COEFFICIENT ESTIMATES FOR RUSCHEWEYH DERIVATIVES

MASLINA DARUS and AJAB AKBARALLY

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We consider functions f , analytic in the unit disc and of the normalized form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For functions $f \in \tilde{R}_{\delta}(\beta)$, the class of functions involving the Ruscheweyh derivatives operator, we give sharp upper bounds for the Fekete-Szegő functional $|a_3 - \mu a_2^2|$.

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1. Introduction. Let S denote the class of normalized analytic univalent functions f defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

in the unit disc $D = \{z : |z| < 1\}$. Suppose that

$$\begin{aligned} S^* &= \left\{ f \in S : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, z \in D \right\}, \\ S^*(\beta) &= \left\{ f \in S : \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\beta\pi}{2}, z \in D \right\} \end{aligned} \quad (1.2)$$

are classes of starlike and strongly starlike functions of order β ($0 < \beta \leq 1$), respectively. Note that $S^*(\beta) \subset S^*$ for $0 < \beta < 1$ and $S^*(1) = S^*$ [5]. Kanas [2] introduced the subclass $\tilde{R}_{\delta}(\beta)$ of function $f \in S$ as the following.

DEFINITION 1.1. For $\delta \geq 0$, $\beta \in (0, 1]$, a function f normalized by (1.1) belongs to $\tilde{R}_{\delta}(\beta)$ if, for $z \in D - \{0\}$ and $D^{\delta}f(z) \neq 0$, the following holds:

$$\left| \arg \frac{z(D^{\delta}f(z))'}{D^{\delta}f(z)} \right| \leq \frac{\beta\pi}{2}, \quad (1.3)$$

where $D^{\delta}f$ denotes the generalized Ruscheweyh derivative which was originally defined as the following.

DEFINITION 1.2 [6]. Let $D^n f$ and f be defined by (1.1). Then for $n \in \mathbb{N} \cup \{0\}$,

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z), \quad (1.4)$$

where $*$ denotes the Hadamard product of two analytic functions and $\mathbb{N}$ is a set of natural numbers.

Later in [1], Al-Amiri generalized the Ruscheweyh derivative D^δ for real numbers $\delta \geq -1$ as a Hadamard product of the functions f and $z/(1-z)^{\delta+1}$.

Note that $\tilde{R}_0(\beta) = S^*(\beta)$ for each $\beta \in (0, 1]$ and $\tilde{R}_0(1) = S^*$. In this note, we obtain sharp estimates for $|a_2|$, $|a_3|$ and the Fekete-Szegő functional for the class $\tilde{R}_\delta(\beta)$. For the subclass S^* , sharp upper bounds for the functional $|a_3 - \mu a_2^2|$ have been obtained for all real μ [3, 4].

2. Preliminary results. In proving our results, we will need the following lemmas. However, we first denote P to be the class of analytic functions with positive real part in D .

LEMMA 2.1. Let $p \in P$ and let it be of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ with $\text{Re}(p(z)) > 0$. Then

- (i) $|c_n| \leq 2$ for $n \geq 1$,
- (ii) $|c_2 - c_1^2/2| \leq 2 - |c_1|^2/2$.

LEMMA 2.2. Let $\delta \geq 0$ and $\beta \in (0, 1]$. If $f \in \tilde{R}_\delta(\beta)$ and is given by (1.1), then

$$|a_2| \leq \frac{2\beta}{\delta+1},$$

$$|a_3| \leq \begin{cases} \frac{2\beta}{(\delta+2)(\delta+1)} & \text{if } \beta \leq \frac{1}{3}, \\ \frac{6\beta^2}{(\delta+2)(\delta+1)} & \text{if } \beta \geq \frac{1}{3}. \end{cases} \quad (2.1)$$

PROOF. Let $F(z) = D^\delta f(z) = z + A_2 z^2 + A_3 z^3 + \dots$. Since $f \in \tilde{R}_\delta(\beta)$ and $D^\delta f(z) \in S^*(\beta)$, then

$$\frac{zF'(z)}{F(z)} = p^\beta(z) \quad (2.2)$$

and so

$$\frac{z(1 + 2A_2 z + 3A_3 z^2 + \dots)}{z + A_2 z^2 + A_3 z^3 + \dots} = (1 + c_1 z + c_2 z^2 + \dots)^\beta, \quad (2.3)$$

which implies that

$$z + 2A_2 z^2 + 3A_3 z^3 + \dots = z + (\beta c_1 + A_2)z^2 + \left(\beta c_2 + \frac{\beta(\beta-1)}{2}c_1^2 + \beta A_2 c_1 + A_3\right)z^3 + \dots \quad (2.4)$$

Equating the coefficients, we have

$$A_2 = \beta c_1, \quad (2.5)$$

$$A_3 = \frac{\beta}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{3}{4} \beta^2 c_1^2. \quad (2.6)$$

Now, for $\delta \geq -1$, $D^\delta f$ has the Taylor expansion

$$D^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\delta)}{(n-1)!\Gamma(1+\delta)} a_n z^n, \quad z \in D, \quad (2.7)$$

where $\Gamma(n+\delta)$ denotes Euler's functions with

$$\Gamma(n+\delta) = \delta(\delta+1) \cdots (\delta+n-1)\Gamma(\delta). \quad (2.8)$$

Then

$$z + A_2 z^2 + A_3 z^3 + \cdots = z + \frac{\Gamma(2+\delta)}{\Gamma(1+\delta)} a_2 z^2 + \frac{\Gamma(3+\delta)}{2\Gamma(1+\delta)} a_3 z^3 + \cdots. \quad (2.9)$$

Equating the coefficients in (2.9), we have

$$a_2 \frac{\Gamma(2+\delta)}{\Gamma(1+\delta)} = a_2(\delta+1) = A_2. \quad (2.10)$$

Then, from (2.5), we obtain

$$a_2 = \frac{\beta c_1}{\delta+1}. \quad (2.11)$$

It follows that from Lemma 2.1(i)

$$|a_2| \leq \frac{2\beta}{\delta+1}, \quad (2.12)$$

whereas the coefficient of z^3 in (2.9) is

$$a_3 \frac{\Gamma(3+\delta)}{2\Gamma(1+\delta)} = a_3 \frac{(\delta+1)(\delta+2)}{2} = A_3. \quad (2.13)$$

From (2.6), we obtain

$$a_3 = \frac{2}{(\delta+1)(\delta+2)} \left[\frac{\beta}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{3}{4} \beta^2 c_1^2 \right]. \quad (2.14)$$

It follows from Lemma 2.1(ii) that

$$|a_3| \leq \frac{2}{(\delta+1)(\delta+2)} \left[\frac{\beta}{2} \left(2 - \frac{|c_1|^2}{2} \right) + \frac{3}{4} \beta^2 |c_1|^2 \right], \quad (2.15)$$

that is,

$$|a_3| \leq \begin{cases} \frac{2\beta}{(\delta+2)(\delta+1)} & \text{if } \beta \leq \frac{1}{3}, \\ \frac{6\beta^2}{(\delta+2)(\delta+1)} & \text{if } \beta \geq \frac{1}{3}. \end{cases} \quad (2.16)$$

□

3. Results. We first consider the functional $|a_3 - \mu a_2^2|$ for complex μ .

THEOREM 3.1. Let $f \in \tilde{R}_\delta(\beta)$ and $\beta \in (0, 1]$. Then for μ complex,

$$|a_3 - \mu a_2^2| \leq \frac{2\beta}{(\delta+1)(\delta+2)} \max \left[1, \frac{|\beta(3(\delta+1) - 2\mu(\delta+2))|}{(\delta+1)} \right]. \quad (3.1)$$

For each μ there is a function in $\tilde{R}_\delta(\beta)$ such that equality holds.

PROOF. From (2.11) and (2.14), we write

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{2}{(\delta+1)(\delta+2)} \left[\frac{\beta}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{3}{4} \beta^2 c_1^2 \right] - \mu \left(\frac{\beta c_1}{\delta+1} \right)^2, \\ &= \frac{1}{(\delta+1)(\delta+2)} \left[\beta \left(c_2 - \frac{c_1^2}{2} \right) \right] + \frac{\beta^2(3(\delta+1) - 2\mu(\delta+2))}{2(\delta+1)^2(\delta+2)} c_1^2. \end{aligned} \quad (3.2)$$

It follows from (3.2) and Lemma 2.1(ii) that

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{\beta}{(\delta+1)(\delta+2)} \left(2 - \frac{|c_1|^2}{2} \right) + \left| \frac{\beta^2(3(\delta+1) - 2\mu(\delta+2))}{2(\delta+1)^2(\delta+2)} \right| |c_1|^2, \\ &= \frac{2\beta}{(\delta+1)(\delta+2)} + \frac{|\beta^2(3(\delta+1) - 2\mu(\delta+2))| - \beta(\delta+1)}{2(\delta+1)^2(\delta+2)} |c_1|^2, \end{aligned} \quad (3.3)$$

which on using Lemma 2.1(i), that is, $|c_1| \leq 2$, gives

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\beta}{(\delta+1)(\delta+2)} & \text{if } \kappa(\delta) \leq \beta(\delta+1), \\ \frac{|\beta^2(6(\delta+1) - 4\mu(\delta+2))|}{(\delta+1)^2(\delta+2)} & \text{if } \kappa(\delta) \geq \beta(\delta+1), \end{cases} \quad (3.4)$$

where $\kappa(\delta) = |\beta^2(3(\delta+1) - 2\mu(\delta+2))|$.

Equality is attained for functions in $\tilde{R}_\delta(\beta)$ given by

$$\frac{z(D^\delta f(z))'}{D^\delta f(z)} = \left(\frac{1+z^2}{1-z^2} \right)^\beta, \quad \frac{z(D^\delta f(z))'}{D^\delta f(z)} = \left(\frac{1+z}{1-z} \right)^\beta, \quad (3.5)$$

respectively. □

We next consider the cases where μ is real and prove the following.

THEOREM 3.2. Let $f \in \bar{R}_\delta(\beta)$ and $\beta \in (0, 1]$. Then for μ real,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\beta^2(6(\delta+1) - 4\mu(\delta+2))}{(\delta+1)^2(\delta+2)} & \text{if } \mu \leq \frac{(6\beta-2)(\delta+1)}{4\beta(\delta+2)}, \\ \frac{2\beta}{(\delta+1)(\delta+2)} & \text{if } \frac{(6\beta-2)(\delta+1)}{4\beta(\delta+2)} \leq \mu \leq \frac{(2+6\beta)(\delta+1)}{4\beta(\delta+2)}, \\ \frac{\beta^2(4\mu(\delta+2) - 6(\delta+1))}{(\delta+1)^2(\delta+2)} & \text{if } \mu \geq \frac{(2+6\beta)(\delta+1)}{4\beta(\delta+2)}. \end{cases} \quad (3.6)$$

For each μ , there is a function in $\bar{R}_\delta(\beta)$ such that equality holds.

PROOF. Here we consider two cases.

Case (i): $\mu \leq 3(\delta+1)/2(\delta+2)$.

In this case, (3.2) and Lemma 2.1(ii) give

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{\beta}{(\delta+1)(\delta+2)} \left(2 - \frac{|c_1|^2}{2} \right) + \frac{\beta^2(6(\delta+1) - 4\mu(\delta+2))}{4(\delta+1)^2(\delta+2)} |c_1|^2, \\ &= \frac{2\beta}{(\delta+1)(\delta+2)} + \frac{\beta^2(6(\delta+1) - 4\mu(\delta+2)) - 2\beta(\delta+1)}{4(\delta+1)^2(\delta+2)} |c_1|^2, \end{aligned} \quad (3.7)$$

and so, using the fact that $|c_1| \leq 2$, we obtain

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\beta^2(6(\delta+1) - 4\mu(\delta+2))}{(\delta+1)^2(\delta+2)} & \text{if } \mu \leq \frac{(6\beta-2)(\delta+1)}{4\beta(\delta+2)}, \\ \frac{2\beta}{(\delta+1)(\delta+2)} & \text{if } \frac{(6\beta-2)(\delta+1)}{4\beta(\delta+2)} \leq \mu \leq \frac{3(\delta+1)}{2(\delta+2)}. \end{cases} \quad (3.8)$$

Equality is attained on choosing $c_1 = c_2 = 2$ and $c_1 = 0, c_2 = 2$, respectively, in (3.2).

Case (ii): $\mu \geq 3(\delta+1)/2(\delta+2)$.

It follows from (3.2) and Lemma 2.1(ii) that

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{\beta}{(\delta+1)(\delta+2)} \left(2 - \frac{|c_1|^2}{2} \right) + \frac{\beta^2(4\mu(\delta+2) - 6(\delta+1))}{4(\delta+1)^2(\delta+2)} |c_1|^2, \\ &= \frac{2\beta}{(\delta+1)(\delta+2)} + \frac{\beta^2(4\mu(\delta+2) - 6(\delta+1)) - 2\beta(\delta+1)}{4(\delta+1)^2(\delta+2)} |c_1|^2, \end{aligned} \quad (3.9)$$

and so, using the fact that $|c_1| \leq 2$, we obtain

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\beta}{(\delta+1)(\delta+2)} & \text{if } \frac{3(\delta+1)}{2(\delta+2)} \leq \mu \leq \frac{(6\beta+2)(\delta+1)}{4\beta(\delta+2)}, \\ \frac{\beta^2(4\mu(\delta+2) - 6(\delta+1))}{(\delta+1)^2(\delta+2)} & \text{if } \mu \geq \frac{(6\beta+2)(\delta+1)}{4\beta(\delta+2)}. \end{cases} \quad (3.10)$$

Equality is attained on choosing $c_1 = 0, c_2 = 2$ and $c_1 = 2i, c_2 = -2$, respectively, in (3.2). Thus the proof is complete. $\square$

THEOREM 3.3. *Let $f \in \bar{R}_\delta(\beta)$ and let it be given by (1.1). Then*

$$|a_3| - |a_2| \leq \frac{2\beta}{(\delta+1)(\delta+2)} \quad \text{if } \beta \leq \frac{3(\delta+1)}{5\delta+1}. \quad (3.11)$$

PROOF. Write

$$|a_3| - |a_2| \leq \left| a_3 - \frac{2}{3}a_2^2 \right| + \frac{2}{3}|a_2|^2 - |a_2|. \quad (3.12)$$

Then since $(6\beta - 2)(\delta + 1)/4\beta(\delta + 2) \leq 2/3$ for $\beta \leq 3(\delta + 1)/(5\delta + 1)$, it follows from Theorem 3.2 that

$$|a_3| - |a_2| \leq \frac{2\beta}{(\delta+1)(\delta+2)} + \frac{2}{3}|a_2|^2 - |a_2| = \lambda(x), \quad (3.13)$$

where $x = |a_2| \in [0, 2\beta/(\delta + 1)]$. Since $\lambda(x)$ attains its maximum value at $x = 0$, the theorem follows. This is sharp. $\square$

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Maslina Darus: School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor Darul Ehsan, Malaysia

E-mail address: maslina@pkriscc.ukm.my

Ajab Akbarally: School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor Darul Ehsan, Malaysia