POWERS OF A PRODUCT OF COMMUTATORS
AS PRODUCTS OF SQUARES

ALIREZA ABDOLLAHI

Received 1 April 2003

We prove that for any odd integer $N$ and any integer $n > 0$, the $N$th power of a product of $n$ commutators in a nonabelian free group of countable infinite rank can be expressed as a product of squares of $2n + 1$ elements and, for all such odd $N$ and integers $n$, there are commutators for which the number $2n + 1$ of squares is the minimum number such that the $N$th power of its product can be written as a product of squares. This generalizes a recent result of Akhavan-Malayeri.

2000 Mathematics Subject Classification: 20F12, 20F99.

1. Introduction. Lyndon et al. [2] have shown that the product of $n$ commutators in a nonabelian free group can be written as a product of $2n + 1$ squares of elements and there are commutators for which the number $2n + 1$ of squares is the minimum number such that the product of these commutators can be written as a product of squares. Recently, Akhavan-Malayeri [1] proved, for an odd integer $n$, that $[x, y]^n$ of two distinct elements of a free generating set of a nonabelian free group is not a product of two squares but it is the product of three squares. We generalize these results in the following theorem.

**Theorem 1.1.** Let $F$ be a free group with a basis of distinct elements $x_1, \ldots, x_{2n}$, and $N$ any odd integer. Then there exist elements $u_1, \ldots, u_m$ in $F$ such that

$$([x_1, x_2] \cdots [x_{2n-1}, x_{2n}])^N = u_1^2 \cdots u_m^2$$

(1.1)

if and only if $m \geq 2n + 1$.

Note that the theorem for even $N$ is not true since the element in the left-hand side of the above equation is actually a square. The proof of this theorem is almost *mutatis mutandis* as the proof of the main result of [2]. Throughout this note, $[x, y] = x^{-1}y^{-1}xy$ and $[x, y, z] = [[x, y], z]$ for all elements $x, y, z$ of a group $G$, and $G'$ denotes the derived subgroup of $G$.

2. Proof of the main result

**Proof of Theorem 1.1.** We show first that this equation has a solution for $m = 2n + 1$, hence trivially for $m \geq 2n + 1$. Since $N$ is odd, there is an integer $k$ such that $N = 2k + 1$. Thus it is enough to show that, for any element $v$ of $F$, we can express the element $v^2[x_1, x_2] \cdots [x_{2n-1}, x_{2n}]$ as a product of $2n + 1$ squares. We argue by
induction on \( n \). If \( n = 1 \), then by the following well-known identity this case is proved:

\[
A^2[B,C] = (A^2B^{-1}A^{-1})^2 (ABA^{-1}C^{-1}A^{-1})^2 (AC)^2. 
\]  

(2.1)

Assume \( n > 1 \) and suppose inductively that

\[
v^2[x_1,x_2] \cdots [x_{2n-3},x_{2n-2}] = u_1^2 \cdots u_{2n-1}^2 
\]

(2.2)

for some elements \( u_1, \ldots, u_{2n-1} \) in \( F \). Now by the identity (2.1) we can write

\[
u_{2n-1}^2 [x_{2n-1},x_{2n}] = U^2V^2W^2 
\]

(2.3)

for some elements \( U, V, \) and \( W \) in \( F \), and so

\[
v^2[x_1,x_2] \cdots [x_{2n-1},x_{2n}] = u_1^2 \cdots u_{2n-2}^2 U^2V^2W^2, 
\]

(2.4)

which completes the induction. This first part of the proof is essentially well known in a topological context: the nonorientable surface formed by attaching one cross-cap and \( n \) handles to a sphere (the connected sum of 1 projective plane and \( n \) tori) is homeomorphic to the surface obtained by attaching \( 2n + 1 \) cross-caps (the connected sum of \( 2n + 1 \) projective planes). In this context, the identity (2.1) is just the handle calculus that says cross-cap + handle = 3 cross-caps.

For the converse, we suppose that the equation holds. Let \( G \) be the group with the following presentation:

\[
\langle y_i \mid y_i^2 = [y_i,y_j,y_k] = 1 \ \forall i,j,k \in \{1,2,\ldots,2n\} \rangle.
\]

(2.5)

The equation would also hold in \( G \) since \( G \) is a quotient of \( F \). So we have

\[
([y_1,y_2] \cdots [y_{2n-1},y_{2n}])^N = v_1^2 \cdots v_m^2
\]

(2.6)

for some elements \( v_1, \ldots, v_m \) in \( G \). Since \( N \) is odd, \( N = 2t + 1 \) for some integer \( t \). Since \( G \) is nilpotent of class 2 and \( y_i^2 = 1 \) for each \( i \), we have \([y_i,y_j]^2 = 1 \) and all the commutators are in the center of \( G \), so the latter equation can be rewritten as

\[
[y_1,y_2] \cdots [y_{2n-1},y_{2n}] = v_1^2 \cdots v_m^2.
\]

(2.7)

Let \( c_{ij} = [y_i,y_j] \). Then each element \( v \) of \( G \) has a unique expression

\[
v = y_1^{a_1} \cdots y_{2n}^{a_{2n}} \prod_{i<j} c_{ij}^{d_{ij}} \text{ for } a_i,d_{ij} \in \mathbb{Z}_2.
\]

(2.8)

Let

\[
v_k = y_1^{a_{1k}} \cdots y_{2n}^{a_{2nk}} z_k,
\]

(2.9)

where \( a_{ik} \in \mathbb{Z}_2 \) and \( z_k \in G' \) for all \( i \in \{1,\ldots,2n\} \) and all \( k \in \{1,\ldots,m\} \). Since \( z_k^2 = 1 \) for all \( k \), we have

\[
v_1^2 \cdots v_m^2 = \prod_{i<j} c_{ij}^{\sum_{k=1}^{m} a_{ik} a_{jk}}.
\]

(2.10)
If $A$ is the matrix $A = (a_{ij})$ over $\mathbb{Z}_2$, and $A_i = (a_{i1}, \ldots, a_{im})$ is the $i$th row of $A$, then from the relation $v_1^2 \cdots v_m^2 = [y_1, y_2] \cdots [y_{2n-1}, y_{2n}]$ we conclude that, taking inner products,

$$A_i \cdot A_j = \begin{cases} 1 & \text{if } \{i, j\} = \{2h - 1, 2h\} \text{ for } 1 \leq h \leq n, \\ 0 & \text{otherwise.} \end{cases}$$ (2.11)

We conclude that $A \cdot A^T = B$, where $A^T$ is the transpose of $A$, and $B$ is the direct sum of $n$ matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and hence has rank $2n$. It follows that $\text{rank}(A) \geq 2n$. But the equation $A_i \cdot A_i = \sum_{j=1}^{m} a_{ij}a_{ij} = 0$ for each $i$ implies that the sum of the columns of $A$ is 0, whence $\text{rank}(A) \leq m - 1$. Therefore $m - 1 \geq 2n$.

**References**


Alireza Abdollahi: Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran

*E-mail address: a.abdollahi@sci.ui.ac.ir*
Submit your manuscripts at http://www.hindawi.com