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SHAPE-PRESERVING MULTIVARIATE POLYNOMIAL APPROXIMATION IN $C[-1,1]^m$

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We construct multivariate polynomials attached to a function f of m variables, $m \geq 2$, which approximate f with Jackson-type rate involving a multivariate Ditzian-Totik ω_2^{φ}-modulus and preserve some natural kinds of multivariate monotonicity and convexity of function. The result extends the bivariate case in Gal (2002), but does not follow straight from it and requires a careful calculation.

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1. Introduction. In a very recent paper [2] the following approximation result for the bivariate case was proved.

Theorem 1.1. *If $f:[-1,1]\times[-1,1]\to\mathbb{R}$ is continuous, then there exists a sequence of bivariate polynomials $\{P_{n_1,n_2}(f)(x,y);\ n_1,n_2\in\mathbb{N}\}$, where degree $(P_{n_1,n_2}(f)(x,y))\leq n_k$ with respect to the kth variable, $k=1,2$, such that*

$$\|f-P_{n_1,n_2}(f)\|\leq C\omega_2^{\varphi}\left(f;\frac{1}{n_1},\frac{1}{n_2}\right),\quad \forall n_1,n_2\in\mathbb{N}, \tag{1.1}$$

where $C>0$ is independent of f, n_1, and n_2, satisfying, moreover, the following shape-preserving properties:

(i) *if f is convex of order $(0,0)$, then so is $P_{n_1,n_1}(f)$;*
(ii) *if f is simultaneously convex of orders $(-1,0)$, $(0,-1)$, and $(0,0)$ (i.e., totally upper monotone), then so is $P_{n_1,n_2}(f)$;*
(iii) *if f is convex of order $(1,1)$, then so is $P_{n_1,n_2}(f)$;*
(iv) *if f is simultaneously convex of orders $(-1,1)$, $(1,-1)$, $(0,1)$, $(1,0)$, and $(1,1)$ (i.e., totally convex), then so is $P_{n_1,n_2}(f)$.*

Here the convexity of order (m,n) for bivariate functions was defined by Popoviciu in [6, page 78] and $\omega_2^{\varphi}(f;\delta_1,\delta_2)$ is the bivariate modulus of smoothness defined by Ditzian and Totik in [1, Chapter 12].

By using the convexity of order $(n_1,\dots,n_m)$ and the Ditzian-Totik modulus of smoothness $\omega_2^{\varphi}(f;\delta_1,\dots,\delta_m)$ for functions of m variables defined on $[-1,1]^m$, in this paper we extend Theorem 1.1 to the case of functions of m variables, $m>2$.

A simple comparison between the statements of Theorem 1.1(ii), (iv) and Theorem 3.1(ii), (iv), shows that Theorem 3.1 cannot be trivially suggested (by mathematical

induction) from Theorem 1.1, that is, we can say that the $m = 2$ case is not representative for the general case $m \in \mathbb{N}$ (as, e.g., would be the $m = 3$ case). Also, the proof of Theorem 3.1 requires more intricate calculation than in the case of Theorem 1.1, which motivates us in Section 3 to prove the main result in its full generalization.

2. Preliminaries. This section contains the concepts of convexity and of modulus of smoothness for multivariate functions, which will be used in Section 3.

Following the ideas in [6, page 78], we can introduce the following definition.

Definition 2.1. The function $f : [-1,1]^m \to \mathbb{R}$, $m \in \mathbb{N}$, is called convex of order $(n_1,\dots,n_m)$, where $n_i \in \{-1,0,1,2,\dots\}$, $i = \overline{1,m}$, if for any $n_i + 2$ distinct points in $[-1,1]$, $x_1^{(i)} < x_2^{(i)} < \cdots < x_{n_i+2}^{(i)}$, $i = \overline{1,m}$, it follows that

$$\begin{bmatrix} x_1^{(1)}, x_2^{(1)}, \dots, x_{n_1+2}^{(1)} \\ x_1^{(2)}, x_2^{(2)}, \dots, x_{n_2+2}^{(2)} \\ \vdots \\ x_1^{(m)}, x_2^{(m)}, \dots, x_{n_m+2}^{(m)} \end{bmatrix} ; f \Bigg] \geq 0, \tag{2.1}$$

where the above symbol $[\cdot; f]$ means the divided difference of the function f and it is defined (by means of the divided difference of univariate functions) as $[x_1^{(1)},\dots,x_{n_1+2}^{(1)}; [x_1^{(2)},\dots,x_{n_2+2}^{(2)};\dots;[x_1^{(m)},\dots,x_{n_m+2}^{(m)};f]]]$ (here each univariate divided difference $[x_1^{(i)}, \dots,x_{n_i+2}^{(i)};\cdot]$ is considered with respect to the x_i variable, respectively, for all $i = \overline{1,m}$).

Remark. (1) For $m = 2$, we get the concept in [6, page 78].

(2) If f is of $C^{n_1+\cdots+n_m+m}$ class on $[-1,1]^m$, then by the mean value theorem, it follows that the condition

$$\frac{\partial^{n_1+\cdots+n_m+m} f(x_1,\dots,x_m)}{\partial x_1^{n_1+1}\cdots\partial x_m^{n_m+1}} \geq 0 \quad \text{on } [-1,1]^m, \tag{2.2}$$

implies that f is convex of order $(n_1,\dots,n_m)$.

The method in [1, Chapter 12] suggests introducing the following definition.

Definition 2.2. If $f : [-1,1]^m \to \mathbb{R}$, then

$$\begin{aligned} \omega_2^{\varphi}(f;\delta_1,\dots,\delta_m) = \sup\Big\{ \big|\Delta^2_{h_1\varphi(x_1),\dots,h_m\varphi(x_m)} f(x_1,\dots,x_m)\big| ; \\ 0 \leq h_i \leq \delta_i,\ i = \overline{1,m},\ x_1,\dots,x_m \in [-1,1]\Big\}, \end{aligned} \tag{2.3}$$

where $\varphi(t) = \sqrt{1-t^2}$,

$$\begin{aligned} &\Delta^2_{h_1\varphi(x),\dots,h_m\varphi(x)} f(x_1,\dots,x_m) \\ &\quad = \sum_{k=0}^{2} C_k^2 (-1)^k f(x_1 + (1-k)h_1\varphi(x_1),\dots,x_m + (1-k)h_m\varphi(x_m)) \end{aligned} \tag{2.4}$$

if $(x_1 \pm h_1\varphi(x_1),\dots,x_m \pm h_m\varphi(x_m)) \in [-1,1]^m$,

$$\Delta^2_{h_1\varphi(x_1),\dots,h_m\varphi(x_m)} f(x_1,\dots,x_m) = 0 \quad \text{elsewhere}, \tag{2.5}$$

where C_k^2 denotes the usual binomial coefficients.

Remark. For $m = 2$, we get the concept in [1, Chapter 12].

3. Main result. Our main result can be stated as follows.

Theorem 3.1. *If $f : [-1,1]^m \to \mathbb{R}$, $m \geq 2$, is continuous, then there exists a sequence of multivariate polynomials $\{P_{n_1,\dots,n_m}(f)(x_1,\dots,x_m);\ n_1,\dots,n_m \in \mathbb{N}\}$, where degree $(P_{n_1,\dots,n_m}(f)(x_1,\dots,x_m)) \leq n_k$ with respect to the kth variable, $k = \overline{1,m}$, such that*

$$\|f - P_{n_1,\dots,n_m}(f)\| \leq C_m \omega^2_\varphi\left(f; \frac{1}{n_1},\dots,\frac{1}{n_m}\right), \quad \forall n_1,\dots,n_m \in \mathbb{N}, \tag{3.1}$$

where $C_m > 0$ is independent of f and n_i, $i = \overline{1,m}$, satisfying moreover the following shape-preserving properties:

(i) *if f is convex of order $(0,\dots,0)$ on $[-1,1]^m$, then so is $P_{n_1,\dots,n_m}(f)$;*
(ii) *if f is simultaneously convex of orders $(s_1,\dots,s_m) \in \{(s_1,\dots,s_m);\ s_i \in \{-1,0\}, \forall i = \overline{1,m}$ and $\exists k$ with $s_k = 0\}$, then so is $P_{n_1,\dots,n_m}(f)$;*
(iii) *if f is convex of order $(1,\dots,1)$ on $[-1,1]^m$, then so is $P_{n_1,\dots,n_m}(f)$;*
(iv) *if f is simultaneously convex of orders $(s_1,\dots,s_m) \in \{(s_1,\dots,s_m); s_i \in \{-1,0,1\}, \forall i = \overline{1,m}$ and $\exists k$ with $s_k = 1\}$, then so is $P_{n_1,\dots,n_m}(f)$.*

Proof. If $g : [-1,1] \to \mathbb{R}$, then, according to [4, relation (5)], the approximation polynomials are given by

$$P_n(g)(x) = g(-1) + \sum_{j=0}^{n-1} s_{j,n}\big(R_{j,n}(x) - R_{j+1,n}(x)\big), \tag{3.2}$$

where $s_{j,n} = (g(\xi_{j+1,n}) - g(\xi_{j,n}))/(\xi_{j+1,n} - \xi_{j,n})$, $\xi_{j,n}$, $j = \overline{0,n}$, are suitable nodes in $[-1,1]$, and $R_{j,n}(x)$ are suitable chosen polynomials of degree less than or equal to n.

According to [4, Theorem 1], we have

$$\|g - P_n(g)\| \leq C\omega^\varphi_2\left(g; \frac{1}{n}\right), \quad \forall n \in \mathbb{N}, \tag{3.3}$$

where $\omega^\varphi_2(g;\delta)$ is the usual Ditzian-Totik modulus of smoothness and $C > 0$ is independent of g and n.

We will construct the polynomials $P_{n_1,\dots,n_m}(f)(x_1,\dots,x_m)$ by applying the tensor-product method (see, e.g., [5, pages 195–296]). We obtain (by mathematical induction)

$$
\begin{aligned}
&P_{n_1,\dots,n_m}(f)(x_1,\dots,x_m)\\
&\quad= f(-1,\dots,-1)+\sum_{k=1}^{m}\left\{\sum_{i_k=0}^{n_k-1}[\cdot;f]_k\cdot\left[R_{i_k,n_k}(x_k)(x_k)-R_{i_k+1,n_k}(x_k)\right]\right\}\\
&\qquad+\sum_{\substack{k,j=1\\k<j}}^{m}\left\{\sum_{i_k=0}^{n_k-1}\sum_{i_j=0}^{n_j-1}[\cdot;f]_{j,k}\cdot\left[R_{i_k,n_k}(x_k)-R_{i_k+1,n_k}(x_k)\right]\right.\\
&\qquad\qquad\left.\cdot\left[R_{i_j,n_j}(x_j)-R_{i_j+1,n_j}(x_j)\right]\right\}\\
&\qquad+\cdots+\sum_{\substack{p_1,\dots,p_{m-1}=1\\p_1<\cdots<p_{m-1}}}^{m}\left\{\sum_{i_{p_1}=0}^{n_{p_1}-1}\cdots\sum_{i_{p_{m-1}}=0}^{n_{p_{m-1}}-1}[\cdot;f]_{p_1,\dots,p_{m-1}}\right.\\
&\qquad\qquad\left.\cdot\prod_{s=1}^{m-1}\left[R_{i_{p_s},n_{p_s}}(x_{p_s})-R_{i_{p_s}+1,n_{p_s}}(x_{p_s})\right]\right\}\\
&\qquad+\sum_{i_1=0}^{n_1-1}\cdots\sum_{i_m=0}^{n_m-1}[\cdot;f]_{1,\dots,m}\cdot\prod_{k=1}^{m}\left[R_{i_k,n_k}(x_k)-R_{i_k+1,n_k}(x_k)\right],
\end{aligned}
\tag{3.4}
$$

where $\xi_{i_k,n_k}^{(k)}$, $R_{i_k,n_k}(x_k)$, $i_k=\overline{0,n_k}$, $k=1,2,\dots,m$, are constructed as in the univariate case in [4]. The value of f on the point $(-1,\dots,-1)$ by definition can be represented as a divided difference by

$$
f(-1,\dots,-1)=\begin{bmatrix}-1\\ \vdots & ;f\\ -1\end{bmatrix},
\tag{3.5}
$$

where the notation in the right-hand side denotes the divided difference of f with m lines (see Definition 2.1) with -1 on each line. Then $[\cdot;f]_k$ denotes the divided difference obtained from the above by replacing the line k (which has only one node, -1) with another one composed of the two nodes $\xi_{i_k,n_k}^{(k)}$ and $\xi_{i_k+1,n_k}^{(k)}$ (the rest of lines remain unchanged), also $[\cdot;f]_{k,j}$, $k\neq j$, denotes the divided difference obtained (from the same divided difference which represents $f(-1,\dots,-1)$) by replacing the lines k and j (which have only the node -1) by lines composed of the pairs of nodes $\xi_{i_k,n_k}^{(k)}$, $\xi_{i_k+1,n_k}^{(k)}$, and $\xi_{i_j,n_j}^{(j)}$, $\xi_{i_j+1,n_j}^{(j)}$, respectively, and so on.

Note that, finally,

$$
[\cdot;f]_{1,\dots,m}=\begin{bmatrix}\xi_{i_1,n_1}^{(1)},\xi_{i_1+1,n_1}^{(1)}\\ \vdots & ;f\\ \xi_{i_m,n_m}^{(m)},\xi_{i_m+1,n_m}^{(m)}\end{bmatrix},
\tag{3.6}
$$

that is, it is a divided difference with m lines, having two nodes on each line.

Obviously, degree $(P_{n_1,\dots,n_m}(f))\le n_k$ with respect to the kth variable, $k=\overline{1,m}$.

Firstly, we prove the estimate in Theorem 3.1.

For any univariate function g, we have

$$\|P_n(g)\| \le \|g\| + \|P_n(g) - g\| \le \|g\| + C\omega_2^{\varphi}\left(g; \frac{1}{n}\right) \le (1+2C)\|g\|, \tag{3.7}$$

that is, passing to supremum with $\|g\| \le 1$, for the linear operator P_n, we obtain

$$\big|\|P_n\|\big| \le (1+2C), \quad \forall n \in \mathbb{N}, \tag{3.8}$$

where $c > 0$ is independent of n.

Now applying [3, Theorem 5], we immediately get

$$\|f - P_{n_1,\dots,n_m}(f)\| \le C \sum_{i=1}^{m} \omega_{2,x_i}^{\varphi}\left(f; \frac{1}{n_i}\right), \tag{3.9}$$

where $\omega_{2,x_i}^{\varphi}(f;\delta_i)$, $i = \overline{1,m}$, are the partial moduli of smoothness defined by

$$\begin{aligned} \omega_{2,x_i}^{\varphi}(f;\delta_i) = \sup\Big\{ \big|\Delta_{h_i\varphi(x_i)}^{2,x_i} f(x_1,\dots,x_m)\big|; \\ 0 \le h_i \le \delta_i, x_i,\dots,x_m \in [-1,1]\Big\}, \end{aligned} \tag{3.10}$$

where $\varphi(t) = \sqrt{1-t^2}$,

$$\begin{aligned} &\Delta_{h_i\varphi(x_i)}^{2,x_i} f(x_i,\dots,x_m) \\ &\quad = \sum_{k=0}^{2} C_k^2 (-1)^k f(x_1,\dots,x_{i-1}, x_i + (1-k)h_i\varphi(x_i), x_{i+1},\dots,x_m) \end{aligned} \tag{3.11}$$

if $x_1,\dots,x_{i-1}, x_i \pm h_i\varphi(x_i), x_{i+1},\dots,x_m \in [-1,1]$,

$$\Delta_{h_i\varphi(x_i)}^{2,x_i} f(x_1,\dots,x_m) = 0 \quad \text{elsewhere}. \tag{3.12}$$

Taking into account that obviously

$$\sum_{i=1}^{m} \omega_{2,x_i}^{\varphi}\left(f; \frac{1}{n_i}\right) \le m\omega_2^{\varphi}\left(f; \frac{1}{n_1},\dots,\frac{1}{n_m}\right), \tag{3.13}$$

we obtain the desired estimate.

In what follows we will prove the shape-preserving properties.

(i) Suppose f is convex of order $(0,\dots,0)$. According to Remark (2) in Section 2, we have to prove that

$$\frac{\partial^m P_{n_1,\dots,n_m}(f)(x_1,\dots,x_m)}{\partial x_1 \cdots \partial x_m} \ge 0 \quad \text{on } [-1,1]^m. \tag{3.14}$$

We get

$$\begin{aligned} &\frac{\partial^m P_{n_1,\dots,n_m}(f)(x_1,\dots,x_m)}{\partial x_1 \cdots \partial x_m} \\ &\quad = \sum_{i_1=0}^{n_1-1} \cdots \sum_{i_m=0}^{n_m-1} [\cdot;f]_{1,\dots,m} \prod_{k=1}^{m} \left[\frac{\partial R_{i_k,n_k}(x_k)}{\partial x_k} - \frac{\partial R_{i_k+1,n_k}(x_k)}{\partial x_k}\right] \\ &\quad \ge 0 \end{aligned} \tag{3.15}$$

because $[\cdot;f]_{1,\dots,m} \geq 0$ (f is convex of order $(0,\dots,0)$), and from the univariate case, each $R_{i_k,n_k}(x_k) - R_{i_k+1,n_k}(x_k)$, $k=\overline{1,m}$, is increasing as a function of $x_k \in [-1,1]$.

(ii) By hypothesis on f, it follows that all the quantities $[\cdot,f]_k, [\cdot,f]_{k,j} \cdots [\cdot;f]_{p_1,\dots,p_{m-1}}$, and $[\cdot;f]_{1,\dots,m}$ in the expression of $P_{n_1,\dots,n_m}(f)$ are greater than or equal to 0.

By $R_{i_k,n_k}(-1)=0$, for all $i_k=\overline{0,n_k}$, $k=\overline{1,m}$, we immediately get

$$R_{i_k,n_k}(x_k) - R_{i_k+1,n_k}(x_k) \geq 0, \quad \forall i_k = \overline{0,n_k-1},\ x_k \in [-1,1],\ k=\overline{1,m}. \tag{3.16}$$

Also, from the univariate case, we have

$$R'_{i_k,n_k}(x_k) - R'_{i_k+1,n_k}(x_k) \geq 0, \quad \forall i_k = \overline{0,n_k-1},\ x_k \in [-1,1],\ k=\overline{1,m}. \tag{3.17}$$

Let $(s_i,\dots,s_m) \in \{(s_1,\dots,s_m),\ s_i \in \{-1,0\},\ \forall i=\overline{1,m},\ \exists k \text{ with } s_k=0\}$. The above hypothesis and simple calculations (similar to those in the bivariate case, see the proof of [2, Theorem 3.1(ii), pages 30-31]) immediately imply that $P_{n_1,\dots,n_m}(f)(x_1,\dots,x_m)$ is convex of order $(s_1,\dots,s_m)$, which proves (ii).

(iii) We have to prove that

$$\frac{\partial^{2m} P_{n_1,\dots,n_m}(f)(x_1,\dots,x_m)}{\partial x_1^2 \cdots \partial x_m^2} \geq 0. \tag{3.18}$$

Applying with respect to each variable the relation in the univariate case (see [4, (5), page 473]), that is,

$$P_n(g)(x) = g(-1) + \sum_{j=0}^{n-1} s_{j,n}\big(R_{j,n}(x) - R_{j+1,n}(x)\big), \tag{3.19}$$

where

$$s_{j,n} = \frac{g(\xi_{j+1,n}) - g(\xi_{j,n})}{\xi_{j+1,n} - \xi_{j,n}}, \tag{3.20}$$

we get

$$\begin{aligned}
&P_{n_1,\dots,n_m}(f)(x_1,\dots,x_m) \\
&\quad = f(-1,\dots,-1) + \sum_{k=1}^{m} (1+x_k)C_k + F(x_1,\dots,x_m) + E(x_1,\dots,x_m) \\
&\qquad + \sum_{i_1=1}^{n_1-1} \cdots \sum_{i_m=1}^{n_m-1} \left(\xi^{(1)}_{i_1+1,n_1} - \xi^{(1)}_{i_1,n_1}\right) \cdots \left(\xi^{(m)}_{i_m+1,n_m} - \xi^{(m)}_{i_m,n_m}\right) \left(\prod_{k=1}^{m} R_{i_k,n_k}(x_k)\right) \\
&\qquad \cdot \begin{bmatrix} \xi^{(1)}_{i_1-1,n_1}, \xi^{(1)}_{i_1,n_1}, \xi^{(1)}_{i_1+1,n_1} \\ \vdots \\ \xi^{(m)}_{i_m-1,n_m}, \xi^{(m)}_{i_m,n_m}, \xi^{(m)}_{i_m+1,n_m} \end{bmatrix} ; f \Bigg],
\end{aligned} \tag{3.21}$$

where $F(x_1,\dots,x_m)$ is a sum of several expressions of the form $(1+x_{i_1})\cdots(1+x_{i_k})C$ with distinct indexes $i_1,\dots,i_k$, $k<m$, C a real constant (which can be different at each occurrence) and $E(x_1,\dots,x_m)$ is a sum of several expressions, each expression being a simple or multiple sum of terms, where each term is represented by product between:

(a) various $R_{i_k,n_k}(x_k)$, or a product of distinct $R_{i_k,n_k}(x_k)$ (having at most $m-1$ terms in that product),

(b) a divided difference of f on one, two, or three nodes with respect to each variable x_k, such that at least with respect to one variable the divided difference is taken on three nodes,

(c) a positive quantity of the form $(\xi^{(k)}_{i_k+1,n_k}-\xi^{(k)}_{i_k,n_k})$ or $(\xi^{(k)}_{i_k+1,n_k}-\xi^{(k)}_{i_k-1,n_k})$ or the product of such distinct quantities.

Moreover, the above-mentioned expressions in $E(x_1,\dots,x_m)$ which depend on the variables x_k, $k=\overline{1,m}$, through $R_{i_k,n_k}(x_k)$, are of two kinds:

(1) expressions which do not depend on all variables x_k, $k=\overline{1,m}$;

(2) expressions which depend on all variables x_k, $k=\overline{1,m}$, but at least one $R_{i_k,n_k}(x_k)=1+x_k$.

We exemplify the passing from $m=2$ to $m=3$. Therefore, let f be a function of three variables, that is, $f=f(x_1,x_2,x_3)$.

Applying the formula in the univariate case (specified at the beginning of (iii)) with respect to the variables x_1 and x_2, and by the formulas in [2, pages 31-32], we immediately get

$$\begin{aligned}
&P_{n_1,n_2}(f)(x_1,x_2,x_3)\\
&\quad=f(-1,-1,x_3)+(1+x_2)\begin{bmatrix}-1\\ \\ \xi^{(2)}_{1,n_2},\xi^{(2)}_{0,n_2}\end{bmatrix};f\Bigg]+(1+x_1)\begin{bmatrix}\xi^{(1)}_{1,n_1},\xi^{(1)}_{0,n_1}\\ \\ -1\end{bmatrix};f\Bigg]\\
&\qquad+\sum_{i_2=1}^{n_2-1}\left(\xi^{(2)}_{i_2+1,n_2}-\xi^{(2)}_{i_2-1,n_2}\right)\begin{bmatrix}-1\\ \\ \xi^{(2)}_{i_2-1,n_2},\xi^{(2)}_{i_2,n_2},\xi^{(2)}_{i_2+1,n_2}\end{bmatrix};f\Bigg]R_{i_2,n_2}(x_2)\\
&\qquad+\sum_{i_1=1}^{n_1-1}\left(\xi^{(1)}_{i_1+1,n_1}-\xi^{(1)}_{i_1-1,n_1}\right)\begin{bmatrix}\xi^{(1)}_{i_1-1,n_1},\xi^{(1)}_{i_1,n_1},\xi^{(1)}_{i_1+1,n_1}\\ \\ -1\end{bmatrix};f\Bigg]R_{i_1,n_1}(x_1)\\
&\qquad+(1+x_1)(1+x_2)S^{*}_{0,0}(f)(x_3)\\
&\qquad+(1+x_1)\sum_{i_2=1}^{n_2-1}\left(\xi^{(2)}_{i_2+1,n_2}-\xi^{(2)}_{i_2-1,n_2}\right)\begin{bmatrix}-1,\xi^{(1)}_{1,n_1}\\ \\ \xi^{(2)}_{i_2-1,n_2},\xi^{(2)}_{i_2,n_2},\xi^{(2)}_{i_2+1,n_2}\end{bmatrix};f\Bigg]R_{i_2,n_2}(x_2)\\
&\qquad+(1+x_2)\sum_{i_1=1}^{n_1-1}\left(\xi^{(1)}_{i_1+1,n_1}-\xi^{(1)}_{i_1-1,n_1}\right)\begin{bmatrix}\xi^{(1)}_{i_1-1,n_1},\xi^{(1)}_{i_1,n_1},\xi^{(1)}_{i_1+1,n_1}\\ \\ -1,\xi^{(2)}_{1,n_2}\end{bmatrix};f\Bigg]R_{i_1,n_1}(x_1)
\end{aligned}$$

$$+\sum_{i_1=1}^{n_1-1}\sum_{i_2=1}^{n_2-1}\left(\xi_{i_1+1,n_1}^{(1)}-\xi_{i_1,n_1}^{(1)}\right)\left(\xi_{i_2+1,n_2}^{(2)}-\xi_{i_2,n_2}^{(2)}\right)\begin{bmatrix}\xi_{i_1-1,n_1}^{(1)},\xi_{i_1,n_1}^{(1)},\xi_{i_1+1,n_1}^{(1)} & \\ & ;f \\ \xi_{i_2-1,n_2}^{(2)},\xi_{i_2,n_2}^{(2)},\xi_{i_2+1,n_2}^{(2)} & \end{bmatrix}$$
$$\cdot R_{i_1,n_1}(x_1)R_{i_2,n_2}(x_2), \tag{3.22}$$

where all the divided differences are considered with respect to the variables x_1, x_2, and x_3 is arbitrarily fixed. Also, recall that the formula for $S_{0,0}^{*}(f)(x_3)$ is given by [2, page 31] and depends on the values of $f(\cdot,\cdot,x_3)$ on some nodes, where f is considered as a function of the variables x_1 and x_2.

Now, applying the formula in the univariate case with respect to x_3 to $P_{n_1,n_2}(f)(x_1, x_2,x_3)$, that is, to each term of it, and taking into account the recurrence formula satisfied by the divided differences, we immediately obtain $P_{n_1,n_2,n_3}(f)(x_1,x_2,x_3)$ of the claimed form.

As a conclusion, all these immediately imply that

$$\begin{aligned}\frac{\partial^{2m}P_{n_1,\dots,n_m}(f)(x_1,\dots,x_m)}{\partial x_1^2\cdots\partial x_m^2} &= \sum_{i_1=1}^{n_1-1}\cdots\sum_{i_m=1}^{n_m-1}\left(\prod_{k=1}^{m}\left(\xi_{i_k+1,n_k}^{(k)}-\xi_{i_k,n_k}^{(k)}\right)\right)\left(\prod_{k=1}^{m}R''_{i_k,n_k}(x_k)\right)\\ &\quad\cdot\begin{bmatrix}\xi_{i_1-1,n_1}^{(1)},\xi_{i_1,n_1}^{(1)},\xi_{i_1+1,n_1}^{(1)} & \\ \vdots & ;f \\ \xi_{i_m-1,n_m}^{(m)},\xi_{i_m,n_m}^{(m)},\xi_{i_m+1,n_m}^{(m)} & \end{bmatrix}\geq 0\end{aligned} \tag{3.23}$$

by the hypothesis on f and by the conditions $R''_{i_k,n_k}(x_k)\geq 0$, for all $i_k=\overline{0,n_k}$, $x_k\in[-1,1]$, $k=\overline{1,m}$ (see [4]).

(iv) Firstly, we recall that by construction we have (see [4])

$$\begin{gathered}R_{i_k,n_k}(x_k)\geq 0,\quad R'_{i_k,n_k}(x_k)\geq 0,\quad R''_{i_k,n_k}(x_k)\geq 0,\\ \forall\, i_k=\overline{0,n_k},\quad x_k\in[-1,1],\quad k=\overline{1,m}.\end{gathered} \tag{3.24}$$

We have to check the inequalities

$$\frac{\partial^{r}P_{n_1,\dots,n_m}(f)(x_1,\dots,x_m)}{\partial x_{i_1}^{r_1}\cdots\partial x_{i_p}^{r_p}}\geq 0\quad\text{on } [-1,1]^m, \tag{3.25}$$

for all $r\in\{2,\dots,m\}$, $p\in\{1,\dots,m\}$, $i_k\neq i_j$, if $i\neq j$, $r=r_1+\cdots+r_p$, where at least one r_l is equal to 2 and $r_k\in\{0,1,2\}$, $k=\overline{1,p}$.

By hypothesis, the divided differences of f which contains, at least on a line, three distinct points, all are greater than or equal to 0. Then, taking into account the forms

of $F(x_1,\dots,x_m)$ and $E(x_1,\dots,x_m)$ described at the previous point (iii), we immediately obtain the required conclusion. □

Remark. (1) For $m=2$, we recall [2, Theorem 3.1].

(2) Since in the univariate case (i.e., $m=1$) the property in Theorem 3.1(i) reduces to the usual increasing monotonicity and in this case by [7] we know that $\omega_\varphi^2(f;\cdot)$ cannot be replaced by higher-order moduli of smoothness $\omega_\varphi^k(f;\cdot)$ with $k\geq 3$, it follows that for arbitrary $m\geq 2$, the same phenomenon is expected.

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