IJMMS 2004:53, 2835–2845
PII. S0161171204311130
http://ijmms.hindawi.com

WEAK INCIDENCE ALGEBRA AND MAXIMAL RING OF QUOTIENTS

SURJEET SINGH and FAWZI AL-THUKAIR

Received 12 November 2003

Let X, X' be two locally finite, preordered sets and let R be any indecomposable commutative ring. The incidence algebra $I(X,R)$, in a sense, represents X, because of the well-known result that if the rings $I(X,R)$ and $I(X',R)$ are isomorphic, then X and X' are isomorphic. In this paper, we consider a preordered set X that need not be locally finite but has the property that each of its equivalence classes of equivalent elements is finite. Define $I^*(X,R)$ to be the set of all those functions $f : X \times X \to R$ such that $f(x,y) = 0$, whenever $x \nleqslant y$ and the set S_f of ordered pairs (x,y) with $x < y$ and $f(x,y) \neq 0$ is finite. For any $f,g \in I^*(X,R)$, $r \in R$, define $f+g$, fg, and rf in $I^*(X,R)$ such that $(f+g)(x,y) = f(x,y)+g(x,y)$, $fg(x,y) = \sum_{x \leqslant z \leqslant y} f(x,z)g(z,y)$, $rf(x,y) = r \cdot f(x,y)$. This makes $I^*(X,R)$ an R-algebra, called the *weak incidence algebra* of X over R. In the first part of the paper it is shown that indeed $I^*(X,R)$ represents X. After this all the essential one-sided ideals of $I^*(X,R)$ are determined and the maximal right (left) ring of quotients of $I^*(X,R)$ is discussed. It is shown that the results proved can give a large class of rings whose maximal right ring of quotients need not be isomorphic to its maximal left ring of quotients.

2000 Mathematics Subject Classification: 16S60, 16S90, 16W20.

1. Introduction. Let X and X' be two locally finite, preordered sets, and let R be a commutative ring. Under what conditions are incidence rings $I(X,R)$ and $I(X',R)$ isomorphic? In particular, under what conditions on R can one conclude that X and X' are isomorphic, when the two incidence rings $I(X,R)$ and $I(X',R)$ are isomorphic? The latter question has been discussed by many authors. One of the earliest results in this direction is by Stanley [9], who proved that if R is a field, then the two incidence rings are isomorphic if and only if X and X' are isomorphic. Froelich [4] extended this result to the case of an indecomposable ring R. Similar questions have been examined in [1, 3, 10] in case R need not be commutative.

Now consider any preordered set X that need not be locally finite. Two elements $x,y \in X$ are said to be *equivalent*, $x \backsim y$, if $x \le y \le x$. In Section 3, the isomorphism problem for weak incidence algebras is discussed. Let X and X' be two preordered sets in each of which every equivalence class is finite, and let R, R' be two commutative rings such that the weak incidence algebras $I^*(X,R)$ and $I^*(X',R')$ are isomorphic as rings. In case R and R' are indecomposable, Theorem 3.10 shows that X, X' are isomorphic and R, R' are isomorphic. The main aim of Section 4 is to prove some results that can help in studying the maximal ring of quotients of an $I^*(X,R)$. Similar work has been done in a recent paper [2] for certain classes of incidence algebras. In [7], Spiegel determines some essential ideals of an incidence algebra of a locally finite, partially ordered set. Here we are in a position to determine all the essential one-sided ideals of an $S = I^*(X,R)$ whenever R is indecomposable. A particular essential right ideal T is isolated and the ring $Q = \mathrm{Hom}_S(T,T)$ is discussed in Theorems 4.8, 4.9, and 4.10. This ring Q is used to give some results on maximal right (left) ring of quotients of S.

2. Preliminaries. All rings considered here are with identity $1 \neq 0$. As the various concepts discussed here for weak incidence algebras are similar to those for incidence algebras, for details on incidence algebras one may consult [8]. We now collect some results on rings and modules.

Lemma 2.1. *For any commutative ring R and any positive integer n, if $M_R = R^{(n)}$ is isomorphic to its summand N, then $M = N$.*

Proof. Now $M = N \oplus K$. For any maximal ideal P of R, the localization $M_P = N_P \oplus K_P$. As the ranks of the free R_P-modules M_P and N_P are the same and finite, $K_P = 0$. Hence $K = 0$. □

Lemma 2.2. *Let R be a commutative ring and let K be any ring such that $M_n(R) \cong M_m(K)$. Then m divides n. If $n = m$, then $R \cong K$.*

Proof. The first part follows from Wedderburn's structure theorem for simple artinian algebras, and the second part is in [6]. □

Lemma 2.3. *Let T be any ring and let e, e', f, f' be any four idempotents in T such that $eT \cong e'T$, $fT \cong f'T$. Then $eTf \neq 0$ if and only if $e'Tf' \neq 0$.*

Proof. The hypothesis gives that $\mathrm{Hom}_T(fT,eT) \cong \mathrm{Hom}_T(f'T,e'T)$, $eTf \cong e'Tf'$, as abelian groups. This proves the result. □

3. Isomorphism. Let X be any preordered set (i.e., X is a set with a relation $\leqslant$ that is reflexive and transitive). For any $x,y \in X$, set $x \backsim y$, if $x \leqslant y \leqslant x$. Then $\backsim$ is an equivalence relation. A preordered set X is said to be a *class finite, preordered set* if, for any $x \in X$, the equivalence class $[x] = \{y \in X : x \leq y \leq x\}$ is finite. Henceforth we take X to be a class finite, preordered set and R a commutative ring. The set $K^*(X,R) = \{f \in I^*(X,R) : f(x,y) = 0 \text{ whenever } x \backsim y\}$ is a nil ideal. Indeed, given $f \in K^*(X,R)$, $f^{m+1} = 0$, for $m = |S_f|$. Indeed, one can see that each member of $K^*(X,R)$ is strongly nilpotent, as defined in [8, page 176], so $K^*(X,R)$ is contained in the lower nil radical of $I^*(X,R)$. Let Y be a representative partially ordered subset of X. For any $x \in X$, let $|[x]| = n_x$. For each $x \in X$, the set $B_x = \{f \in I^*(X,R) : f(u,v) = 0 \text{ whenever } u \nsim x \text{ or } v \nsim x\}$, is a ring with δ_x as identity, where $\delta_x(u,v) = 0$, whenever $u \nsim x$, $v \nsim x$, or $u \neq v$, and $\delta_x(u,u) = 1$ whenever $u \backsim x$. Let δ denote the identity element of $I^*(X,R)$. For any $x,y \in X$, with $x \leqslant y$, let $e_{xy} \in I^*(X,R)$ be such that $e_{xy}(u,v) = 0$, for $(u,v) \neq (x,y)$, and $e_{xy}(x,y) = 1$. Each of e_{xy} is called a *matrix unit* of $I^*(X,R)$. We write $e_x = e_{xx}$. Then B_x is the $n_x \times n_x$ full matrix ring over R with $\{e_{uv} : u \backsim x,\ v \backsim x\}$ as its set of matrix units. Let $M_n(R)$ denote the $n \times n$ full matrix ring over R. Further, $D^*(X,R) = \{f \in I^*(X,R) : f(u,v) = 0 \text{ whenever } u \nsim v\}$ is a subring of $I^*(X,R)$, each B_x is an ideal of $D^*(X,R)$. Set $S = I^*(X,R)$, $K = K^*(X,R)$, $D = D^*(X,R)$. For any subset Z of X, let $E_Z \in S$ be such that $E_Z(u,u) = 1$ for $u \in Z$, and $E_Z(x,y) = 0$ otherwise. For any

$f \in S$, *support* of f, denoted by $\mathrm{suppt}(f)$, equals $\{(x,y) : f(x,y) \neq 0\}$, the cardinality of $\mathrm{suppt}(f)$ is called the *weight* of f and we denote it by $wt(f)$. Let X' be another class finite, preordered set. Let R' be another commutative ring. We use the same symbols for the matrix units of $I^*(X,R)$ or $I^*(X',R')$ and so on, but $S' = I^*(X',R')$, $K' = K^*(X',R')$, and $D' = D^*(X',R')$. Let Y and Y' be fixed *representative partially ordered subsets* of X and X', respectively. For any two distinct members y, z of Y, δ_y, δ_z are orthogonal idempotents. Any $f \in S$ will be sometimes denoted by the formal sum $\sum_{x,y} f(x,y)e_{xy}$ (or by the matrix $[f(x,y)]$ indexed by X). The following is obvious.

Lemma 3.1. (i) *$I^*(X,R) = D^*(X,R) \oplus K^*(X,R)$ as abelian groups.*

(ii) *$D^*(X,R) = \Pi_{y\in Y} B_y$, where Y is any representative partially ordered subset of X.*

(iii) *$I^*(X,R)/K^*(X,R) \cong \Pi_{y\in Y} M_{n_y}(R) \cong D^*(X,R)$, where Y is any representative partially ordered subset of X.*

(iv) *For any $f, e_{xy} \in I^*(X,R)$, $wt(fe_{xy})$ is finite, that is, $fe_{xy} = \sum_{u\leq y} a_{uy}e_{uy}$, with finitely many $a_{ux} \neq 0$.*

It follows from (ii) that $K^*(X,R)$ does not equal the Jacobson radical of S, unless the Jacobson radical of R is zero. For any $f \in S$, we write $f = f_D + f_K$ with $f_D \in D$ and $f_K \in K$; f_D is called the *diagonal* of f. The following is obvious.

Lemma 3.2. *For any nonempty subset Z of X, $E_Z S E_Z \cong I^*(Z,R)$.*

Lemma 3.3. *For any two idempotents $f, g \in S$, $fSg \neq 0$ if and only if $f_D S g_D \neq 0$.*

Proof. In $\overline{S} = S/K$, $f + K = f_D + K$. As K is nil, we get $fS \cong f_D S$. After this, Lemma 2.3 completes the proof. □

Lemma 3.4. *Let $0 \neq e = e^2 \in S$.*

(i) *e_D is a nonzero idempotent and $e_D\delta_y = \delta_y e_D$ for any $y \in Y$.*

(ii) *There exists $y \in Y$ such that $e_D\delta_y = \delta_y e_D \neq 0$.*

(iii) *For any $y \in Y$, $e' = ee_D\delta_y e$ is an idempotent such that $e'(u,v) = \sum e(u,w_1)e(w_1,w_2)e(w_2,v)$, where the summation runs over w_1, w_2 in $[y] \cap [u,v]$. Further, $e - e'$, e' are orthogonal idempotents. If $e_D\delta_y \neq 0$, then $e' \neq 0$.*

Proof. (i) is obvious. Now $S/K = \overline{D} = \Pi_{y\in K}\overline{B_y} \cong D$, $\overline{\delta} = \Pi\overline{\delta_y}$, and $\overline{e} = \overline{e_D}$. It follows that for some $y \in Y$, $\overline{e}\,\overline{\delta_y} = \overline{e_D}\,\overline{\delta_y} \neq 0$. This proves (ii). Consider any $y \in Y$ and $e' = ee_D\delta_y e$. The definition of the product of two members of S gives that $e'(u,v) = \sum e(u,w_1)e(w_1,w_2)e(w_2,v)$, where the summation runs over all w_1, w_2 in $[y] \cap [u,v]$. Then we have $(e')^2(u,v) = \sum_{u\leqslant w\leqslant v} e'(u,w)e'(w,v) = \sum e(u,w_1)e(w_1,w_2)e(w_2,w)e(w,w_3)e(w_3,w_4)e(w_4,v)$, where summation runs over all w_i, w in $[y] \cap [u,v]$ such that $w_2 \leqslant w \leqslant w_3$. Thus $(e')^2(u,v) = \sum e(u,w_1)e(w_1,w_4)e(w_4,v) = e'(u,v)$. Hence e' is an idempotent. As $ee' = e' = e'e$, it follows that $e - e'$ is an idempotent orthogonal to e'. If $e_D\delta_y \neq 0$, as obviously $\overline{e'} = \overline{e_D\delta_y}$ in S/K, we get $e' \neq 0$. □

Lemma 3.5. (i) *If $e \in S$ is an indecomposable idempotent, then there exists a unique $y \in Y$ such that $e = ee_D\delta_y e$.*

(ii) *Let $e \in S$ be a nonzero idempotent such that $e_D \in B_y$ for some $y \in Y$. Then $e = ee_D\delta_y e$; this y is uniquely determined by e.*

Proof. (i) In $\overline{S} = S/K$, $\overline{e} = \overline{e_D}$ is an indecomposable idempotent. So there exists a unique $y \in Y$ such that $\overline{e} = \overline{e_D\delta_y}$. By Lemma 3.4(iii), $e' = ee_D\delta_y e$ is a nonzero idempotent. As $e - e'$ is orthogonal to e' and e is indecomposable, $e = e'$.

(ii) The hypothesis gives $\overline{e} = \overline{ee_D\delta_y e}$. Then Lemma 3.4(iii) gives $e = ee_D\delta_y e$. □

Theorem 3.6. *Let R be any indecomposable commutative ring and X any class finite, preordered set. Then for any automorphism σ of $S = I^*(X,R)$, $\sigma(K) = K$.*

Proof. Consider any $f \in S\backslash K$. For some $x \backsim y$, $f(x,y) \neq 0$. Then $g = e_x f e_{yx}$ is such that $g(x,x) \neq 0$ and $g = e_x g e_x$. So $\sigma(g) = e\sigma(g)e$, where $e = \sigma(e_x)$ is an indecomposable idempotent. Let Y be a representative partially ordered subset of X. By Lemma 3.5, there exists unique $z \in Y$ such that $e = ee_D\delta_z e$, $e_D \in B_z$. Thus $\sigma(g) = ee_D\delta_z e\sigma(g)ee_D\delta_z e \neq 0$, $\delta_z e\sigma(g)ee_D\delta_z \neq 0$, so for some $u,v \in [z]$, $\sigma(g)(u,v) \neq 0$. Hence $\sigma(g) \notin K$. Consequently, $\sigma(f) \notin K$. This proves the result. □

Lemma 3.7. *For some $y, y' \in Y$, let there exist idempotents $e \in B_y$, $f \in B_{y'}$ such that $eSf \neq 0$. Then $e_y S e_{y'} \neq 0$.*

Proof. The hypothesis gives that $\delta_y S\delta_{y'} \neq 0$, so there exist $u \in [y]$, $v \in [y']$ such that $e_u S e_v \neq 0$. After this, Lemma 2.3 completes the proof. □

Lemma 3.8. *If for some idempotent $f \in S$, $fS \cong \delta_y S$ for some $y \in Y$, then $f_D = \delta_y$.*

Proof. We have $f_D S \cong \delta_y S$. In $\overline{S} = S/K$, $\overline{f_D S} \cong \overline{\delta_y S}$, so $f_D \in B_y$ and $f_D B_y \cong B_y$. By Lemma 2.1, $f_D = \delta_y$. □

Lemma 3.9. *Let R, R' be indecomposable and $\sigma : S \to S'$ an isomorphism.*

There exists a one-to-one mapping η of Y onto Y' such that $\sigma(\delta_y) = \delta_{\eta(y)} + g_{\eta(y)}$ for some $g_y \in K'$, $|[y]| = |[\eta(y)]|$, and $R \cong R'$.

Proof. The hypothesis gives that for any $x \in X$, e_x is an indecomposable idempotent in S. Now $\sigma(\delta_y)S' = \oplus\sum_{u\backsim y}\sigma(e_u)S'$. As these $\sigma(e_u)S'$ are indecomposable and isomorphic right ideals, there exist unique $\eta(y) \in Y'$ such that each $\sigma(e_u)_D \in B'_{\delta_{\eta(y)}}$. Consequently, $\sigma(\delta_y)_D \in B'_{\eta(y)}$ and $\sigma(\delta_y)_D\delta_{\eta(y)} = \delta_{\eta(y)}\sigma(\delta_y)_D$. By Lemma 3.5(ii), $\sigma(\delta_y) = \sigma(\delta_y)\sigma(\delta_y)_D\delta_{\eta(y)}\sigma(\delta_y)$. Similarly,

$$\sigma^{-1}(\delta_{\eta(y)}) = \sigma^{-1}(\delta_{\eta(y)})(\sigma^{-1}(\delta_{\eta(y)}))_D\delta_z\sigma^{-1}(\delta_{\eta(y)}) \tag{3.1}$$

for some $z \in Y$. So, $\delta_{\eta(y)} = \delta_{\eta(y)}\sigma((\sigma^{-1}(\delta_{\eta(y)}))_D)\sigma(\delta_z)\delta_{\eta(y)}$. Thus, in $\overline{S'} = S'/K'$,

$$\overline{\sigma(\delta_y)} = \overline{\sigma(\delta_y)\sigma(\delta_y)_D\delta_{\eta(y)}\sigma((\sigma^{-1}(\delta_{\eta(y)}))_D)\sigma(\delta_z)\delta_{\eta(y)}\sigma(\delta_y)}. \tag{3.2}$$

In $\overline{S'}$, $\overline{\delta_{\eta(y)}}$ is a central idempotent. Thus

$$\overline{\sigma(\delta_y)} = \overline{\sigma(\delta_y)\sigma((\sigma^{-1}(\delta_{\eta(y)}))_D)\sigma(\delta_z\delta_y)\delta_{\eta(y)}}, \tag{3.3}$$

which equals zero, if $z \neq y$. Hence $z = y$ and η is a bijection from Y onto Y'. We get $\overline{\sigma(\delta_y)} = \overline{\delta_{\eta(y)}\sigma((\sigma^{-1}(\delta_{\eta(y)}))_D\delta_y)}$ and $\overline{\delta_{\eta(y)}} = \overline{\delta_{\eta(y)}\sigma((\sigma^{-1}(\delta_{\eta(y)}))_D\delta_y)}$. Hence $\overline{\sigma(\delta_y)} = \overline{\delta_{\eta(y)}}$. This shows that $\sigma(\delta_y) = \delta_{\eta(y)} + g_{\eta(y)}$ for some $g_{\eta(y)} \in K'$. Now $\delta_y S \delta_y = B_y$. As $\sigma(\delta_y)S' \cong \delta_{\eta(y)}S'$, it follows that $B_y \cong B'_{\eta(y)}$. By Lemma 2.2, $|[y]| = |[\eta(y)]|$ and $R \cong R'$. $\square$

Theorem 3.10. *Let X and X' be two class finite, preordered sets. Let R and R' be any two indecomposable commutative rings. If there exists an isomorphism of $I^*(X,R)$ onto $I^*(X',R')$, then X, X' are isomorphic and the rings R, R' are isomorphic.*

Proof. We use the terminology developed before Theorem 3.10. Consider any u, $v \in Y$ such that $u \leqslant v$. Then $e_u S e_v \neq 0$, $\sigma(e_u)S'\sigma(e_v) \neq 0$. It follows from Lemma 3.9 that $\sigma(e_u)_D \in B'_{\eta(u)}$, $\sigma(e_v)_D \in B'_{\eta(v)}$. By Lemma 3.3, $\sigma(e_u)_D S' \sigma(e_v)_D \neq 0$, $e_{\eta(u)}S'e_{\eta(v)} \neq 0$, hence $\eta(u) \leqslant \eta(v)$. Thus η is an isomorphism of Y onto Y'. Also by Lemma 3.9, $|[y]| = |[\eta(y)]|$, hence it follows that X and X' are isomorphic. By Lemma 3.9, R and R' are isomorphic. $\square$

Lemma 3.11. *For any commutative ring T and any class finite, preordered set X, the following hold.*

(i) *A central idempotent $e \in I^*(X,T)$ is centrally indecomposable if and only if $e = gE_Z$ for some indecomposable idempotent $g \in T$ and a connected component Z of X.*

(ii) *Let g and h be two indecomposable idempotents in T and let Z, Z' be two connected components of X; the rings $gE_Z I^*(X,T)$, $hE_Z I^*(X,T)$ are isomorphic if and only if the rings gT, hT are isomorphic and Z, Z' are isomorphic.*

Proof. (i) Consider any central idempotent $e \in I^*(X,T)$. On the same lines as for incidence algebras, it can be easily seen that $e(x,y) = 0$, whenever $x \neq y$. For any connected component Z of X, there exists an idempotent $g_Z \in T$ such that $e(x,x) = g_Z$ for every $x \in X$. Using this, (i) follows. (ii) As $gE_Z I^*(X,T) \cong I^*(Z,gT)$ and $hE_{Z'} I^*(X,T) \cong I^*(Z' \cdot hT)$, the result follows from Theorem 3.10. $\square$

Let T be any ring. Let $\mathrm{In}(T)$ be the set of all *centrally indecomposable central idempotents* of T. Two central idempotents g, h of T are said to be *equivalent* if the rings gT and hT are isomorphic. For any central idempotent $g \in T$, $[g]$ denotes the set of central idempotents in T equivalent to g.

Theorem 3.12. *Let R and R' be any two commutative rings and let X, X' be two class finite, preordered sets. Let $\sigma : I^*(X,R) \to I^*(X',R')$ be a ring isomorphism. Let $g \in \mathrm{In}(R)$ and let Z be a connected component of X.*

(i) *There exist unique $g' \in \mathrm{In}(R')$ and unique connected component Z' of X' such that $\sigma(gE_Z) = g'E_{Z'}$; further, $Z \cong Z'$, $|[g]||[Z]| = |[gE_Z]| = |[g'E_{Z'}]| = |[g']||[Z']|$.*

(ii) *If the cardinalities of $[g]$ and $[g']$ are finite and equal, then X and X' are isomorphic.*

Proof. (i) The first part follows from Lemma 3.11(i); the second part follows from Lemma 3.11(ii). (ii) If $|[g]| = |[g']|$ and they are finite, if follows from (i) that, given any connected component Z of X, there exists a connected component Z' of X' isomorphic to Z, and $[Z]$, $[Z']$ have the same cardinalities. Consequently, X and X' are isomorphic. □

The following is immediate from Theorem 3.12.

Corollary 3.13. *Let R be any commutative ring such that R admits an indecomposable idempotent g for which the equivalence class $[g]$ is finite. Let X and X' be any two class finite, preordered sets. If the rings $I^*(X,R)$ and $I^*(Y,R)$ are isomorphic, then X and X' are isomorphic.*

4. Essential right ideals and maximal ring of quotients. Throughout $S = I^*(X,R)$, where X is a class finite, preordered set and R is a commutative ring in which 1 is indecomposable. Any $x \in X$ is said to be a *maximal element* if the equivalence class $[x]$ is maximal in the partially ordered set of the equivalence classes in X. For any $x, y \in X$, we say $x < y$, if $x \le y$ but $[x] \ne [y]$. Set $X_0 = \{x \in X : x \text{ is maximal}\}$, $Y_0 = \{(x,y) \in X \times X_0 : x \le y\}$, $Y_1 = \{(x,y) : x < y$ and there does not exist any $z \in X_0$ such that $y \le z\}$, $Y_2 = \{(x,y) : x < y$ and there exists a $z \in X_0$ such that $y < z\}$, and $Y_3 = \{(x,y) \in X_0 \times X_0 : [x] = [y]\}$. Further, $K = K^*(X,R)$. Now $L = \sum_{(x,y)\in Y_3} e_{xy}R$ is a right ideal of S. In [2], maximal rings of quotients of certain incidence algebras have been discussed. Here we intend to prove some results that can help in studying the maximal rings of quotients of S. Spiegel [7] has determined certain classes of essential ideals of an incidence algebra of a locally finite, preordered set. Here we determine all essential one-sided ideals of S. For the definitions of an *essential submodule, dense submodule*, and *singular submodule* of a module, one may refer to [5]. Let M be any module, then $N \subset_e M$ ($N \subset_d M$) denotes that N is an *essential* (*dense*) submodule of M, and $Z(M)$ denotes the singular submodule of M. The concept of the *maximal right ring of quotients* of a ring is discussed in [5, Section 13].

Lemma 4.1. *Let $K_1 = K + L$. Then K_1 is an essential right ideal of S and $l\cdot\operatorname{ann}(K_1) = 0$. Indeed for any $0 \ne f \in S$, there exists $e_{xw} \in K_1$ such that $0 \ne f e_{xw} \in K_1$.*

Proof. Let $0 \ne f \in S$. Then $f(u,v) \ne 0$ for some $u \le v$. Suppose $fK_1 = 0$. If v is not maximal in X, there exists $e_{vz} \in K$, and $fe_{vz} \ne 0$, which is a contradiction. Hence v is maximal. Then $e_v \in K_1$ with $fe_v \ne 0$, which is again a contradiction. Hence $l\cdot\operatorname{ann}(K_1) = 0$. In any case there exists $e_{xy} \in K_1$ such that $fe_{xy} \ne 0$. By applying induction on $wt(fe_{xy})$, we prove that for some $g \in S$, $0 \ne fe_{xy}g \in K_1$, which will prove that $K_1 \subset_e S_S$. Suppose $wt(fe_{xy}) = 1$. Then $fe_{xy} = ae_{uy}$, for some $0 \ne a \in R$. If y is not maximal, for any $z > y$, $fe_{xy}e_{yz} = ae_{uz} \in K_1$. If y is maximal, then $e_y \in K_1$, so $fe_{xy}e_y = ae_{uy} \in K_1$. To apply induction, suppose that $wt(fe_{xy}) = n > 1$, and for any $h \in S$, if for some $e_{uv} \in K_1$, $wt(he_{uv}) < n$ and $he_{uv} \ne 0$, then for some $e_{vz} \in S$, $0 \ne he_{uv}e_{vz} \in K_1$. We can write $fe_{xy} = ae_{uy} + h$, where $wt(h) = n-1$ and $h(u,y) = 0$. For some $e_{ys} \in K_1$, $ae_{uy}e_{ys} = ae_{us} \in K_1$. Then $fe_{xs} = ae_{us} + he_{ys}$ with $wt(he_{ys}) = n-1$. By the induction hypothesis, there exists $e_{sw} \in K_1$ such that $0 \ne he_{ys}e_{sw} \in K_1$. Then $0 \ne fe_{xw} \in K_1$. Hence $K_1 \subset_e S_S$. □

We call a subset B of R an *essential subset* of R if, for each $0 \neq r \in R$, there exists an $s \in R$ such that $0 \neq rs \in B$. Clearly the ideal of R generated by an essential subset is an essential ideal.

Lemma 4.2. *Let $E \subset_e S_S$. For any $x \leq y$ in X, let $A_{xy} = \{r \in R : re_{xy} \in E\}$, $B_{xy} = \cup_{y \leq z} A_{xz}$.*

(i) *$A_{xy} \subseteq A_{xw}$ whenever $x \leq y \leq w$.*

(ii) *B_{xy} is an essential subset of R.*

Proof. (i) is trivial. Let $0 \neq r \in R$. Then for some $g \in S$, $0 \neq re_{xy}g \in E$. For some $y \leq w$, $rg(y,w) \neq 0$. This gives $re_{xy}ge_w = rg(y,w)e_{xw} \in E$, $rg(y,w) \in B_{xy}$. This proves that B is an essential subset of R. □

Lemma 4.3. *Let $\{A_{xy}$: either $x < y$, or $x \leq y$ and y is maximal in $X\}$ be a family of ideals in R such that (i) $A_{xy} \subseteq A_{xz}$ whenever $y \leq z$, and (ii) for any $x \leq y$ in X, $B_{xy} = \cup_{y \leq z} A_{xz}$ is an essential subset of R. Then $E = \sum_{x,y} A_{xy}e_{xy}$ is an essential right ideal of S and $E \subseteq K_1$.*

Proof. It is easy to verify that E is a right ideal of S contained in K_1. Let $0 \neq f \in K_1$. By induction on $wt(f)$, we prove that $0 \neq fre_{xy} \in E$ for some $e_{xy} \in K_1$, $r \in R$, which will prove that $E \subset_e S_S$. Suppose $f = ae_{xy}$. As $a \neq 0$, there exists a $z \geq y$ and an $r \in R$ such that $0 \neq ar \in A_{xz}$. Then $0 \neq fre_{yz} = are_{xz} \in E$. Here, if y is not maximal, choose $z > y$; if y is maximal, choose $y = z$; in any case $e_{xz} \in K_1$. Thus the result holds for $wt(f) = 1$. To apply induction, let $wt(f) = n > 1$, and let the result hold for any positive integer less than n. We write $f = ae_{xy} + h$, with $0 \neq a \in R$, $e_{xy} \in K_1$, $wt(h) = n-1$, and $h(x,y) = 0$. There exists an $re_{yz} \in K_1$ such that $0 \neq ae_{xy}re_{yz} = are_{xz} \in E$. Then $0 \neq fre_{xz} = are_{xz} + hre_{yz}$. If $hre_{yz} = 0$, $fre_{xz} = are_{xz} \in E$ and we finish. Suppose $hre_{yz} \neq 0$. By the induction hypothesis, there exists $be_{zw} \in K_1$, with $b \in R$, such that $0 \neq hre_{yz}be_{zw} \in E$. Then $0 \neq frbe_{xw} \in E$. □

Let $\mathrm{Minness}(S)$ be the set of all essential right ideals of the form given in Lemma 4.3.

Lemma 4.4. *$Z(S) = \{f \in S : fE = 0$ for some $E \in \mathrm{Minness}(S)\}$.*

Proof. Let $f \in Z(S)$. For some $E \subset_e S_S$, $fE = 0$. By Lemmas 4.2 and 4.3, there exists an $E' \in \mathrm{Minness}(S)$ such that $E' \subseteq E$. Then $fE' = 0$. This proves the result. □

Theorem 4.5. *$Z(S_S) = 0$ if and only if $Z(R) = 0$.*

Proof. Let $Z(R) \neq 0$. For some $r \neq 0$ and an essential ideal A of R, $rA = 0$. In Lemma 4.3, by taking every $A_{xy} = A$, we get an $E \subset_e S_S$ such that $rIE = 0$. Thus $Z(S) \neq 0$. Conversely, let $Z(S) \neq 0$. Consider any $0 \neq f \in Z(S)$. For some $E \in \mathrm{Minness}(S)$, $fE = 0$. Now $f(u,v) \neq 0$ for some $u \leq v$. Then $0 \neq e_uf \in Z(S)$. Suppose there exists a maximal $z \geq v$. As z is maximal, it follows from Lemma 4.3(i) that $B_{vz} = A_{vz}$, so $e_vfe_{vz}A_{vz} = 0$, $f(u,v)A_{vz} = 0$, $f(u,v) \in Z(R)$. Hence $Z(R) \neq 0$. □

Proposition 4.6. *For any $(x,y) \in Y_0$, set $A_{xy} = R$, for $(x,y) \in Y_1$, set $A_{xy} = R$, and for $(x,y) \in Y_2$, set $A_{xy} = 0$. Let $T = \sum_{x,y} e_{xy}A_{xy}$.*

(i) *Then T is an ideal of S, $T \subset_e S_S$, and $l \cdot \mathrm{ann}(T) = 0$.*

(ii) *S embeds in the ring $Q = \mathrm{Hom}(T_S, T_S)$ such that S_S is dense in Q_S.*

Proof. That T is an essential right ideal in S follows from Lemma 4.3. Suppose that $0 \neq f \in l\cdot\mathrm{ann}(T)$. Then $f(u,v) \neq 0$ for some $u \leq v$. Suppose there exists no maximal $z \geq v$. Choose any $w > v$. Then $e_{vw} \in T$ but $fe_{vw} \neq 0$, which is a contradiction. Hence there exists a maximal $z \geq v$. Then $e_{vz} \in T$ and $fe_{vz} \neq 0$, which is also a contradiction. Hence $l\cdot\mathrm{ann}(T) = 0$. Consider any $e_{xy} \in T$. By Lemma 3.1, $wt(fe_{xy})$ is finite, so $fe_{xy} = \sum_{u\leq y} a_{uy}e_{uy}$, a finite sum. By definition, the following two cases arise.

Case 1. y is maximal. Then every $e_{uy} \in T$, so $fe_{xy} \in T$.

Case 2. There does not exist any maximal $z \geq y$. Then $u < y$, $A_{uy} = R$, $e_{uy} \in T$, hence $fe_{xy} \in T$.

This proves that T is an ideal in S. For each $f \in S$, let $\lambda(f)$ be the left multiplication on T by f. Then λ is an embedding of S in Q. Consider any $\sigma,\eta \in Q$, with $\sigma \neq 0$. Then for some $f \in T$, $\sigma(f) \neq 0$. We see that $\sigma\cdot\lambda(f) = \lambda(\sigma(f)) \neq 0$ and $\eta\cdot\lambda(f) = \lambda(\eta(f)) \in \lambda(S)$. Hence S_S is dense in Q_S. □

For each $x_0 \in X_0$, set $T_{[x_0]} = \sum\{e_{xy}R : (x,y) \in Y_3$ and $[x_0] = [y]\}$, and set $T' = \sum\{e_{xy}R : (x,y) \in Y_1\}$. Observe that $T_{[x_0]} = T_{[x_1]}$ if and only if $[x_0] = [x_1]$. Each of $T_{[x_0]}$, T' is a right ideal of S contained in T, and T is a direct sum of these right ideals. Let Z_0 be the set of equivalence classes in X given by the members of X_0. For any ring P, let $\hat{P}$ be the *maximal right ring of quotients* of P [5, Section 13]. The following result can be easily deduced from various results and exercises given in [5, Sections 8 and 13].

Theorem 4.7. (I) *For any family of rings* $\{P_\alpha : \alpha \in \Lambda\}$, $P = \prod_{\alpha\in\Lambda} P_\alpha$, $\hat{P} = \prod_{\alpha\in\Lambda}\widehat{P_\alpha}$.

(II) *For any two subrings* A, B *of a ring* P, *if* $A_A \subset_d B_A$, $B_B \subset_d P_B$, *then* $\hat{A} = \hat{P}$.

(III) *For any positive integer* n *and any ring* P, $\widehat{M_n(P)} = M_n(\hat{P})$.

Theorem 4.8. (i) $Q = \mathrm{Hom}(T_S,T_S) \cong (\Pi\{\mathrm{Hom}_S(T_{[x_0]},T_{[x_0]}) : [x_0] \in Z_0\}) \times \mathrm{Hom}_s(T', T')$.

(ii) *Maximal right rings of quotients of* S *and* Q *are the same.*

(iii) *Let* $P_{[x_0]} = \mathrm{Hom}_S(T_{[x_0]},T_{[x_0]})$ *and* $P' = \mathrm{Hom}(T',T')$. *Then* $\hat{S} \cong (\Pi\{\widehat{P_{[x_0]}} : [x_0] \in Z_0\}) \times \widehat{P'}$.

Proof. To prove (i) it is enough to prove that $\mathrm{Hom}_S(T_{[x_0]},T_{[x_1]}) = 0$ whenever $[x_0] \neq [x_1]$, $\mathrm{Hom}_S(T_{[x_0]},T') = 0 = \mathrm{Hom}_S(T',T_{[x_0]})$. Consider $\sigma \in \mathrm{Hom}_S(T_{[x_0]},T_{[x_1]})$. For any $e_{xy} \in T_{[x_0]}$, $[x_0] = [y]$, so $e_{uy} \notin T_{[x_1]}$, but $\sigma(e_{xy}) = \sum_{u\leq y} a_{uy}e_{uy}$, $a_{uy} \in R$. Thus $\sigma(e_{xy}) = 0$, $\sigma = 0$. Similarly, we can prove that the others are also zero. As S_S is dense in Q_S, $\hat{S} = \hat{Q}$. Because of (i) and Theorem 4.7, we get $\hat{S} \cong (\Pi\{\widehat{P_{[x_0]}} : [x_0] \in Z_0\}) \times \widehat{P'}$. □

We now discuss matrix representations of $\mathrm{Hom}_S(T_{[x_0]},T_{[x_0]})$ and $\mathrm{Hom}_s(T',T')$.

Theorem 4.9. *Let* x_0 *be a maximal member of* X, $U_{x_0} = \{x \in X : x \leq x_0\}$. *Then* $\mathrm{Hom}_S(T_{[x_0]},T_{[x_0]})$ *is isomorphic to the ring of column-finite matrices over* R *indexed by* U_{x_0}.

Proof. Let $\sigma \in \mathrm{Hom}_S(T_{[x_0]},T_{[x_0]})$. For $e_{xy} \in T_{[x_0]}$, $y \backsim x_0$. If $\sigma(e_{xy}) = \sum_{u\leq y} a_{uy}e_{uy}$, then for any other $e_{xz} \in T_{[x_0]}$, $\sigma(e_{xz}) = \sum_{u\leq z} a_{uz}e_{uz} = \sigma(e_{xy})e_{yz} = \sum_{u\leq y} a_{uy}e_{uz}$, $a_{uy} = a_{uz}$. Conversely, any $\sigma \in \mathrm{Hom}_R(T_{[x_0]},T_{[x_0]})$, such that if $\sigma(e_{xy}) = \sum_{u\leq y} a_{uy}e_{uy}$, then $\sigma(e_{xz}) = \sum_{u\le y} a_{uy}e_{uz}$ for $y \backsim z$, is in $\mathrm{Hom}_S(T_{[x_0]}, T_{[x_0]})$. Now $V_{x_0} = \{e_{xy} : x \in U_{x_0},\ y \backsim x_0\}$ is an R-basis of $T_{[x_0]}$. We write $\sigma(e_{xy}) = \sum_{u,v} a_{uvxy}e_{uv}, e_{uv} \in V_{x_0}$. Then $a_{uvxy} = 0$, for $v \ne y$, $a_{uyxy} = a_{uzxz}$, whenever $y \backsim z$. We write $b_{ux} = a_{uyxy}$ and $b_{ux} = 0$ otherwise. We get matrix $[b_{ux}]$ over R indexed by U_{x_0}. This matrix is column finite; $\sigma \leftrightarrow [b_{ux}]$ gives the desired isomorphism. □

Theorem 4.10. *Let $X' = \{y \in X :$ there exist no maximal $z \ge y\}$. Let G be the set of arrays $[a_{vxy}]$ over R indexed by $X' \times X' \times X'$ such that it has following properties:*

(i) *$a_{vxy} = 0$, whenever $x \not< y$, $v \not< y$, or $x < v < y$,*

(ii) *for any fixed pair (x,y) with $x < y$, the number of v for which $a_{vxy} \ne 0$ is finite,*

(iii) *for $y \le z$, $a_{vxy} = a_{vxz}$ if $v < y$, and $a_{vxz} = 0$ if $v \not< y$ and $v < z$.*

In G, define addition componentwise and the product by $[a_{vxy}][b_{vxy}] = [c_{vxy}]$ such that $c_{vxy} = \sum_w a_{vwy}b_{wxy}$. Then $\mathrm{Hom}_S(T', T') \cong G$.

In case X' has the property that for every pair of elements u, v in X' there exists a $w \in X'$ such that $u \le w$, $v \le w$, then any array $[a_{vxy}] \in G$ has the following additional properties:

(iv) *if $u, v \in X'$ are not comparable, then $a_{uxv} = 0$,*

(v) *for $x < y$, $x < z$, $a_{vxy} = a_{vxz}$.*

Put $b_{vx} = a_{vxy}$. Then $[b_{vx}]$ is a column finite matrix indexed by X' with the property that $b_{vx} = 0$ if $v > x$, or there exists $y > x$ such that $v \not< y$. Set $b_{vx} = 0$ in all other cases. Let B be the set of all such matrices. Then B is a ring isomorphic to $\mathrm{Hom}_S(T', T')$.

Proof. Let $\sigma \in \mathrm{Hom}_S(T', T')$. For any $x < y \le z \in X'$, we have $\sigma(e_{xy}) = \sum c_{uvxy}e_{uv}$, $c_{uvxy} \in R$, $(u,v) \in Y_1$, with $c_{uvxy} = 0$, whenever $v \ne y$. So we can write $\sigma(e_{xy}) = \sum_{v<y} e_{vy}a_{vxy}$, a finite sum. For $y \le z$, $\sigma(e_{xz}) = \sigma(e_{xy})e_{yz}$ gives $a_{vxy} = a_{vxz}$ for $v < y$ and $a_{vxz} = 0$ whenever $v \not< y$, $v < z$. Suppose we have some $x < v < y$, by considering $\sigma(e_{xy}) = \sigma(e_{xv})e_{vy}$ it follows that $a_{vxy} = 0$. For any other $(v,x,y) \in X' \times X' \times X'$, set $a_{vxy} = 0$. We get an array $[a_{vxy}]$ with the desired properties. Conversely, any such array gives an S-endomorphism of T'. This gives the desired isomorphism.

Suppose every pair of elements in X' have a common upper bound. Consider any $v, w \in X'$ that are not comparable. By (i), $a_{vxw} = 0$ for any x; this proves (iv). Suppose $x < y$, $x < z$. There exists $w \in X'$ such that $y < w$, $z < w$. Then $\sigma(e_{xz})e_{zw} = \sigma(e_{xy})e_{yw} = \sigma(e_{xw})$ gives (v). Set $b_{vx} = a_{vxy}$. Because of (v), b_{vx} is well defined. It gives a matrix $[b_{vx}]$ indexed by X', which is column finite and has the property that $b_{vx} = 0$ if either $v > x$, or there exists $y > x$ such that $v \not< y$. Let B be the set of all column-finite matrices $[b_{vx}]$ over R indexed by $X' \times X'$ with $b_{vx} = 0$, whenever either $v > x$ or there exists a $y > x$ such that $v \not< y$. Then $\mathrm{Hom}_S(T', T')$ is isomorphic to the ring B. □

Remark 4.11. Let X be any locally finite, preordered set and let R be any indecomposable commutative ring. Obviously, $S = I^*(X,R)$ is a subring of $S' = I(X,R)$. But S_S need not be dense or essential in S'_S. So the maximal right rings of quotients of S and S' need not be the same; in fact, they need not be isomorphic (see the example given below). In case S_S is dense in S', the two rings will have the same maximal right ring of quotients. In that case, S can help in studying S'.

Theorem 4.12 [2]. *Let X be any partially ordered set such that for any $x \in X$, there exists a maximal element $z \geq x$ and $L_z = \{y \in X : y \leq z\}$ is finite. Let X_0 be the set of maximal elements of X. For each $z \in X_0$, let n_z be the number of elements $y \leq z$. For the ring $S = I(X,R)$, $\widehat{S} \cong \Pi\{M_{n_z}(\widehat{R})$: where z runs over representatives of equivalence classes in $X_0\}$.*

Proof. Let $f, g \in S' = I(X,R)$ with $g \neq 0$. For some $u, v \in X$, $g(u,v) \neq 0$. Then $ge_v \neq 0$. At the same time the hypothesis on X gives that the support of fe_v is finite, so $fe_v \in S = I(X,R)$. Hence S_S is dense in S'. After this, Theorems 4.7, 4.8, and 4.9 complete the proof. □

Example 4.13. Let $X = \mathbb{N}$ be the set of natural numbers and let R be any indecomposable commutative ring. Consider $S = I^*(\mathbb{N},R)$ and $S' = I(\mathbb{N},R)$. Let $0 \neq f \in S'$. For some $r \in \mathbb{N}$, $fe_r \neq 0$. Clearly, the support of fe_r is finite. Hence S_S is dense in S'. So the maximal right quotient rings of S and S' are the same. Consider $g \in S'$ for which $g(1,n) = 1$ for every n, and $g(n,m) = 0$ otherwise. Then for any $h \in S$, $hg = 0$ or $hg = kg$ for some $0 \neq k \in \mathbb{N}$, so ${}_SS$ is not dense in S'. Thus maximal left rings of quotients of S and S' are not the same. We now show that they need not be isomorphic. Consider $R = F$ a countable field. As $\mathbb{N}$ has no maximal element, $K = T = T'$, $Q = \mathrm{Hom}_S(T',T')$. By Theorem 4.10, Q is isomorphic to S'. But S', as a right S'-module, is dense in the ring L of all column-finite matrices over F, indexed by $\mathbb{N}$. It is well known that the ring of all column-finite matrices over a field, indexed by any set, is right self-injective. Hence L is the maximal right ring of quotients of S and S'. Let $\mathbb{N}'$ be the set of natural numbers with reverse ordering. As $\mathbb{N}'$ has unique maximal element 0, $\mathbb{N}' = T_0$, by Theorem 4.9, the corresponding Q' is isomorphic to the ring of all column-finite matrices over F, indexed by $\mathbb{N}$. So Q' is right self-injective. However $S = I^*(\mathbb{N},F)$ is anti-isomorphic to $S_1 = I^*(\mathbb{N}',F)$. So Q'', the maximal left ring of quotients of S, is isomorphic to the ring of all row-finite matrices over F, indexed by $\mathbb{N}$. Now $e_{00}Q''$ is a countable set, and any minimal left ideal of Q'' is generated by an element of $e_{00}Q''$, so the left socle of $e_{00}Q''$ is of countable rank. For S', the left socle is $e_{00}S'$, which is of uncountable rank. Also S' is left nonsingular. So L', the maximal left ring of quotients of S', is such that its left socle is of uncountable rank. This proves that the maximal left rings of quotients of S and S' are not isomorphic.

References

[1] G. Abrams, J. Haefner, and A. del Río, *The isomorphism problem for incidence rings*, Pacific J. Math. **187** (1999), no. 2, 201–214.

[2] F. Al-Thukair, S. Singh, and I. Zaguia, *Maximal ring of quotients of an incidence algebra*, Arch. Math. (Basel) **80** (2003), no. 4, 358–362.

[3] S. Dăscălescu and L. van Wyk, *Do isomorphic structural matrix rings have isomorphic graphs?* Proc. Amer. Math. Soc. **124** (1996), no. 5, 1385–1391.

[4] J. Froelich, *The isomorphism problem for incidence rings*, Illinois J. Math. **29** (1985), no. 1, 142–152.

[5] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999.

[6] U. Leron and A. Vapne, *Polynomial identities of related rings*, Israel J. Math. **8** (1970), 127–137.

[7] E. Spiegel, *Essential ideals of incidence algebras*, J. Austral. Math. Soc. Ser. A **68** (2000), no. 2, 252–260.

[8] E. Spiegel and C. J. O'Donnell, *Incidence Algebras*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 206, Marcel Dekker, New York, 1997.

[9] R. P. Stanley, *Structure of incidence algebras and their automorphism groups*, Bull. Amer. Math. Soc. **76** (1970), 1236–1239.

[10] E. R. Voss, *On the isomorphism problem for incidence rings*, Illinois J. Math. **24** (1980), no. 4, 624–638.

Surjeet Singh: Department of Mathematics, King Saud University, PO Box 2455, Riyadh 11451, Kingdom of Saudi Arabia

E-mail address: ssingh@ksu.edu.sa

Fawzi Al-Thukair: Department of Mathematics, King Saud University, PO Box 2455, Riyadh 11451, Kingdom of Saudi Arabia