We prove that if a one-to-one mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ ($n \geq 2$) preserves the unit circles, then $f$ is a linear isometry up to translation.

2000 Mathematics Subject Classification: 51K05.

1. Introduction. Let $X$ and $Y$ be normed spaces. A mapping $f : X \to Y$ is called an isometry if $f$ satisfies the equality

$$\|f(x) - f(y)\| = \|x - y\|$$

for all $x, y \in X$. A distance $r > 0$ is said to be preserved (conserved) by a mapping $f : X \to Y$ if

$$\|f(x) - f(y)\| = r \quad \forall x, y \in X \text{ with } \|x - y\| = r. \quad (1.2)$$

If $f$ is an isometry, then every distance $r > 0$ is conserved by $f$, and vice versa. We can now raise a question whether each mapping that preserves certain distances is an isometry. Indeed, Aleksandrov [1] had raised a question whether a mapping $f : X \to X$ preserving a distance $r > 0$ is an isometry, which is now known to us as the Aleksandrov problem. Without loss of generality, we may assume $r = 1$ when $X$ is a normed space (see [16]).

Beckman and Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces $X = \mathbb{R}^n$ (see also [3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 17, 18, 19, 20]).

**Theorem 1.1** (Beckman and Quarles). *If a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ ($2 \leq n < \infty$) preserves a distance $r > 0$, then $f$ is a linear isometry up to translation.*

Recently, Zaks [25] proved the rational analogues of the Beckman-Quarles theorem. Indeed, he assumes that $n = 4k(k + 1)$ for some $k \geq 1$ or $n = 2m^2 - 1$ for some $m \geq 3$, and he proves that if a mapping $f : \mathbb{Q}^n \to \mathbb{Q}^n$ preserves the unit distance, then $f$ is an isometry (see also [21, 22, 23, 24]).

It seems interesting to investigate whether the “distance $r > 0$” in the Beckman-Quarles theorem can be replaced by some properties characterized by “geometrical figures” without loss of its validity.

In [9], the first author proved that if a one-to-one mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ ($n \geq 2$) maps every regular triangle (quadrilateral or hexagon) of side length $a > 0$ onto a figure of
the same type with side length $b > 0$, then there exists a linear isometry $I : \mathbb{R}^n \to \mathbb{R}^n$ up to translation such that

$$f(x) = \frac{b}{a} I(x).$$

Furthermore, the first author proved that if a one-to-one mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ maps every unit circle onto a unit circle, then $f$ is a linear isometry up to translation (see [10]).

In this connection, we will extend the result of [10] to the $n$-dimensional cases; more precisely, we prove in this paper that if a one-to-one mapping $f : \mathbb{R}^n \to \mathbb{R}^n (n \geq 2)$ maps every unit circle onto a unit circle, then $f$ is a linear isometry up to translation.

2. Preliminaries. We start with any two distinct points $a$ and $b$ in $\mathbb{R}^n$ with the distance between the two less than $2$. Let their distance be

$$2c = 2\sin \varphi_0 \quad \text{with } 0 < \varphi_0 < \frac{\pi}{2}, \quad 0 < c < 1.$$  

(2.1)

Given such two distinct points whose distance is less than $2$, we can choose a coordinate $(y_1, \ldots, y_n)$ for $\mathbb{R}^n$ such that

$$a = (0, \ldots, 0, \sin \varphi_0), \quad b = (0, \ldots, 0, -\sin \varphi_0).$$  

(2.2)

Let the $(n-2)$-dimensional unit sphere contained in the space orthogonal to the $y_n$-direction be

$$Y = \{(y_1, \ldots, y_{n-1}, 0) \mid y_1^2 + \cdots + y_{n-1}^2 = 1\}.$$  

(2.3)

If we call the center of any unit circle passing through the two points $(a$ and $b$) $o'$ and the origin of the coordinate $o$, then the vector $oo'$ is perpendicular to the $y_n$-axis and its length must be $\cos \varphi_0$ and therefore $oo' \in \tilde{Y} = \cos \varphi_0 Y$, see Figure 2.1. It means that any unit circle passing through the points $a$ and $b$ has its center in $\tilde{Y} = \cos \varphi_0 Y$. Let $T$ be the set of union of all the unit circles passing through the points $a$ and $b$. More precisely, if we define the following set:

$$T = \{(\cos \varphi + \cos \varphi_0) y + (0, \ldots, 0, \sin \varphi) \mid y \in Y, \ 0 \leq \varphi < 2\pi\},$$

(2.4)
then it is clear that this is the set of union of all the unit circles which are centered at 
\(\cos \varphi_0 y\) for each fixed \(y \in Y\) and which pass through \(a\) and \(b\) when \(\varphi = \pi \mp \varphi_0\) (see Figure 2.1).

The intersection of \(T\) and the \(y_1 \cdot y_n\) plane consists of two circles, say \(C_1\) (when \(y_1 = 1\), i.e., \(y = (1,0,\ldots,0)\)) and \(C_2\) (when \(y_1 = -1\), i.e., \(y = (-1,0,\ldots,0)\), see Figure 2.1). In the following contexts, we will consider the cases \(y_1 = 1\) and \(-1\) in connection with \(T\) as the circles \(C_1\) and \(C_2\), respectively. Call \(S_1\) the \((n-1)\)-dimensional unit sphere containing the circle \(C_1\). If we let the center of \(C_1\) be \(O\) and the center of \(S_1\) be \(\tilde{O}\), then it is obvious that \(O = \tilde{O}\).

(To see this, choose any point \(A \in C_1\) and its antipodal point \(B\) in \(C_1\). Then, by the definition of the antipodal points that they lie exactly the opposite with respect to the center of the circle \(C_1\) whose center is at \(O\), and because they are of the same length 1, we have the following condition that
\[
\overrightarrow{OA} = -\overrightarrow{OB}, \quad \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = 2 \overrightarrow{OB}.
\] (2.5)

On the other hand, we have, since the two points \(A\) and \(B\) lie also on the unit sphere \(S_1\) with its center at \(\tilde{O}\),
\[
2 = |\overrightarrow{AB}| = |\overrightarrow{A\tilde{O}} + \overrightarrow{\tilde{O}B}| \leq |\overrightarrow{A\tilde{O}}| + |\overrightarrow{\tilde{O}B}| = 1 + 1 = 2.
\] (2.6)

Therefore, by the Cauchy-Schwarz inequality, \(\overrightarrow{A\tilde{O}}\) is a positive multiple of \(\overrightarrow{\tilde{O}B}\), which means \(A\tilde{O} = \tilde{O}B\) because their lengths are both 1. So,
\[
\overrightarrow{AB} = \overrightarrow{A\tilde{O}} + \overrightarrow{\tilde{O}B} = 2 \overrightarrow{\tilde{O}B},
\] (2.7)

and therefore \(\tilde{O} = O\.)

Now, we first show that \(S_1\) and \(T\) intersect only at \(C_1\). To make computation simpler we use a new coordinate \(x\) for \(\mathbb{R}^n\), where
\[
x = y - (\cos \varphi_0,0,\ldots,0).
\] (2.8)

In this coordinate (see Figure 2.2), \(S_1\) becomes the unit sphere \(S\) centered at the origin,
\[
S_1 = S = \{(x_1,\ldots,x_n) \mid x_1^2 + \cdots + x_n^2 = 1\},
\] (2.9)
\[
T = \{x = (\cos \varphi + \cos \varphi_0)y + (0,\ldots,0,\sin \varphi) - (\cos \varphi_0,0,\ldots,0) \mid y \in Y, \ 0 \leq \varphi < 2\pi\}.
\] (2.10)

With the help of this coordinate we show the following lemma.

**Lemma 2.1.** \(T \cap S_1 = C_1\).
PROOF. If any element in $T$ has distance 1 from the origin of the $x$-coordinate, then we have

$$1 = [(\cos \varphi + \cos \varphi_0) y_1 - \cos \varphi_0]^2 + (\cos \varphi + \cos \varphi_0)^2 y_2^2 + \cdots + (\cos \varphi + \cos \varphi_0)^2 y_{n-1}^2 + \sin^2 \varphi$$

$$= (\cos \varphi + \cos \varphi_0)^2 - 2 \cos \varphi_0 (\cos \varphi + \cos \varphi_0) y_1 + \cos^2 \varphi_0 + \sin^2 \varphi$$

$$= 1 + 2 \cos^2 \varphi_0 (1 - y_1) + 2 \cos \varphi_0 \cos \varphi (1 - y_1).$$

Therefore, we have

$$0 = 2 \cos \varphi_0 (1 - y_1) (\cos \varphi + \cos \varphi_0).$$

With $y_1 = 1$, $T$ in (2.10) represents the unit circle $C_1$ in the $x_1$-$x_n$ plane. If

$$\cos \varphi = -\cos \varphi_0, \quad \text{i.e., } \varphi = \pi \mp \varphi_0,$$

then it follows from (2.10) that

$$T = \{x = (-\cos \varphi_0, 0, \ldots, 0, \pm \sin \varphi_0)\} = \{a, b\}$$

which also belong to $C_1$. \qed

Now, consider, as in Figure 2.3, the origin $e$ and $\tilde{e} = (-2, 0, \ldots, 0)$ in the $x$-coordinate and the unit circle $C_1$ passing through $e$ and $\tilde{e}$ in the $x_1$-$x_n$ plane. Choose a point $d \in C_1$, $d \notin \{e, \tilde{e}\}$. We parameterize all the unit circles passing through the points $e$ and $d$. We assume the $x_n$-coordinate of $d$ is negative.

By triangle inequality, the distance between $e$ and $d$ is less than 2, say $2 \sin \varphi_0$, with $0 < \varphi_0 < \pi/2$. Choose a new coordinate $\gamma'$ for $\mathbb{R}^n$ and consider two points

$$e' = (0, \ldots, 0, \sin \varphi_0), \quad d' = (0, \ldots, 0, -\sin \varphi_0),$$

(see Figure 2.4).
To get a parameterization of the unit circles passing through e and d, we consider the mapping $M$ defined by

$$x = M y = \begin{bmatrix}
\cos \varphi_0 & 0 & \cdots & 0 & \sin \varphi_0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
-\sin \varphi_0 & 0 & \cdots & 0 & \cos \varphi_0
\end{bmatrix} [y + (\cos \varphi_0, 0, \ldots, 0)]^t - (1, 0, \ldots, 0)^t.$$

(2.16)

This transformation $M$ is an isometry (since it is a composition of a rotation and translations) and sends

$$\{y = (0, \ldots, 0, \pm \sin \varphi_0)\} = \{e', d'\}$$

(2.17)

to

$$\{x = (0, \ldots, 0), x = (\cos (-2\varphi_0) - 1, 0, \ldots, 0, \sin (-2\varphi_0))\} = \{e, d\}$$

(2.18)

and therefore it sends any unit circle passing through $e'$ and $d'$ to a unit circle passing through $e$ and $d$. 
Therefore, by comparing Figure 2.4 with Figure 2.1 and considering (2.4), all the unit circles passing through \( e \) and \( d \) can be parameterized as

\[
\{ x = My \mid y = (\cos \varphi + \cos \varphi_0) y' + (0, \ldots, 0, \sin \varphi), \ y' \in Y, \ 0 \leq \varphi < 2\pi \}. \tag{2.19}
\]

With the help of this parameterization, we are ready to show the following lemma.

**Lemma 2.2.** For \( d \in C_1, d \notin \{ e, \bar{e} \} \), any unit circle in \( \mathbb{R}^n \), which passes through \( d \) and \( e \), has some point whose \( x_1 \)-coordinate is positive, except the circle \( C_1 \).

**Proof.** Without loss of generality, we can assume the \( x_n \)-coordinate of \( d \) is negative.

Note that with \( \varphi = \pi \mp \varphi_0 \) in (2.19), \( y = (0, \ldots, 0, \pm \sin \varphi_0) \) are the points \( e' \) or \( d' \) in the \( y \)-coordinate and further \( \varphi = \pi \mp \varphi_0 \) means that

\[
x = (0, \ldots, 0) = e, \quad x = (\cos (-2\varphi_0) - 1, 0, \ldots, 0, \sin (-2\varphi_0)) = d \tag{2.20}
\]

in the \( x \)-coordinate, regardless of \( y' \in Y \). Any unit circle passing through \( e \) and \( d \) is given as \( x = My \) with \( y \) given as in (2.19), that is,

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_{n-1} \\
    x_n
\end{bmatrix}
= \begin{bmatrix}
    \cos \varphi_0 & 0 & \cdots & 0 & \sin \varphi_0 \\
    0 & 1 & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 1 & 0 \\
    -\sin \varphi_0 & 0 & \cdots & 0 & \cos \varphi_0
\end{bmatrix}
\begin{bmatrix}
    (\cos \varphi + \cos \varphi_0) y'_1 + \cos \varphi_0 \\
    (\cos \varphi + \cos \varphi_0) y'_2 \\
    \vdots \\
    (\cos \varphi + \cos \varphi_0) y'_{n-1} \\
    \sin \varphi
\end{bmatrix}
\begin{bmatrix}
    1 \\
    0 \\
    \vdots \\
    0
\end{bmatrix} \tag{2.21}
\]

The first coordinate is

\[
x_1 = \cos \varphi_0 (\cos \varphi + \cos \varphi_0) y'_1 + \cos^2 \varphi_0 + \sin \varphi_0 \sin \varphi - 1. \tag{2.22}
\]

We show that for \( y'_1 \neq -1 \) (\( y'_1 = -1 \) means the circle \( C'_1 \) in the \( y \)-coordinate and the circle \( C_1 \) in the \( x \)-coordinate, see Figure 2.4), there is always some \( \varphi \) near \( \pi - \varphi_0 \) (i.e., near the point \( e \)) such that the above \( x_1 \) becomes positive.

Let

\[
\theta = (\pi - \varphi_0) - \varphi = \pi - (\varphi + \varphi_0), \tag{2.23}
\]

and so

\[
\varphi = \pi - (\theta + \varphi_0). \tag{2.24}
\]
Then, the above is
\[
x_1 = -\cos \varphi_0 \cos (\theta + \varphi_0) y'_1 + \cos^2 \varphi_0 (1 + y'_1) + \sin \varphi_0 \sin (\theta + \varphi_0) - 1
\]
\[
= -\cos \varphi_0 [\cos \theta \cos \varphi_0 - \sin \theta \sin \varphi_0] y'_1 + \sin \varphi_0 [\sin \theta \cos \varphi_0 + \cos \theta \sin \varphi_0]
\]
\[
- 1 + \cos^2 \varphi_0 (1 + y'_1)
\]
\[
= \sin \theta \sin \varphi_0 \cos \varphi_0 (1 + y'_1) + \cos \theta \sin^2 \varphi_0 - \cos \theta \cos^2 \varphi_0 y'_1
\]
\[
- 1 + \cos^2 \varphi_0 (1 + y'_1)
\]
\[
= \sin \theta \sin \varphi_0 \cos \varphi_0 (1 + y'_1) + \cos \theta - \cos \theta \cos^2 \varphi_0 (1 + y'_1)
\]
\[
- [1 - \cos^2 \varphi_0 (1 + y'_1)]
\]
\[
= \sin \theta \sin \varphi_0 \cos \varphi_0 (1 + y'_1) - [1 - \cos^2 \varphi_0 (1 + y'_1)] (1 - \cos \theta).
\]
\[
(2.25)
\]

\[
\theta = 0 (\varphi = \pi - \varphi_0) \text{ means the intersection point e and the above } x_1 \text{ becomes 0 as it should. Assume}
\]
\[
\theta \neq 0 \quad (\pi - \varphi_0 < \theta < 0, \quad 0 < \theta \leq \pi - \varphi_0).
\]
\[
(2.26)
\]

Then, \( x_1 \) is positive if and only if
\[
\sin \theta \sin \varphi_0 \cos \varphi_0 (1 + y'_1) > [1 - \cos^2 \varphi_0 (1 + y'_1)] (1 - \cos \theta),
\]
\[
(2.27)
\]

that is,
\[
\frac{\sin \theta}{1 - \cos \theta} > \frac{1 - \cos^2 \varphi_0 (1 + y'_1)}{\sin \varphi_0 \cos \varphi_0 (1 + y'_1)}
\]
\[
(2.28)
\]

(recall \( y'_1 \neq -1 \) and \( 0 < \varphi_0 < \pi/2 \)). In other words, the \( x_1 \)-coordinate is positive if and only if
\[
\cot \frac{\theta}{2} > \frac{1 - \cos^2 \varphi_0 (1 + y'_1)}{\sin \varphi_0 \cos \varphi_0 (1 + y'_1)}.
\]
\[
(2.29)
\]

Therefore, for \( y'_1 \neq -1 \) (i.e., except the circle \( C_1 \)), the \( x_1 \)-coordinate is positive for small enough \( \theta > 0 \). \qed

3. Main theorem. In the previous section, we introduced all preliminary lemmas for the main result of this paper. Now, we prove our main theorem.

**Theorem 3.1.** If a one-to-one mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) maps every unit circle onto a unit circle, then \( f \) is a linear isometry up to translation.

**Proof.** We show \( f \) preserves the distance 2. Suppose the distance between \( a = f(A) \) and \( b = f(B) \) is less than 2, while the distance between \( A \) and \( B \) is 2—see Figure 3.1. Then, we show it leads to a contradiction.

Let the distance between \( a \) and \( b \) be \( 2c \) \((0 < c < 1)\). Choose any unit circle \( C \) passing through \( A \) and \( B \) and let \( f(C) = C_1 \). Choose a coordinate for \( a \) and \( b \) as in Figure 3.1 such that \( C_1 \) lies in the \( x_1 \)-plane and
\[
a = \left(-1 - \sqrt{1-c^2}, 0, \ldots, 0, c\right), \quad b = \left(-1 - \sqrt{1-c^2}, 0, \ldots, 0, -c\right).
\]
\[
(3.1)
\]
Let

\[ e = (0, \ldots, 0), \quad \tilde{e} = (-2, 0, \ldots, 0). \]  

(3.2)

Let \( f(E) = e \) and \( \tilde{E} \) the antipodal point (in C) of E and let \( f(\tilde{E}) = d \). Let the union of all the unit circles passing through \( a \) and \( b \) be \( T \) and the \((n-1)\)-dimensional unit sphere passing through \( A \) and \( B \) be \( S \) and the \((n-1)\)-dimensional unit sphere passing through \( e \) and \( \tilde{e} \) be \( S_1 \).

Then, it is clear that any point \( P \) on \( S \) (\( P \notin \{A, B\} \)) lies in some unit circle determined by the three points \( A, B, \) and \( P \). To see this, if we call \( O \) the common center of \( C \) and \( S \), and let

\[ \langle \vec{OP}, \vec{OA} \rangle = \sin \varphi_0 \quad \left( -\frac{\pi}{2} < \varphi_0 < \frac{\pi}{2} \right), \]  

(3.3)

then the unit circle determined by these three points is parameterized as

\[ \overrightarrow{OV}(\varphi) = \cos \varphi \left( \frac{\overrightarrow{OP} - \sin \varphi_0 \overrightarrow{OA}}{\cos \varphi_0} \right) + \sin \varphi \overrightarrow{OA} \quad (-\pi < \varphi \leq \pi). \]  

(3.4)

Note that

\[ \left\{ \left( \frac{\overrightarrow{OP} - \sin \varphi_0 \overrightarrow{OA}}{\cos \varphi_0} \right), \overrightarrow{OA} \right\}, \]  

(3.5)

are orthonormal to each other and

\[ \overrightarrow{OV}(\varphi_0) = \overrightarrow{OP}, \quad \overrightarrow{OV}\left(\frac{\pi}{2}\right) = \overrightarrow{OA}, \]  

\[ \overrightarrow{OV}\left(-\frac{\pi}{2}\right) = -\overrightarrow{OA} = \overrightarrow{OB}. \]  

(3.6)

Since the image of this unit circle lies in \( T \), it follows that the image of the whole \( S \) under \( f \) lies in \( T \).
It is also obvious that the \( x_1 \)-coordinate of any point in \( T \) is nonpositive. (Note that the center of any unit circle passing through \( a \) and \( b \) has coordinate
\[
\sqrt{1-c^2}y - \left(1 + \sqrt{1-c^2},0,...,0\right)
\]
for some \( y \in Y \), (3.7) and the distance between this center and any \( x = (x_1,...,x_n) \) is
\[
\sqrt{(x_1 + 1 + \sqrt{1-c^2}(1-y_1))^2 + \cdots}
\]
(3.8) and because
\[
\sqrt{1-c^2}(1-y_1) \geq 0,
\]
(3.9) positive \( x_1 \) makes the distance larger than 1, which means that if \( x_1 > 0 \), we have \( x \notin T \).)

Now, if \( d = \tilde{e} \), then the image of any unit circle passing through \( E \) and \( \tilde{E} \) lies in both \( T \) and \( S_1 \). However, by Lemma 2.1, \( T \cap S_1 = C_1 \) and this fact contradicts the injectivity of \( f \).

On the other hand, if \( d \neq \tilde{e} \), the image of any unit circle, except the circle \( C \), passing through \( E \) and \( \tilde{E} \) is a unit circle passing through \( e \) and \( d \). This unit circle is not \( C_1 \) since \( f \) is one-to-one, and by Lemma 2.2 it cannot stay completely in \( T \), a contradiction.

Consequently, \( f \) preserves the distance 2. According to the well-known theorem of Beckman and Quarles, \( f \) is a linear isometry up to translation. \( \square \)

**ACKNOWLEDGMENT.** This work was supported by Korea Research Foundation Grant KRF-2003-015-C00023.

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