POWERS OF A PRODUCT OF COMMUTATORS AS PRODUCTS OF SQUARES

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We prove that for any odd integer N and any integer $n > 0$, the N th power of a product of n commutators in a nonabelian free group of countable infinite rank can be expressed as a product of squares of $2n + 1$ elements and, for all such odd N and integers n , there are commutators for which the number $2n + 1$ of squares is the minimum number such that the N th power of its product can be written as a product of squares. This generalizes a recent result of Akhavan-Malayeri.

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1. Introduction. Lyndon et al. [2] have shown that the product of n commutators in a nonabelian free group can be written as a product of $2n + 1$ squares of elements and there are commutators for which the number $2n + 1$ of squares is the minimum number such that the product of these commutators can be written as a product of squares. Recently, Akhavan-Malayeri [1] proved, for an odd integer n , that $[x, y]^n$ of two distinct elements of a free generating set of a nonabelian free group is not a product of two squares but it is the product of three squares. We generalize these results in the following theorem.

THEOREM 1.1. *Let F be a free group with a basis of distinct elements $x_1, \dots, x_{2n}$, and N any odd integer. Then there exist elements $u_1, \dots, u_m$ in F such that*

$$([x_1, x_2] \cdots [x_{2n-1}, x_{2n}])^N = u_1^2 \cdots u_m^2 \quad (1.1)$$

if and only if $m \geq 2n + 1$.

Note that the theorem for even N is not true since the element in the left-hand side of the above equation is actually a square. The proof of this theorem is almost *mutatis mutandis* as the proof of the main result of [2]. Throughout this note, $[x, y] = x^{-1}y^{-1}xy$ and $[x, y, z] = [[x, y], z]$ for all elements x, y, z of a group G , and G' denotes the derived subgroup of G .

2. Proof of the main result

PROOF OF THEOREM 1.1. We show first that this equation has a solution for $m = 2n + 1$, hence trivially for $m \geq 2n + 1$. Since N is odd, there is an integer k such that $N = 2k + 1$. Thus it is enough to show that, for any element v of F , we can express the element $v^2[x_1, x_2] \cdots [x_{2n-1}, x_{2n}]$ as a product of $2n + 1$ squares. We argue by

induction on n . If $n = 1$, then by the following well-known identity this case is proved:

$$A^2[B, C] = (A^2B^{-1}A^{-1})^2(ABA^{-1}C^{-1}A^{-1})^2(AC)^2. \quad (2.1)$$

Assume $n > 1$ and suppose inductively that

$$v^2[x_1, x_2] \cdots [x_{2n-3}, x_{2n-2}] = u_1^2 \cdots u_{2n-1}^2 \quad (2.2)$$

for some elements $u_1, \dots, u_{2n-1}$ in F . Now by the identity (2.1) we can write

$$u_{2n-1}^2[x_{2n-1}, x_{2n}] = U^2V^2W^2 \quad (2.3)$$

for some elements U, V , and W in F , and so

$$v^2[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] = u_1^2 \cdots u_{2n-2}^2 U^2 V^2 W^2, \quad (2.4)$$

which completes the induction. This first part of the proof is essentially well known in a topological context: the nonorientable surface formed by attaching one cross-cap and n handles to a sphere (the connected sum of 1 projective plane and n tori) is homeomorphic to the surface obtained by attaching $2n + 1$ cross-caps (the connected sum of $2n + 1$ projective planes). In this context, the identity (2.1) is just the handle calculus that says cross-cap + handle = 3 cross-caps.

For the converse, we suppose that the equation holds. Let G be the group with the following presentation:

$$\langle \gamma_i \mid \gamma_i^2 = [\gamma_i, \gamma_j, \gamma_k] = 1 \ \forall i, j, k \in \{1, 2, \dots, 2n\} \rangle. \quad (2.5)$$

The equation would also hold in G since G is a quotient of F . So we have

$$([\gamma_1, \gamma_2] \cdots [\gamma_{2n-1}, \gamma_{2n}])^N = v_1^2 \cdots v_m^2 \quad (2.6)$$

for some elements $v_1, \dots, v_m$ in G . Since N is odd, $N = 2t + 1$ for some integer t . Since G is nilpotent of class 2 and $\gamma_i^2 = 1$ for each i , we have $[\gamma_i, \gamma_j]^2 = 1$ and all the commutators are in the center of G , so the latter equation can be rewritten as

$$[\gamma_1, \gamma_2] \cdots [\gamma_{2n-1}, \gamma_{2n}] = v_1^2 \cdots v_m^2. \quad (2.7)$$

Let $c_{ij} = [\gamma_i, \gamma_j]$. Then each element v of G has a unique expression

$$v = \gamma_1^{a_1} \cdots \gamma_{2n}^{a_{2n}} \prod_{i < j} c_{ij}^{d_{ij}} \quad \text{for } a_i, d_{ij} \in \mathbb{Z}_2. \quad (2.8)$$

Let

$$v_k = \gamma_1^{a_{1k}} \cdots \gamma_{2n}^{a_{2nk}} z_k, \quad (2.9)$$

where $a_{ik} \in \mathbb{Z}_2$ and $z_k \in G'$ for all $i \in \{1, \dots, 2n\}$ and all $k \in \{1, \dots, m\}$. Since $z_k^2 = 1$ for all k , we have

$$v_1^2 \cdots v_m^2 = \prod_{i < j} c_{ij}^{\sum_{k=1}^m a_{ik} a_{jk}}. \quad (2.10)$$

If A is the matrix $A = (a_{ij})$ over $\mathbb{Z}_2$, and $A_i = (a_{i1}, \dots, a_{im})$ is the i th row of A , then from the relation $v_1^2 \cdots v_m^2 = [\gamma_1, \gamma_2] \cdots [\gamma_{2n-1}, \gamma_{2n}]$ we conclude that, taking inner products,

$$A_i \cdot A_j = \begin{cases} 1 & \text{if } \{i, j\} = \{2h-1, 2h\} \text{ for } 1 \leq h \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (2.11)$$

We conclude that $A \cdot A^T = B$, where A^T is the transpose of A , and B is the direct sum of n matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and hence has rank $2n$. It follows that $\text{rank}(A) \geq 2n$. But the equation $A_i \cdot A_i = \sum_{j=1}^m a_{ij}a_{ij} = 0$ for each i implies that the sum of the columns of A is 0, whence $\text{rank}(A) \leq m-1$. Therefore $m-1 \geq 2n$. $\square$

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