

## ON THE PUTNAM-FUGLEDE THEOREM

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We extend the Putnam-Fuglede theorem and the second-degree Putnam-Fuglede theorem to the nonnormal operators and to an elementary operator under perturbation by quasinilpotents. Some asymptotic results are also given.

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**1. Introduction.** Let  $H$  be a complex Hilbert space and let  $B(H)$  be the Banach algebra consisting of all the bounded linear operators on  $H$ . For the normal operators, we have the following well-known Putnam-Fuglede (PF) theorem [7].

**THEOREM 1.1.** *If  $N, M$  are normal operators in  $B(H)$ , and if  $X \in B(H)$  such that  $NX = XM$ , then  $N^*X = XM^*$ .*

Putnam [7] also obtained another important result that we call the second-degree PF (SPF) theorem.

**THEOREM 1.2.** *If  $N, M$  are normal operators in  $B(H)$ , and if  $X \in B(H)$  such that  $N(NX - XM) = (NX - XM)M$ , then  $NX = XM$ .*

If we let  $\mathbf{A} = (N_1, N_2)$  and  $\mathbf{B} = (M_1, M_2)$  denote tuples of commuting operators in  $B(H)$ , and define the elementary operators  $\Delta_{(\mathbf{A}, \mathbf{B})}$  and  $\Delta_{(\mathbf{A}^*, \mathbf{B}^*)} \in B(B(H))$  by

$$\begin{aligned}\Delta_{(\mathbf{A}, \mathbf{B})}(X) &= N_1 X N_2 - M_1 X M_2, \\ \Delta_{(\mathbf{A}^*, \mathbf{B}^*)}(X) &= N_1^* X N_2^* - M_1^* X M_2^*,\end{aligned}\tag{1.1}$$

then an extension of the classical PF theorem, [Theorem 1.1](#), is obtained as follows (see [\[4, 5\]](#)).

**THEOREM 1.3.** *If the operators  $N_i, M_i \in B(H)$ ,  $i = 1, 2$ , are normal, then  $\Delta_{(\mathbf{A}, \mathbf{B})}(X) = 0$  for some  $X \in B(H)$  implies  $\Delta_{(\mathbf{A}^*, \mathbf{B}^*)}(X) = 0$ .*

Let  $\mathbf{A} = (N_1, N_2)$  and  $\mathbf{B} = (M_1, M_2)$ . For  $n = 2, 3, \dots$ , we define the high-order elementary operator  $\Delta_{(\mathbf{A}, \mathbf{B})}^{(n)}$  by

$$\Delta_{(\mathbf{A}, \mathbf{B})}^{(n)}(X) = \Delta_{(\mathbf{A}, \mathbf{B})}(\Delta_{(\mathbf{A}, \mathbf{B})}^{(n-1)}(X)), \quad X \in B(H).\tag{1.2}$$

### 2. Putnam-Fuglede theorem under perturbation by quasinilpotents

**THEOREM 2.1.** *Let  $A, B$  be normal operators, and let  $C, D$  be quasinilpotents such that  $AC = CA, BD = DB$ . If  $(A + C)X = X(B + D)$  for some  $X \in B(H)$ , then  $AX = XB$ .*

**PROOF.** If  $(A + C)X = X(B + D)$ , then  $AX - XB = -(CX - XD)$ . For any  $N, M \in B(H)$ , denote by  $\delta_{NM}$  the linear operator on  $B(H)$ :

$$\delta_{NM}(X) = NX - XM; \tag{2.1}$$

then  $\delta_{AB}(X) = -\delta_{CD}(X)$ , so

$$\delta_{AB}^{(n)}(X) = (-1)^n \delta_{CD}^{(n)}(X). \tag{2.2}$$

Since  $\sigma(\delta_{CD}) = \sigma(C) - \sigma(D) = \{0\}$  (see [6]), we have  $\sqrt[n]{\|\delta_{CD}^{(n)}\|} \rightarrow 0$ . But

$$\sqrt[n]{\|\delta_{AB}^{(n)}(X)\|} \leq \sqrt[n]{\|\delta_{CD}^{(n)}\|} \sqrt[n]{\|X\|}, \tag{2.3}$$

so  $\sqrt[n]{\|\delta_{AB}^{(n)}(X)\|} \rightarrow 0$ . The theorem follows by a result of Anderson and Foias [1] which says that if  $A, B$  are normal operators, and  $\sqrt[n]{\|\delta_{AB}^{(n)}(X)\|} \rightarrow 0$ , then  $AX - XB = 0$ .  $\square$

**REMARK 2.2.** With the operators  $A$  and  $B$  being normal, it follows from [Theorem 2.1](#) that  $(A + C)X = X(B + D) \Rightarrow (A^* + C)X = X(B^* + D)$ . It is, however, not true in general that  $(A + C)^*X = X(B + D)^*$  (see [9]).

We give now a simple application of [Theorem 2.1](#).

**COROLLARY 2.3.** *Let  $N$  be a normal operator and let  $C$  be a quasinilpotent that commutes with  $N$ . If  $f$  is a polynomial of degree  $n$  such that  $f(N + C) = 0$ , then  $f^{(k)}(N)C^k = 0$  for  $k = 0, 1, \dots, n$ . So  $C$  is nilpotent of order at most  $n$ . Moreover, if  $f$  has no multiple root, then  $C = 0$ .*

**PROOF.** It is easy to see that

$$f(N + C) = f(N) + f'(N)C + \frac{f''(N)}{2!}C^2 + \dots + \frac{f^{(n)}(N)}{n!}C^n. \tag{2.4}$$

Applying [Theorem 2.1](#) to (2.4), we have  $f(N) = 0$  and

$$f'(N)C + \frac{f''(N)}{2!}C^2 + \dots + \frac{f^{(n)}(N)}{n!}C^n = 0, \tag{2.5}$$

or

$$\left(f'(N) + \frac{f''(N)}{2!}C + \dots + \frac{f^{(n)}(N)}{n!}C^{n-1}\right)C = 0. \tag{2.6}$$

Applying [Theorem 2.1](#) again to (2.6) yields  $f'(N)C = 0$  and

$$\left(\frac{f''(N)}{2!} + \dots + \frac{f^{(n)}(N)}{n!}C^{n-2}\right)C^2 = 0. \tag{2.7}$$

So we have  $(f''(N)/2!)C^2 = 0, \dots, (f^{(n)}(N)/n!)C^n = 0$ .

If  $f$  has no multiple root, then it follows from  $f(N) = 0$  that  $f'(N)$  is invertible. As  $f'(N)C = 0$ , we know immediately that  $C = 0$ .  $\square$

**LEMMA 2.4.** *Let  $C, M \in B(H)$ . If  $C$  is quasinilpotent, then the only solution  $X \in B(H)$  of  $X = CXM$  is  $X = 0$ .*

**PROOF.** If  $X = CXM$ , we have, for  $n = 2, 3, \dots$ ,  $X = C^n X M^n$ , so

$$\|X\| \leq \|C^n\| \|X\| \|M^n\| \leq \|C^n\| \|X\| \|M\|^n. \tag{2.8}$$

But with  $C$  being quasinilpotent, it follows that

$$\sqrt[n]{\|C^n\| \|M\|^n} = \sqrt[n]{\|C^n\|} \|M\| \rightarrow 0, \quad n \rightarrow \infty. \tag{2.9}$$

Thus  $\|C^n\| \|M\|^n \rightarrow 0$ , so  $X = 0$  by (2.8). □

**LEMMA 2.5.** *Let  $N$  be a normal operator and let  $C, D$  be quasinilpotents such that  $N, C, D$  mutually commute. If  $M \in B(H)$ , and  $(N + C)X(N + C) = MXD$  for some  $X \in B(H)$ , then  $NXN = 0$ .*

**PROOF.** Suppose that  $X \in B(H)$  such that  $(N + C)X(N + C) = MXD$ . If the kernel  $\text{Ker}(N) \neq \{0\}$ , then letting  $P$  be the project from  $H$  to  $\text{Ker}(N)$ , we have  $NPXN = 0, NXP N = 0$ . Therefore, to prove  $NXN = 0$ , it is sufficient to prove  $NP^\perp X P^\perp N = 0$ . Thus we can assume that  $\text{Ker}(N) = \{0\}$ . Let

$$N = \int_{\sigma(N)} \lambda dE_\lambda \tag{2.10}$$

be the spectral decomposition of  $N$ . Define  $\Delta_\epsilon = \{z \mid |z| \leq \epsilon\}$ ,  $\Delta_\epsilon^c = \mathbf{C} \setminus \Delta_\epsilon$ , and  $T_\epsilon = E(\Delta_\epsilon^c)T|_{E(\Delta_\epsilon^c)H}$  for any  $T \in B(H)$ , then we have

$$(N_\epsilon + C_\epsilon)X_\epsilon(N_\epsilon + C_\epsilon) = M_\epsilon X_\epsilon D_\epsilon, \tag{2.11}$$

but  $N_\epsilon$  is invertible, so

$$(N_\epsilon + C_\epsilon)^{-1} = N_\epsilon^{-1} + C_\epsilon^o, \tag{2.12}$$

where  $C_\epsilon^o$  is also quasinilpotent, and

$$X_\epsilon = (N_\epsilon + C_\epsilon)^{-1} M_\epsilon X_\epsilon D_\epsilon (N_\epsilon + C_\epsilon)^{-1}. \tag{2.13}$$

Because  $D_\epsilon(N_\epsilon + C_\epsilon)^{-1}$  is quasinilpotent, by Lemma 2.4, we have  $X_\epsilon = 0$ . Letting  $\epsilon \rightarrow 0$ , we have  $X = 0$ , so  $NXN = 0$ . This completes the proof. □

**LEMMA 2.6.** *Let  $N$  be a normal operator and let  $C$  be quasinilpotent such that  $NC = CN$ . If  $(N + C)X(N + C) = X$  for some  $X \in B(H)$ , then  $NXN = X$ .*

**PROOF.** If  $\text{Ker}(N) \neq \{0\}$ , then let  $P$  be the project  $H \rightarrow \text{Ker}(N)$ . If  $(N + C)X(N + C) = X$  for some  $X \in B(H)$ , then  $P(N + C)X(N + C) = PX$ , so  $CPX(N + C) = PX$ , but since  $C$  is quasinilpotent, by Lemma 2.4, we have  $PX = 0$ . The same way shows that  $XP = 0$ . Therefore, we may assume  $\text{Ker}(N) = \{0\}$ .

Let  $N = \int_{\sigma(N)} \lambda dE_\lambda$  be the spectral decomposition of  $N$ . Define  $\Delta_\epsilon$ ,  $\Delta_\epsilon^\epsilon$ , and  $T_\epsilon$  to be the same as in [Lemma 2.5](#). Then

$$(N_\epsilon + C_\epsilon)X_\epsilon(N_\epsilon + C_\epsilon) = X_\epsilon \tag{2.14}$$

or

$$(N_\epsilon + C_\epsilon)X_\epsilon = X_\epsilon(N_\epsilon + C_\epsilon)^{-1} = X_\epsilon(N_\epsilon^{-1} + C_\epsilon^o), \tag{2.15}$$

where  $C_\epsilon^o$  is quasinilpotent. So by [Theorem 2.1](#),  $N_\epsilon X_\epsilon = X_\epsilon N_\epsilon^{-1}$ , or  $X_\epsilon = N_\epsilon X_\epsilon N_\epsilon$ . Letting  $\epsilon \rightarrow 0$ , we have  $NXN = X$ . □

Using the same technique as in the proof of [Lemma 2.6](#), we are able to obtain the following theorem.

**THEOREM 2.7.** *Let  $N, M$  be normal operators and let  $C, D$  be quasinilpotents such that  $NC = CN$  and  $MD = DM$ . If  $(N + C)X(N + C) = (M + D)X(M + D)$  for some  $X \in B(H)$ , then  $NXN = MXM$ .*

**PROOF.** If  $\text{Ker}(N) \neq \{0\}$ , then let  $P$  be the project:  $H \mapsto \text{Ker}(N)$ . If  $(N + C)X(N + C) = (M + D)X(M + D)$  for some  $X \in B(H)$ , then  $P(N + C)X(N + C) = P(M + D)X(M + D)$ , that is,  $CPX(N + C) = (M + D)PX(M + D)$ . Since  $C$  is quasinilpotent, by [Lemma 2.5](#), we have  $MPXM = 0$ . The same method shows that  $MXPM = 0$ . Therefore, we can assume that  $\text{Ker}(N) = \{0\}$ .

Let  $N = \int_{\sigma(N)} \lambda dE_\lambda$  be the spectral decomposition of  $N$ . Define  $\Delta_\epsilon$ ,  $\Delta_\epsilon^\epsilon$ , and  $T_\epsilon$  to be the same as in [Lemma 2.5](#). Then

$$(N_\epsilon + C_\epsilon)X_\epsilon(N_\epsilon + C_\epsilon) = (M_\epsilon + D_\epsilon)X_\epsilon(M_\epsilon + D_\epsilon). \tag{2.16}$$

If we write  $(N_\epsilon + C_\epsilon)^{-1} = N_\epsilon^{-1} + C_\epsilon^o$ , where  $C_\epsilon^o$  is quasinilpotent, then the above equation becomes

$$X_\epsilon = (N_\epsilon^{-1} + C_\epsilon^o)(M_\epsilon + D_\epsilon)X_\epsilon(M_\epsilon + D_\epsilon)(N_\epsilon^{-1} + C_\epsilon^o) \tag{2.17}$$

or

$$X_\epsilon = (N_\epsilon^{-1}M_\epsilon + F_\epsilon)X_\epsilon(N_\epsilon^{-1}M_\epsilon + F_\epsilon), \tag{2.18}$$

where  $F_\epsilon$  is quasinilpotent. Applying [Lemma 2.6](#) to the equation yields  $X_\epsilon = N_\epsilon^{-1}M_\epsilon X_\epsilon N_\epsilon^{-1}M_\epsilon$  or  $N_\epsilon X_\epsilon N_\epsilon = M_\epsilon X_\epsilon M_\epsilon$ . Letting  $\epsilon \rightarrow 0$ , we have  $NXN = MXM$ . □

More generally, using Berberian’s trick, we obtain the PF theorem under perturbation by quasinilpotents for the elementary operators.

**THEOREM 2.8.** *Let  $N_1, N_2, M_1, M_2$  be normal operators and let  $C_1, C_2, D_1, D_2$  be quasinilpotents such that  $N_i, M_i, C_i, D_i$  mutually commute for  $i = 1, 2$ . If  $(N_1 + C_1)X(N_2 + C_2) = (M_1 + D_1)X(M_2 + D_2)$  for some  $X \in B(H)$ , then  $N_1XN_2 = M_1XM_2$ .*

**PROOF.** Let

$$\tilde{T} = \begin{pmatrix} T_1 & \\ & T_2 \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \tag{2.19}$$

where  $T = N, M, C, D$ ; then  $\tilde{N}, \tilde{M}$  are normal, and  $\tilde{C}, \tilde{D}$  are quasinilpotents in  $B(H \oplus H)$ . If  $(N_1 + C_1)X(N_2 + C_2) = (M_1 + D_1)X(M_2 + D_2)$ , then  $(\tilde{N} + \tilde{C})\tilde{X}(\tilde{N} + \tilde{C}) = (\tilde{M} + \tilde{D})\tilde{X}(\tilde{M} + \tilde{D})$ , so  $\tilde{N}\tilde{X}\tilde{N} = \tilde{M}\tilde{X}\tilde{M}$  by [Theorem 2.7](#), that is,  $N_1XN_2 = M_1XM_2$ .  $\square$

**3. Second-degree PF theorem.** First we will extend [Theorem 1.2](#) to the more general case.

**THEOREM 3.1.** *Let  $N_1, N_2, M_1, M_2$  be normal operators such that  $N_1M_1 = M_1N_1, N_2M_2 = M_2N_2$ . If  $N_1(N_1XN_2 - M_1XM_2)N_2 = M_1(N_1XN_2 - M_1XM_2)M_2$  for some  $X \in B(H)$ , then  $N_1XN_2 - M_1XM_2 = 0$ .*

**PROOF.** First we will prove that if  $N, M$  are normal operators, then  $N(NXN - MXM)N = M(NXN - MXM)M$  implies  $NXN = MXM$ .

If  $\text{Ker}(N) \neq \{0\}$ , then letting  $P$  be the project  $H \mapsto \text{Ker}(N)$ , we have  $PN(NXN - MXM)N = PM(NXN - MXM)M$ . That is,  $0 = -M^2PXM^2$  or  $M(M(PXM^2) - (PXM^2)0) = (M(PXM^2) - (PXM^2)0)0$ . By the SPF theorem ([Theorem 1.2](#)),  $MPXM^2 = 0$ . By the same way, we have  $MPXM = 0$ . Similarly,  $MXPM = 0$ . So we may assume that  $\text{Ker}(N) = \{0\}$ .

Let  $T_\epsilon$  be the same as in [Lemma 2.5](#). If  $X \in B(H)$  such that

$$N(NXN - MXM)N = M(NXN - MXM)M, \tag{3.1}$$

then

$$N_\epsilon(N_\epsilon X_\epsilon N_\epsilon - M_\epsilon X_\epsilon M_\epsilon)N_\epsilon = M_\epsilon(N_\epsilon X_\epsilon N_\epsilon - M_\epsilon X_\epsilon M_\epsilon)M_\epsilon \tag{3.2}$$

or

$$X_\epsilon - N_\epsilon^{-1}M_\epsilon X_\epsilon N_\epsilon^{-1}M_\epsilon = N_\epsilon^{-1}M_\epsilon(X_\epsilon - N_\epsilon^{-1}M_\epsilon X_\epsilon N_\epsilon^{-1}M_\epsilon)N_\epsilon^{-1}M_\epsilon. \tag{3.3}$$

Since  $N_\epsilon^{-1}M_\epsilon$  is normal, by [\[2\]](#), we have

$$X_\epsilon - N_\epsilon^{-1}M_\epsilon X_\epsilon N_\epsilon^{-1}M_\epsilon = 0 \tag{3.4}$$

or

$$N_\epsilon X_\epsilon N_\epsilon = M_\epsilon X_\epsilon M_\epsilon. \tag{3.5}$$

Letting  $\epsilon \rightarrow 0$ , we have  $NXN = MXM$ .

In general, let

$$\tilde{N} = \begin{pmatrix} N_1 & \\ & N_2 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}. \tag{3.6}$$

If

$$N_1(N_1XN_2 - M_1XM_2)N_2 = M_1(N_1XN_2 - M_1XM_2)M_2, \tag{3.7}$$

then

$$\tilde{N}(\tilde{N}\tilde{X}\tilde{N} - \tilde{M}\tilde{X}\tilde{M})\tilde{N} = \tilde{M}(\tilde{N}\tilde{X}\tilde{N} - \tilde{M}\tilde{X}\tilde{M})\tilde{M}; \tag{3.8}$$

so  $\tilde{N}\tilde{X}\tilde{N} = \tilde{M}\tilde{X}\tilde{M}$ , that is,  $N_1XN_2 = M_1XM_2$ . □

Let  $\mathbf{A} = (N_1, N_2)$ ,  $\mathbf{B} = (M_1, M_2)$  be tuples of commuting operators in  $B(H)$ . We say that  $(\mathbf{A}, \mathbf{B})$  has the SPF theorem if for any  $X \in B(H)$  and for some  $n \geq 2$  such that  $\Delta_{(\mathbf{A}, \mathbf{B})}^{(n)}(X) = 0$ , we have  $\Delta_{(\mathbf{A}, \mathbf{B})}(X) = 0$ .

**THEOREM 3.2.** *Let  $N, M, D \in B(H)$  such that  $N$  commutes with  $D$  and  $M$ . If  $N$  is invertible and  $D$  is quasinilpotent, then  $((N, N), (M, D))$  has the SPF theorem.*

**PROOF.** If

$$N(NXN - MXD)N = M(NXN - MXD)D, \tag{3.9}$$

then

$$X - N^{-1}MXN^{-1}D = N^{-1}M(X - N^{-1}MXN^{-1}D)N^{-1}D. \tag{3.10}$$

Note that  $N^{-1}D$  is quasinilpotent; so by applying [Lemma 2.4](#) to  $X - N^{-1}MXN^{-1}D$ , we have  $X - N^{-1}MXN^{-1}D = 0$ , that is,  $NXN - MXD = 0$ . □

**THEOREM 3.3.** *Let  $N, M \in B(H)$  such that  $N$  commutes with  $M$ . If  $M$  is invertible and  $\|N\|\|M^{-1}\| \leq 1$ , then  $((N, N), (M, M))$  has the SPF theorem.*

**PROOF.** If (3.1) holds for some  $X \in B(H)$ , then

$$NM^{-1}XNM^{-1} - X = NM^{-1}(NM^{-1}XNM^{-1} - X)NM^{-1}. \tag{3.11}$$

Since  $\|N\|\|M^{-1}\| \leq 1$ , by [\[2\]](#), we have  $NM^{-1}XNM^{-1} - X = 0$ , that is,  $NXN = MXM$ . □

The next theorem establishes the relationship between the SPF theorem and the PF theorem under perturbation by nilpotents.

**THEOREM 3.4.** *Let  $N_i, M_i \in B(H)$  and let  $C_i, D_i$  be nilpotents such that  $C_i, D_i, N_i, M_i$  mutually commute for  $i = 1, 2$ . If  $((N_1, N_2), (M_1, M_2))$  has the SPF theorem, then  $(N_1 + C_1)X(N_2 + C_2) = (M_1 + D_1)X(M_2 + D_2)$  implies that  $N_1XN_2 = M_1XM_2$ .*

**PROOF.** If

$$(N_1 + C_1)X(N_2 + C_2) = (M_1 + D_1)X(M_2 + D_2), \tag{3.12}$$

then by expanding both sides of the equation and moving  $M_1XM_2$  to the left-hand side and moving all the terms in the left-hand side to the right-hand side except  $N_1XN_2$ , we have

$$N_1XN_2 - M_1XM_2 = S(X), \tag{3.13}$$

where  $S$  is a linear operator on  $B(H)$  defined by

$$S(X) = -N_1XC_2 - C_1XN_2 - C_1XC_2 + M_1XD_2 + D_1XM_2 + D_1XD_2. \tag{3.14}$$

It is clear that  $S^{(2)}(X) = S(S(X))$  consists of  $6^2$  terms like

$$(-1)^l N_1^{m_1} M_1^{n_1} C_1^{s_1} D_1^{t_1} X N_2^{m_2} M_2^{n_2} C_2^{s_2} D_2^{t_2}, \quad \text{where } s_1 + t_1 + s_2 + t_2 \geq 2, \dots, \tag{3.15}$$

$S^{(n)}(X)$  consists of  $6^n$  terms like  $(-1)^l N_1^{m_1} M_1^{n_1} C_1^{s_1} D_1^{t_1} X N_2^{m_2} M_2^{n_2} C_2^{s_2} D_2^{t_2}$ , where  $s_1 + t_1 + s_2 + t_2 \geq n$ .

Since  $C_1, C_2, D_1, D_2$  are all nilpotents, we have  $n_0$  such that  $C_1^{n_0} = D_1^{n_0} = C_2^{n_0} = D_2^{n_0} = 0$ . Thus for each term of  $S^{(4n_0+1)}(X)$ , as  $s_1 + t_1 + s_2 + t_2 \geq 4n_0 + 1$ , we have at least one integer among  $s_1, s_2, t_1, t_2$  greater than  $n_0$ , so every term of  $S^{(4n_0+1)}(X)$  is 0. Therefore,  $S^{(4n_0+1)}(X) = 0$ . But

$$\Delta_{((N_1, N_2), (M_1, M_2))}^{(4n_0+1)}(X) = S^{(4n_0+1)}(X) = 0, \tag{3.16}$$

and  $((N_1, N_2), (M_1, M_2))$  has the SPF theorem; so it follows that

$$\Delta_{((N_1, N_2), (M_1, M_2))}(X) = 0, \tag{3.17}$$

or  $N_1XN_2 = M_1XM_2$ . □

By Theorems 3.3 and 3.4, it is easy to see the following.

**THEOREM 3.5.** *Let  $N, M \in B(H)$  and let  $C, D$  be nilpotents such that  $N, M, C, D$  mutually commute. If  $M$  is invertible and  $\|N\| \|M^{-1}\| \leq 1$ , then  $(N + C)X(N + C) = (M + D)X(M + D)$  implies  $NXN = MXM$ .*

Moreover, if the strict inequality in Theorem 3.5 holds, then Theorem 3.5 is true even for the quasinilpotent operators.

**THEOREM 3.6.** *Let  $N, M \in B(H)$  and let  $C, D$  be quasinilpotents such that  $N, M, C, D$  mutually commute. If  $M$  is invertible and  $\|N\| \|M^{-1}\| < 1$ , then  $(N + C)X(N + C) = (M + D)X(M + D)$  implies  $X = 0$ .*

**PROOF.** If  $D$  is quasinilpotent and  $M$  is invertible, then  $M + D$  is invertible. If  $(N + C)X(N + C) = (M + D)X(M + D)$  for some  $X \in B(H)$ , then

$$(N + C)(M + D)^{-1}X(N + C)(M + D)^{-1} = X \tag{3.18}$$

or

$$(NM^{-1} + F)X(NM^{-1} + F) = X, \tag{3.19}$$

where  $F$  is quasinilpotent. By [3],

$$\sigma(\Delta_{((NM^{-1}+F, NM^{-1}+F), (I, I))}) = \sigma(NM^{-1})\sigma(NM^{-1}) - 1. \tag{3.20}$$

Since  $\|N\|\|M^{-1}\| < 1$ ,  $0$  is not in

$$\sigma(\Delta_{((NM^{-1}+F, NM^{-1}+F), (I, I))}), \tag{3.21}$$

and therefore  $\Delta_{((NM^{-1}+F, NM^{-1}+F), (I, I))}$  is invertible. It follows from the equation

$$\Delta_{((NM^{-1}+F, NM^{-1}+F), (I, I))}(X) = 0 \tag{3.22}$$

that  $X = 0$ . □

The following results show that even if  $((A, A), (B, B))$  has the SPF theorem, we still do not know if  $((A^2, A^2), (B^2, B^2))$  has the SPF theorem.

**THEOREM 3.7.** *Let  $A, B \in B(H)$ . Let  $\omega$  be an  $n$ th root of 1, but  $\omega^k \neq 1$  for  $k$  such that  $1 \leq k \leq n-1$ . If for any  $k$  such that  $0 \leq k \leq n-1$ ,  $((A, A), (B, \omega^k B))$  has the SPF theorem, then  $((A^n, A^n), (B^n, B^n))$  has the SPF theorem too.*

**PROOF.** By induction, we can prove that

$$\begin{aligned} &\Delta_{((A^n, A^n), (B^n, B^n))}(X) \\ &= \Delta_{((A, A), (B, B))}(\Delta_{((A, A), (B, \omega B))}(\cdots(\Delta_{((A, A), (B, \omega^{(n-1)B})}(X))\cdots))). \end{aligned} \tag{3.23}$$

Now if

$$\Delta_{((A^n, A^n), (B^n, B^n))}^{(2)}(X) = 0, \tag{3.24}$$

then

$$\Delta_{((A, A), (B, B))}^{(2)}(\Delta_{((A, A), (B, \omega B))}^{(2)}(\cdots(\Delta_{((A, A), (B, \omega^{(n-1)B})}^{(2)}(X))\cdots)) = 0. \tag{3.25}$$

Since  $((A, A), (B, B))$  has the SPF theorem, it follows that

$$\Delta_{((A, A), (B, B))}(\Delta_{((A, A), (B, \omega B))}^{(2)}(\cdots(\Delta_{((A, A), (B, \omega^{(n-1)B})}^{(2)}(X))\cdots)) = 0. \tag{3.26}$$

or

$$\Delta_{((A, A), (B, \omega B))}^{(2)}(\Delta_{((A, A), (B, B))}(\cdots(\Delta_{((A, A), (B, \omega^{(n-1)B})}^{(2)}(X))\cdots)) = 0, \tag{3.27}$$

and therefore

$$\Delta_{((A, A), (B, \omega B))}(\Delta_{((A, A), (B, B))}(\cdots(\Delta_{((A, A), (B, \omega^{(n-1)B})}^{(2)}(X))\cdots)) = 0. \tag{3.28}$$

Proceeding in this way, we have finally

$$\Delta_{((A, A), (B, B))}(\Delta_{((A, A), (B, \omega B))}(\cdots(\Delta_{((A, A), (B, \omega^{(n-1)B})}^{(2)}(X))\cdots)) = 0, \tag{3.29}$$

that is, by (3.23),

$$\Delta_{((A^n, A^n), (B^n, B^n))}(X) = 0. \tag{3.30}$$

□

The following result says that the converse of Theorem 3.8 is also true.

**THEOREM 3.8.** *Let  $A, B \in B(H)$ . Let  $\omega$  be an  $n$ th root of 1, but  $\omega^k \neq 1$  for  $k$  such that  $1 \leq k \leq n - 1$ . If  $A$  or  $B$  is invertible and  $((A^n, A^n), (B^n, B^n))$  has the SPF theorem, then for any  $k$  such that  $0 \leq k \leq n - 1$ ,  $((A, A), (B, \omega^k B))$  has the SPF theorem too.*

**PROOF.** It is sufficient to prove that if  $(A^n, B^n)$  has the SPF theorem and  $B$  is invertible, then  $((A, A), (B, B))$  has the SPF theorem. Now if

$$A(AXA - BXB)A = B(AXA - BXB)B, \tag{3.31}$$

then

$$A^n(AXA - BXB)A^n = B^n(AXA - BXB)B^n \tag{3.32}$$

or

$$A^n(A^n X A^n - B^n X B^n)A^n = B^n(A^n X A^n - B^n X B^n)B^n; \tag{3.33}$$

so (3.24) holds. Since  $((A^n, A^n), (B^n, B^n))$  has the SPF theorem, we have (3.30). It follows from (3.23) that (3.29) holds. From (3.31), we see that

$$\Delta_{((A, A), (B, B))}^{(2)}(\Delta_{((A, A), (B, \omega^2 B))}(\cdots(\Delta_{((A, A), (B, \omega^{(n-1) B})}(X))\cdots)) = 0. \tag{3.34}$$

Note that

$$\Delta_{((A, A), (B, B))}(Y) - \Delta_{((A, A), (B, \omega B))}(Y) = (\omega - 1)BYB. \tag{3.35}$$

Since  $B$  is invertible, (3.29) and (3.34) will give

$$\Delta_{((A, A), (B, B))}(\Delta_{((A, A), (B, \omega^2 B))}(\cdots(\Delta_{((A, A), (B, \omega^{(n-1) B})}(X))\cdots)) = 0. \tag{3.36}$$

From (3.31), we see also that

$$\Delta_{((A, A), (B, B))}^{(2)}(\Delta_{((A, A), (B, \omega^3 B))}(\cdots(\Delta_{((A, A), (B, \omega^{(n-1) B})}(X))\cdots)) = 0; \tag{3.37}$$

then (3.36) and (3.37) yields

$$\Delta_{((A, A), (B, B))}(\Delta_{((A, A), (B, \omega^3 B))}(\cdots(\Delta_{((A, A), (B, \omega^{(n-1) B})}(X))\cdots)) = 0. \tag{3.38}$$

Proceeding in this way, we have finally

$$\Delta_{((A, A), (B, B))}(\Delta_{((A, A), (B, \omega^{(n-1) B})}(X)) = 0. \tag{3.39}$$

Now (3.31) and (3.39) will give the desired equation:  $AXA - BXB = 0$ .

□

**THEOREM 3.9.** *If  $C, D$  are nilpotents such that  $CD = DC$  but  $C^2 \neq D^2$ , then  $((C, C), (D, D))$  does not have the SPF theorem.*

**PROOF.** It is not difficult to see that

$$\Delta_{((C,C),(D,D))}^{(n)}(I) = \sum_{k=0}^n (-1)^k C_n^k C^{2n-2k} D^{2k}, \tag{3.40}$$

where  $I$  is the identity operator.

If  $C, D$  are nilpotents, then there exists an  $n_0$  such that  $C^{n_0} = 0, D^{n_0} = 0$ . For any  $k$  such that  $1 \leq k \leq n_0$ , at least one of  $2n_0 + 2 - 2k$  and  $2k$  is greater than  $n_0$ . So by (3.40), we have

$$\Delta_{((C,C),(D,D))}^{(n_0+1)}(I) = 0. \tag{3.41}$$

But  $\Delta_{((C,C),(D,D))}(I) = C^2 - D^2 \neq 0$ . This completes the proof. □

**4. Asymptotic PF theorem and compact operators.** We now give a theorem about the compact operators, which generalizes the relative result in [2].

**THEOREM 4.1.** *Let  $\mathbf{A} = (N_1, N_2)$  and  $\mathbf{B} = (M_1, M_2)$  be tuples of commuting normal operators in  $B(H)$ . If  $X \in B(H)$  such that  $\Delta_{(\mathbf{A},\mathbf{B})}^{(n)}(X)$  is compact for some  $n \geq 2$ , then  $\Delta_{(\mathbf{A},\mathbf{B})}(X)$  is compact too.*

**PROOF.** Let  $K(H)$  be the ideal of  $B(H)$  consisting of all compact operators on  $H$ , let  $B(H)/K(H)$  be the Calkin algebra, and let  $\pi$  be the Calkin map from  $B(H)$  to  $B(H)/K(H)$ . It is clear that

$$\pi(\Delta_{((N_1,N_2),(M_1,M_2))}^{(n)}(X)) = \Delta_{((\pi(N_1),\pi(N_2)),(\pi(M_1),\pi(M_2)))}^{(n)}(\pi(X)). \tag{4.1}$$

If  $\Delta_{((N_1,N_2),(M_1,M_2))}^{(n)}(X)$  is compact, then  $\pi(\Delta_{((N_1,N_2),(M_1,M_2))}^{(n)}(X)) = 0$ . It follows that

$$\Delta_{((\pi(N_1),\pi(N_2)),(\pi(M_1),\pi(M_2)))}^{(n)}(\pi(X)) = 0. \tag{4.2}$$

Since  $\pi(N_i), \pi(M_i)$  are normal, for  $i = 1, 2$ , applying Theorem 3.1, we have

$$\Delta_{((\pi(N_1),\pi(N_2)),(\pi(M_1),\pi(M_2)))}(\pi(X)) = 0. \tag{4.3}$$

Therefore,  $\Delta_{((N_1,N_2),(M_1,M_2))}(X)$  is compact. □

The following theorem is an asymptotic version of the SPF theorem. It generalizes the corresponding result in [10].

**THEOREM 4.2.** *Let  $\mathbf{A} = (N_1, N_2)$  and  $\mathbf{B} = (M_1, M_2)$  be tuples of commuting normal operators in  $B(H)$ . Let  $K$  be any positive real number and let  $n$  be an integer greater than 1. Then for every neighborhood  $U$  of 0 in  $B(H)$  (under uniform, strong or weak topology), a neighborhood  $V$  of 0 under the same topology is obtained such that if  $\Delta_{(\mathbf{A},\mathbf{B})}^{(n)}(X) \in V$  and  $\|X\| \leq K$ , then  $\Delta_{(\mathbf{A},\mathbf{B})}(X) \in U$ .*

**PROOF.** We first consider the following particular case:  $N_1 = N_2 = N, M_1 = M_2 = M$ . Assume that  $\|N\|$  and  $\|M\|$  are not greater than 1 (if not, we can replace  $N$  and  $M$  by  $N/(\|N\| + \|M\|)$  and  $M/(\|N\| + \|M\|)$ , resp.).

Let  $K > 0$  and let  $U$  be any neighborhood of 0 in  $B(H)$  under uniform (or strong or weak) topology. Let  $U_{ij}, i, j = 1, 2, 3, 4$ , be neighborhoods of 0 in  $B(H)$  under the same topology such that

$$\sum_{i=1}^4 \sum_{j=1}^4 U_{ij} \subset U. \tag{4.4}$$

Suppose that  $N, M$  have the following spectral decomposition:

$$N = \int_{\sigma(N)} \lambda dE_\lambda, \quad M = \int_{\sigma(M)} \lambda dF_\lambda. \tag{4.5}$$

For any  $\epsilon > 0$ , define  $\Delta_\epsilon = \{z \mid |z| \leq \epsilon\}, \Delta_\epsilon^c = \mathbb{C} \setminus \Delta_\epsilon$ , and

$$\begin{aligned} H_1(\epsilon) &= E(\Delta_\epsilon)F(\Delta_\epsilon)H, \\ H_2(\epsilon) &= E(\Delta_\epsilon)F(\Delta_\epsilon^c)H, \\ H_3(\epsilon) &= E(\Delta_\epsilon^c)F(\Delta_\epsilon)H, \\ H_4(\epsilon) &= E(\Delta_\epsilon^c)F(\Delta_\epsilon^c)H. \end{aligned} \tag{4.6}$$

Then  $H$  can be written as  $H = H_1(\epsilon) \oplus H_2(\epsilon) \oplus H_3(\epsilon) \oplus H_4(\epsilon)$ . Under this decomposition, we have

$$\begin{aligned} N &= \begin{pmatrix} N_1(\epsilon) & & & \\ & N_2(\epsilon) & & \\ & & N_3(\epsilon) & \\ & & & N_4(\epsilon) \end{pmatrix}, \\ M &= \begin{pmatrix} M_1(\epsilon) & & & \\ & M_2(\epsilon) & & \\ & & M_3(\epsilon) & \\ & & & M_4(\epsilon) \end{pmatrix}, \end{aligned} \tag{4.7}$$

where  $\|N_1(\epsilon)\|, \|N_2(\epsilon)\|, \|M_1(\epsilon)\|, \|M_3(\epsilon)\|$  are not greater than  $\epsilon$ , and  $N_3(\epsilon), N_4(\epsilon), M_2(\epsilon)$ , and  $M_4(\epsilon)$  are invertible.

Let  $X = ((X_{ij}(\epsilon)))_{i,j=1,2,3,4}$  and let  $Z$  denote the set

$$Z = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1)\}. \tag{4.8}$$

If  $(k, l) \in Z$ , then at least one operator in each pair of  $(N_k, N_l), (M_k, M_l)$  has norm less

than  $\epsilon$ . Hence

$$\|N_k(\epsilon)X_{kl}(\epsilon)N_l(\epsilon) - M_k(\epsilon)X_{kl}(\epsilon)M_l(\epsilon)\| \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \tag{4.9}$$

Therefore, we are able to choose a fixed number  $\epsilon_0 > 0$  such that for each pair  $(k, l) \in Z$ ,

$$(\delta_{ij}(k, l)\Delta_{((N_i(\epsilon_0), N_j(\epsilon_0)), (M_i(\epsilon_0), M_j(\epsilon_0)))}(X_{ij}(\epsilon_0)))_{4 \times 4} \in U_{kl}, \tag{4.10}$$

where  $\delta_{ij}(k, l)$  equals 1 if  $i = k, j = l$  and 0 otherwise. Set  $V_{kl} = U_{kl}$ .

For the sake of simplicity, we will omit  $\epsilon_0$  in the notations of each component in the decompositions of  $H, N, M, X$ .

It is easy to see that  $\Delta_{(A, B)}^{(n)}(X)$  has the following decomposition:

$$\Delta_{((N, N), (M, M))}^{(n)}(X) = (\Delta_{((N_i, N_j), (M_i, M_j))}^{(n)}(X_{ij}))_{4 \times 4}. \tag{4.11}$$

If  $(k, l)$  is not in  $Z$ , then at least one pair of  $(N_k, N_l)$  and  $(M_k, M_l)$  has two invertible operators. We assume that  $N_k$  and  $N_l$  are invertible (we can follow the same way if  $M_k, M_l$  are invertible).

Let

$$O_{kl} = \{o_{kl} : (\delta_{ij}(k, l)o_{ij})_{4 \times 4} \in U_{ij}\}. \tag{4.12}$$

Then  $O_{kl}$  is a neighborhood of 0 in  $B(H_l, H_k)$ .

Since  $N_k, N_l$  are invertible, we can see that

$$\Delta_{((N_k, N_l), (M_k, M_l))}^{(n)}(X_{kl}) = N_k^n \Delta_{((I_k, I_l), (N_k^{-1}M_k, N_l^{-1}M_l))}^{(n)}(X_{kl})N_l^n, \tag{4.13}$$

where  $I_k, I_l$  are identities on  $H_k, H_l$ . It follows from the asymptotic PF theorem in [2] that there is the neighborhood  $P_{kl}$  of 0 in  $B(H_l, H_k)$  such that for  $\|X_{kl}\| \leq K$ , if

$$\Delta_{((I_k, I_l), (N_k^{-1}M_k, N_l^{-1}M_l))}^{(n)}(X_{kl}) \in P_{kl}, \tag{4.14}$$

then

$$\Delta_{((I_k, I_l), (N_k^{-1}M_k, N_l^{-1}M_l))}^{(n)}(X_{kl}) \in N_k^{-1}O_{kl}N_l^{-1}. \tag{4.15}$$

Set  $V_{kl} = N_k^n P_{kl} N_l^n$ . If

$$\Delta_{((N_k, N_l), (M_k, M_l))}^{(n)}(X_{kl}) \in V_{kl}, \tag{4.16}$$

then

$$\Delta_{((N_k, N_l), (M_k, M_l))}^{(n)}(X_{kl}) \in O_{kl}. \tag{4.17}$$

Let

$$V = \{(v_{ij})_{4 \times 4} : v_{ij} \in V_{ij}\}. \tag{4.18}$$

Then  $V$  is a neighborhood of 0. If  $\|X\| \leq K$  and  $\Delta_{(A,B)}^{(n)}(X) \in V$ , then for each pair  $(k, l)$ ,  $\|X_{kl}\| \leq K$  and (4.16) holds; so it follows that (4.17) holds, that is,

$$(\delta_{ij}(k, l)\Delta_{((N_k, N_l), (M_k, M_l))}(X_{kl}))_{4 \times 4} \in U_{kl}, \tag{4.19}$$

but

$$\Delta_{(A,B)}(X) = \sum_{k=1}^4 \sum_{l=1}^4 (\delta_{ij}(k, l)\Delta_{((N_k, N_l), (M_k, M_l))}(X_{kl}))_{4 \times 4}, \tag{4.20}$$

which is in  $U$  by (4.4).

In general, let

$$\tilde{N} = \begin{pmatrix} N_1 & \\ & N_2 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix}. \tag{4.21}$$

Then  $\tilde{N}, \tilde{M}$  are normal in  $B(H \oplus H)$ . Let

$$\tilde{U} = \left\{ \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} : u_i \in U, i = 1, 2, 3, 4 \right\}. \tag{4.22}$$

$\tilde{U}$  is a neighborhood of 0 in  $B(H \oplus H)$ . So there is a neighborhood  $\tilde{V}$  of 0 in  $B(H \oplus H)$  such that if  $\|\tilde{X}\| \leq K$ ,  $\Delta_{(\tilde{A}, \tilde{B})}^{(n)}(\tilde{X}) \in \tilde{V}$ , then  $\Delta_{(\tilde{A}, \tilde{B})}(\tilde{X}) \in \tilde{U}$ . Let

$$\tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad V = \left\{ v : \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in \tilde{V} \right\}. \tag{4.23}$$

$V$  is a neighborhood of 0 in  $B(H)$ . If  $\|X\| \leq K$ ,  $\Delta_{(A,B)}^{(n)}(X) \in V$ , then  $\|\tilde{X}\| \leq K$  and  $\Delta_{(\tilde{A}, \tilde{B})}^{(n)}(\tilde{X}) \in \tilde{V}$ ; so  $\Delta_{(\tilde{A}, \tilde{B})}(\tilde{X}) \in \tilde{U}$ , which means that

$$\begin{pmatrix} 0 & \Delta_{(A,B)}(X) \\ 0 & 0 \end{pmatrix} \in \tilde{U} \tag{4.24}$$

or  $\Delta_{(A,B)}(X) \in U$ . □

Using the same technique, we are able to generalize the asymptotic PF theorems obtained by Moore [6] and Rogers [8].

**THEOREM 4.3.** *Let  $N_1, N_2, M_1, M_2, k$  be the same as in Theorem 4.2. Then for any neighborhood  $U$  of 0 in  $B(H)$  (under uniform, strong or weak topology), a neighborhood  $V$  of 0 under the same topology is obtained such that if  $N_1^* X N_2^* - M_1^* X M_2^* \in V$  and  $\|X\| \leq K$ , then  $N X N - M X M \in U$ .*

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## REFERENCES

- [1] J. Anderson and C. Foiaş, *Properties which normal operators share with normal derivations and related operators*, Pacific J. Math. **61** (1975), no. 2, 313–325.
- [2] Y. Chen, *Kernels of generalized derivation operators*, Chinese Ann. Math. Ser. A **13** (1992), no. 5, 580–588.
- [3] L. A. Fialkow, *Essential spectra of elementary operators*, Trans. Amer. Math. Soc. **267** (1981), no. 1, 157–174.
- [4] T. Furuta, *Generalized Fuglede-Putnam theorem and Hilbert-Schmidt norm inequality*, Proc. Japan Acad. Ser. A Math. Sci. **58** (1982), no. 2, 55–57.
- [5] ———, *A Hilbert-Schmidt norm inequality associated with the Fuglede-Putnam theorem*, Ark. Mat. **20** (1982), no. 1, 157–163.
- [6] R. L. Moore, *An asymptotic Fuglede theorem*, Proc. Amer. Math. Soc. **50** (1975), 138–142.
- [7] C. R. Putnam, *Commutation Properties of Hilbert Space Operators and Related Topics*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 36, Springer-Verlag, New York, 1967.
- [8] D. D. Rogers, *On Fuglede's theorem and operator topologies*, Proc. Amer. Math. Soc. **75** (1979), no. 1, 32–36.
- [9] K. Takahashi, *On the converse of the Fuglede-Putnam theorem*, Acta Sci. Math. (Szeged) **43** (1981), no. 1-2, 123–125.
- [10] Y. S. Tong, *Higher-order generalized derivations*, Chinese Ann. Math. Ser. A **6** (1985), no. 5, 565–572.

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