each finite family of minimal idempotents \( (e_k)_{k \in K} \), we have

\[
\mathcal{A} = \bigoplus_{k \in K} \mathcal{A}e_k \bigoplus \bigcap_{k \in K} \mathcal{A}(1_{\mathcal{A}} - e_k).
\] (3.4)

It follows from (3.3) and (3.4) that \( \text{Soc}(\mathcal{A}) \) is dense in \( \mathcal{A}/\text{Rad}(\mathcal{A}) \). This shows that \( \mathcal{A}/\text{Rad}(\mathcal{A}) \) is finite-dimensional. Therefore

\[
\mathcal{A} = \text{Rad}(\mathcal{A}) \bigoplus \text{Soc}(\mathcal{A}).
\] (3.5)
If $\text{Rad}(\mathcal{A}) \neq \{0\}$, this would mean that $\text{Rad}(\mathcal{A})$ has an identity, which is impossible. So, $\mathcal{A} = \text{Soc}(\mathcal{A})$, and then it is a finite direct sum of certain full matrix algebras.

**Corollary 3.2.** Every commutative \langle contractible, reflexive amenable \rangle Banach algebra $\mathcal{A}$ is finite-dimensional and semisimple.

**Corollary 3.3.** Let $\mathcal{A}$ be a contractible or reflexive amenable Banach algebra such that every irreducible representation of $\mathcal{A}$ is finite-dimensional. Then $\mathcal{A}$ is finite-dimensional and semisimple.

**Proof.** It is easy to check that every primitive ideal of a Banach algebra is finite-codimensional if and only if each of its maximal left ideals is finite-codimensional. So, the corollary follows.

It should be emphasized that the following result appears in [9] or [5, Corollary in page 212].

**Corollary 3.4.** Every \langle contractible, reflexive amenable \rangle $C^*$-algebra $\mathcal{A}$ is finite-dimensional and semisimple.

**Proof.** Suppose that $\mathcal{A}$ is a contractible or reflexive amenable $C^*$-algebra. Let $\mathcal{M}$ be a maximal left ideal. By [7, Theorems 5.3.5 and 5.2.4], the space $\mathcal{A}/\mathcal{M}$ is a Hilbert space. It follows that the short exact sequence

$$\Sigma : 0 \rightarrow \mathcal{M} \rightarrow \mathcal{A} \rightarrow \mathcal{A}/\mathcal{M} \rightarrow 0 \quad (3.6)$$

is admissible, and thus $\mathcal{M}$ has a Banach space complement. By Theorem 3.1, $\mathcal{A}$ is isomorphic to a finite direct sum of full matrix algebras.

**Remark 3.5.** Recall that a simple algebra is an algebra which has no proper ideals other than the zero ideal. To show that every \langle contractible, reflexive amenable \rangle Banach algebra is finite-dimensional and semisimple, it suffices to prove that every \langle contractible, reflexive amenable \rangle simple contractible Banach algebra is finite-dimensional. Indeed, let $\mathcal{A}$ be a contractible Banach algebra. Let $\mathcal{P}$ be a primitive ideal of $\mathcal{A}$. Then the algebra $\mathcal{A}/\mathcal{P}$ is a \langle contractible, reflexive amenable \rangle Banach algebra. Put $\mathcal{B} = \mathcal{A}/\mathcal{P}$ and consider some maximal two-sided ideal $\mathcal{M}$ of $\mathcal{B}$. Since $\mathcal{B}/\mathcal{M}$ is a \langle contractible, reflexive amenable \rangle simple Banach algebra, it is finite-dimensional. There exists then a closed two-sided ideal $\mathcal{J}$ such that $\mathcal{B} = \mathcal{M} \oplus \mathcal{J}$. Recall that in a primitive algebra, every nonzero ideal is essential, that is, it has a nonzero intersection with every nonzero ideal of the algebra. It follows that $\mathcal{M} = 0$, and so $\mathcal{B}$ is finite-dimensional. Using Corollary 3.2, $\mathcal{A}$ must be a finite-dimensional and semisimple algebra. This completes the proof.

**Proposition 3.6.** Let $\mathcal{A}$ be a \langle contractible, reflexive amenable \rangle simple contractible Banach algebra having a maximal left ideal complemented as a Banach space. Then $\mathcal{A}$ is finite-dimensional.

**Proof.** If $\mathcal{A}$ is an infinite-dimensional simple algebra, then $\text{Soc}(\mathcal{A}) = 0$. Moreover, if $\mathcal{A}$ is \langle contractible, reflexive amenable \rangle with a maximal left ideal complemented as a Banach space, then $\mathcal{A}$ has a nontrivial minimal left ideal. This is a contradiction.
Now, assume that $\mathcal{A}$ is a unital Banach $*$-algebra which admits at least one state $\tau$. Then there exists a $*$-representation $\pi_\tau$ of $\mathcal{A}$ on a Hilbert space $H_\tau$, with a cyclic vector $\zeta$ of norm 1 in $H_\tau$ such that $\tau(a) = \langle \pi_\tau(a)\zeta, \zeta \rangle$, for all $a \in \mathcal{A}$, $\langle \cdot, \cdot \rangle$ being the inner product in $H_\tau$.

**Theorem 3.7.** A Hermitian Banach $*$-algebra $\mathcal{A}$ is contractible if and only if there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that (3.1) holds.

**Proof.** It suffices to show the “only if” part. Suppose that a Hermitian Banach algebra $\mathcal{A}$ is contractible. Let $T(\mathcal{A})$ be the set of all states of $\mathcal{A}$ and let $R^*(\mathcal{A})$ be the $*$-radical of $\mathcal{A}$, that is, the intersection of the kernels of all $*$-representations of $\mathcal{A}$ on Hilbert spaces. Since $\mathcal{A}$ is Hermitian and has an identity, $T(\mathcal{A}) \neq \emptyset$, and so $R^*(\mathcal{A}) \neq \emptyset$. Let $\pi = \bigoplus_{\tau \in T(\mathcal{A})} \pi_\tau$ and $H = \bigoplus_{\tau \in T(\mathcal{A})} H_\tau$. Then $\pi$ is a $*$-representation of $\mathcal{A}$ on $H$.

Consider

$$\|\pi(a)\| = \sup_{\tau \in T(\mathcal{A})} \|\pi_\tau(a)\|.$$  \hspace{1cm} (3.7)

Then $\| \cdot \|$ is a $C^*$-norm on $\pi(A)$. Let $\mathcal{B}$ denote the closure of $(\pi(\mathcal{A}), \| \cdot \|)$. Moreover, $\pi : \mathcal{A} \to \mathcal{B}$ is a continuous mapping into a $C^*$-algebra $\mathcal{B}$ such that $\ker(\pi) = R^*(\mathcal{A})$. As $\mathcal{A}$ is contractible, $\mathcal{B}$ is also contractible. Using Corollary 3.4, the algebra $\mathcal{B}$ has to be finite-dimensional. Notice that $\mathcal{A}/R^*(\mathcal{A})$ is isometric with the $*$-subalgebra $\pi(\mathcal{A})$ of $\mathcal{B}$. Thus, it follows that $\mathcal{A}/R^*(\mathcal{A})$ is finite-dimensional. Since $R^*(\mathcal{A})$ is a finite-codimensional closed two-sided $*$-ideal, there exists a closed two-sided ideal $\mathcal{K}$ such that

$$\mathcal{A} = R^*(\mathcal{A}) \oplus \mathcal{K}. \hspace{1cm} (3.8)$$

Next, note that $\|\pi(a)\|^2 = \sup\{\tau(a^*a) : \tau \in T(\mathcal{A})\} \geq |a^*a|_\sigma$, where $|a|_\sigma$ is the spectral radius of $a \in \mathcal{A}$. By Pták [8], we obtain $\|\pi(a)\|^2 \geq |a|_\sigma^2$. So, if $a \in R^*(\mathcal{A})$, then $|a|_\sigma = 0$. Therefore, every element of $R^*(\mathcal{A})$ is quasinilpotent. Notice that in general $\text{Rad}(\mathcal{A}) \subset R^*(\mathcal{A})$. Since $R^*(\mathcal{A})$ is a closed two-sided $*$-ideal, we have $R^*(\mathcal{A}) = \text{Rad}(\mathcal{A})$, and so $\mathcal{A}$ is finite-dimensional and semisimple. \hfill $\Box$

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